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# FUNDAMENTAL PSEUDO $BCK$-ALGEBRAS

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Abstract. In this paper, we define the relations $\beta$ and $\beta^*$ on hyper pseudo $BCK$-algebras and investigate some related properties. We give a necessary and sufficient condition for $\beta^*$ to be regular. By using $\beta^*$, we make the quotient hyper pseudo $BCK$-algebra. Finally, by applying the concept of fundamental on pseudo $BCK$-algebra, we prove that any pseudo $BCK$-algebra is fundamental.

**Key Words:** Pseudo $BCK$-algebra, Regular congruence, Strongly congruence, Fundamental pseudo $BCK$-algebra.
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## 1. Introduction

The study of $BCK$-Algebras was initiated by Y. Imai and K. Iséki [9] in (1966) as a generalization of the concept of set theoretic difference and propositional calculi. Pseudo $BCK$-algebras were introduced by G. Georgescu and A. Iorgulescu [5] as a generalization of $BCK$-algebras in order to give a structure corresponding to pseudo $MV$-algebras, since the bounded commutative $BCK$-algebras correspond to $MV$-algebras. Hyperstructures (also called multi algebras) were introduced in 1934 by F. Marty [11] at the 8th congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyperstructures and developed it. Hyperstructures have many applications to several sectors of both pure and applied sciences. For example, a recent book [2] contains a wealth of applications. In this book, Corsini and Leoreanu

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presented some of the numerous applications of algebraic hyperstructure, especially those from last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. In [1, 10] R.A. Borzooei et al. applied the hyperstructures to (pseudo) $BCK$-algebra, and investigated some related properties. In this paper, we define the relations $\beta$ and $\beta^*$ on hyper pseudo $BCK$-algebras and investigate some related properties. Then we obtain a necessary and sufficient condition for the relation $\beta^*$ to be regular. By applying $\beta^*$ on hyper pseudo $BCK$-algebra, we make the quotient of hyper pseudo $BCK$-algebra. Finally, by considering the concept of fundamental on pseudo $BCK$-algebra, we define the fundamental pseudo $BCK$-algebra and show that any pseudo $BCK$-algebra is fundamental.

## 2. Preliminary

**Definition 2.1.** [5] *A pseudo $BCK$-algebra is a structure $(X;*,\diamond,0)$, where $*$ and $\diamond$ are binary operations on $X$ and $0$ is a constant element of $X$ that satisfies the following:*

(a1) $(x*y)\diamond(x*z)\preceq z*y$ , $(x\diamond y)*(x\diamond z)\preceq z\diamond y$,
(a2) $x*(x\diamond y)\preceq y$, $x\diamond(x*y)\preceq y$,
(a3) $x\preceq x$,
(a4) $0\preceq x$,
(a5) $x\preceq y$, $y\preceq x$ *implies* $x=y$,
(a6) $x\preceq y\Leftrightarrow x*y=0\Leftrightarrow x\diamond y=0$,

*for all $x,y,z\in X$.*

**Definition 2.2.** [1] *A hyper pseudo $BCK$-algebra is a structure $(H;\circ,*,0)$ where $\circ$ and $*$ are hyper operations on $H$ and $0$ is a constant element that satisfies the following axioms:*

*(PHK1)* $(x\circ z)\circ(y\circ z)\ll x\circ y$, $\quad(x*z)*(y*z)\ll x*y$,
*(PHK2)* $(x\circ y)*z=(x*z)\circ y$,
*(PHK3)* $x\circ y\ll x$, $\quad x*y\ll x$,
*(PHK4)* $x\ll y$ *and* $y\ll x$ *imply* $x=y$,

*for all $x,y,z\in H$, where $x\ll y\Leftrightarrow 0\in x\circ y\Leftrightarrow 0\in x*y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A,\exists b\in B$ such that $a\ll b$.*

**Proposition 2.3.** [1] *In any hyper pseudo $BCK$-algebra $H$, the following hold:*

*(i)* $0\circ 0=0$, $\quad 0*0=0\quad x\circ 0=x$, $\quad x*0=x$,

*(ii)* $0 \ll x, \quad x \ll x, \quad A \ll A,$
*(iii)* $0 \circ x = 0, \quad 0 * x = 0, \quad 0 \circ A = 0 , \quad 0 * A = 0,$
*(iv)* $A \subseteq B$ *implies* $A \ll B,$
*(v)* $A \ll 0$ *implies* $A = \{0\},$
*(vi)* $(A \circ c) \circ (B \circ c) \ll A \circ B, \quad (A * c) * (B * c) \ll A * B,$
*for all* $x, y, z, c \in H$ *and* $A, B \subseteq H$.

**Definition 2.4.** [1, 7] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra. For any subset* $I$ *of* $H$ *and any element* $y \in H$*, we denote,*
*(1)* $*(y, I)^{\ll} = \{x \in H | x * y \ll I\}$, *(2)* $\circ(y, I)^{\ll} = \{x \in H | x * y \ll I\}$,
*(3)* $*(y, I)^{\cap} = \{x \in H | x * y \cap I \neq \emptyset\}$, *(4)* $\circ(y, I)^{\cap} = \{x \in H | x \circ y \cap I \neq \emptyset\}$,
*(5)* $*(y, I)^{\subseteq} = \{x \in H | x * y \subseteq I\}$, *(6)* $\circ(y, I)^{\subseteq} = \{x \in H | x * y \subseteq I\}$.

**Definition 2.5.** [1] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra,* $I \subseteq H$ *and* $0 \in I$*, then* $I$ *is called a pseudo* $BCK$*-ideal of type 1 of* $H$ *if*

$$(\forall y \in I) \ *(y, I)^{\ll} \subseteq I \text{ and } \circ(y, I)^{\ll} \subseteq I.$$

**Definition 2.6.** [7] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra,* $I \subseteq H$ *and* $0 \in I$*, then* $I$ *is called a strong hyper pseudo* $BCK$*-ideal of* $H$ *if*

$$(\forall y \in I) \ *(y, I)^{\cap} \subseteq I \textit{ and } \circ (y, I)^{\cap} \subseteq I.$$

**Theorem 2.7.** [7] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra and* $I \subseteq H$*, then* $I$ *is a strong hyper pseudo* $BCK$*-ideal of* $H$ *if and only if the following hold:*

*(i)* $0 \in I$,
*(ii) for any* $y \in I$, $*(y, I)^{\cap} \subseteq I$ *or for any* $y \in I$, $\circ(y, I)^{\cap} \subseteq I$.

**Definition 2.8.** [7] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra and* $I$ *be a subset of* $H$*, then* $I$ *is called reflexive if* $x * x \subseteq I$ *and* $x \circ x \subseteq I$ *for all* $x \in H$.

**Definition 2.9.** [7] *Let* $H$ *be a hyper pseudo* $BCK$*-algebra,* $\rho$ *be a binary relation on* $H$ *and* $A, B \subseteq H$*, then*
*(i)* $A \rho B$ *means that there exist* $a \in A$ *and* $b \in B$ *such that* $a \rho b$*;*
*(ii)* $A \overline{\rho} B$ *means that for any* $a \in A$ *there exists* $b \in B$ *such that* $a \rho b$*, and for any* $b \in B$ *there exists* $a \in A$ *such that* $a \rho b$*;*
*(iii)* $\rho$ *is called a right* $*$*- congruence ( right* $\circ$*-congruence) relation on* $H$ *if* $a \rho b$ *implies* $(a * u) \overline{\rho} (b * u)$ $((a \circ u) \overline{\rho} (b \circ u))$ *for all* $u \in H$*;*
*(iv)* $\rho$ *is called a left* $*$*- congruence ( left* $\circ$*-congruence) on* $H$ *if* $a \rho b$ *implies* $(u * a) \overline{\rho} (u * b)$ $(u \circ a \overline{\rho} u \circ b)$ *for all* $u \in H$*;*
*(v)* $\rho$ *is called a* $*$*-congruence (*$\circ$*-congruence) on* $H$ *if it is a right and left* $*$*-congruence (a right and left* $\circ$*-congruence);*

*(vi) $\rho$ is called a left congruence on $H$ if it is a left $*$-congruence and a left $\circ$-congruence on $H$;*
*(vii) $\rho$ is called a right congruence on $H$ if it is a right $*$-congruence and a right $\circ$-congruence on $H$;*
*(viii) $\rho$ is called a congruence on $H$ if it is a $*$-congruence and a $\circ$-congruence on $H$.*

From now on, $H$ stands for a hyper pseudo $BCK$-algebra unless otherwise state.

**Definition 2.10.** [8] *Let $\rho$ be a congruence on $H$ and $\frac{H}{\rho} = \{[x]_\rho \,|\, x \in H\}$. We define the hyperoperations $*$ and $\circ$ and the relation $\ll$ on $\frac{H}{\rho}$ as follows:*

$$[x]_\rho * [y]_\rho = \{[z]_\rho | z \in x * y\}, \quad [x]_\rho \circ [y]_\rho = \{[z]_\rho | z \in x \circ y\},$$
$$[x]_\rho \ll [y]_\rho \Leftrightarrow [0]_\rho \in [x]_\rho \circ [y]_\rho \Leftrightarrow [0]_\rho \in [x]_\rho * [y]_\rho.$$

**Theorem 2.11.** [8] *Let $\rho$ be a congruence on $H$, then the following are equivalent:*
*(i) $(x * y)\rho 0$ and $(y * x)\rho 0 \Rightarrow x\rho y$,*
*(ii) $(x \circ y)\rho 0$ and $(y \circ x)\rho 0 \Rightarrow x\rho y$,*
*(iii) $(\frac{H}{\rho}; *, \circ, [0]_\rho)$ is a hyper pseudo $BCK$-algebra.*

**Definition 2.12.** [8] *Let $\rho$ be an equivalence relation on $H$, then $\rho$ is called regular on $H$ if it satisfies one of the conditions of Theorem 2.11.*

**Theorem 2.13.** [8] *If $\rho$ is a regular congruence on $H$, then $[0]_\rho$ is a hyper pseudo $BCK$-ideal of type 1.*

**Theorem 2.14.** [8] *Let $\rho$ be a regular congruence on $H$, then*

$$[0]_\rho \text{ is a reflexive hyper pseudo } BCK\text{-ideal of type 1}$$
$$\Leftrightarrow \frac{H}{\rho} \text{is a pseudo } BCK\text{-algebra.}$$

**Theorem 2.15.** [8] *Let $(H_1; *_1, \circ_1, 0_1)$ and $(H_2; *_2, \circ_2, 0_2)$ be two hyper pseudo $BCK$-algebras and $H = H_1 \times H_2$. We define the hyperoperations $*$ and $\circ$ and the relation $\ll$ on $H$ as follows:*

$$(a_1, a_2) \circ (b_1, b_2) = (a_1 \circ_1 b_1, a_2 \circ_2 b_2) = \{(x, y) | x \in a_1 \circ b_1 \text{ and } y \in a_2 \circ b_2\},$$
$$(a_1, a_2) * (b_1, b_2) = (a_1 *_1 b_1, a_2 *_2 b_2) = \{(x, y) | x \in a_1 * b_1 \text{ and } y \in a_2 * b_2\},$$
$$(a_1, a_2) \ll (b_1, b_2) \text{ if and only if } a_1 \ll b_1 \text{ and } a_2 \ll b_2,$$

*for all $(a_1, a_2), (b_1, b_2) \in H$, then $(H; \circ, *, (0, 0))$ is a hyper pseudo $BCK$-algebra, which is called hyper product of $H_1$ and $H_2$.*

## 3. Relations $\beta$ and $\beta^*$

The relations $\beta$ and $\beta^*$ have been defined on hyperstructure [3, 15]. In this section, we apply these kind of relations to hyper pseudo $BCK$-algebra and investigate some related properties.

**Definition 3.1.** *Let $H$ be a hyper pseudo $BCK$-algebra and $U$ be the set of all finite combinations of elements of $H$ with operation $\otimes$ where $\otimes$ stands for $\circ$ or $*$. We define the relations $\beta$ and $\beta^*$ as follows:*

(i) *$x\beta y \Leftrightarrow \{x, y\} \subseteq u$ for some $u \in U$;*
(ii) *$x\beta^* y$ if and only if there exist $z_1, z_2, ..., z_{n+1} \in H$, where $z_1 = x, z_{n+1} = y$ and $u_i \in U$, $1 \leq i \leq n$, such that $\{z_i, z_{i+1}\} \subseteq u_i$ and $x = z_1\beta z_2\beta z_3...\beta z_n\beta z_{n+1} = y$.*

The following example shows that the relation $\beta$ is not necessarily transitive, in general.

*Example* 3.1. Let $H = \{0, a, b, c, d\}$. Hyperoperations $\circ$ and $*$ on $H$ given by the following tables:

| $\circ$ | $0$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0,a\}$ | $\{a\}$ | $\{a\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0\}$ | $\{b\}$ | $\{0\}$ |
| $c$ | $\{c\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $d$ | $\{d\}$ | $\{d\}$ | $\{a\}$ | $\{d\}$ | $\{0\}$ |

| $*$ | $0$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0,a\}$ | $\{a\}$ | $\{a\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0\}$ | $\{b\}$ | $\{0\}$ |
| $c$ | $\{c\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $d$ | $\{d\}$ | $\{b\}$ | $\{d\}$ | $\{d\}$ | $\{0,d\}$ |

Then $(H; *, \circ, 0)$ is a hyper pseudo $BCK$-algebra. Since $\{0, a\} \subseteq a * a$, it follows that $a\beta 0$. Similarly, we have $0\beta d$. But $(a, d) \notin \beta$ because there is not a combination of elements of $H$ containing $a$ and $d$.

**Theorem 3.2.** *$\beta^*$ is the smallest equivalence relation on $H$ containing $\beta$.*

*Proof.* It is clear that $\beta^*$ is reflexive and symmetric. In order to show that $\beta^*$ is transitive, assume that $a\beta^* b$ and $b\beta^* c$. Thus there exist $x_1, x_2, ..., x_{n+1}$,
$y_1, ...y_{m+1} \in H$ and $u_i$, $v_j \in U, 1 \leq i \leq n+1$, $1 \leq j \leq m+1$ such that $\{x_i, x_{i+1}\} \subseteq u_i, \{y_j, y_{j+1}\} \subseteq v_j$ and

$$a = x_1\beta x_2\beta x_3...\beta x_n\beta x_{n+1} = b = y_1\beta y_2\beta y_3...\beta y_m\beta y_{m+1} = c.$$

Hence $a\beta^* c$. Therefore $\beta^*$ is transitive and so $\beta^*$ is an equivalence relation on $H$. Clearly, by Definition 3.1, $\beta \subseteq \beta^*$. Now let $\rho$ be any other equivalence relation on $H$ such that $\beta \subseteq \rho$. Assume that $a\beta^*b$ for some $a, b \in H$. Thus using Definition 3.1 and the transitivity of $\rho$, we get $a\rho b$. Therefore $\beta^* \subseteq \rho$, which completes the proof. □

**Definition 3.3.** *Let $\rho$ be an equivalence relation on $H$. If $A$ and $B$ are non-empty subsets of $H$, then*
*(i) $A\bar{\bar{\rho}}B$ means that for all $a \in A, b \in B$, we have $a\rho b$;*
*(ii) $\rho$ is called a strongly right $*$-congruence (strongly left $*$-congruence) if for all $x \in H$, $a\rho b$ implies that $(a*x)\bar{\bar{\rho}}(b*x)((x*a)\bar{\bar{\rho}}(x*b))$;*
*(iii)$\rho$ is called a strongly right $\circ$-congruence (strongly left $\circ$-congruence) if for all $x \in H$, $a\rho b$ implies that $(a \circ x)\bar{\bar{\rho}}(b \circ x)((x \circ a)\bar{\bar{\rho}}(x \circ b))$;*
*(iv)$\rho$ is called a strongly $*$-congruence (strongly $\circ$-congruence) if it is left and right strongly $*$-congruence (strongly $\circ$-congruence);*
*(v) $\rho$ is called a strongly congruence if it is strongly $*$-congruence and strongly $\circ$-congruence.*

**Theorem 3.4.** *$\beta^*$ is a strongly congruence on $H$.*

*Proof.* We first show that $\beta$ is a strongly congruence on $H$. For this, let $x\beta y$ and $a \in H$, then there exists $u \in U$ such that $\{x, y\} \subseteq u$. Hence $(a*x) \subseteq (a*u) = v$ and $(a*y) \subseteq (a*u) = v$ for some $v \in U$ and so $(a*x) \cup (a*y) \subseteq (a*u) = v$. Therefore $t\beta t'$ for all $t \in (a*x)$ and $t' \in (a*y)$, that is, $(a*x)\bar{\bar{\beta}}(a*y)$. Therefore $\beta$ is a right strongly $*$-congruence on $H$. Similarly, we can show that $\beta$ is a strongly left $*$-congruence on $H$. Also, by the same argument we can show that $\beta$ is a left and right strongly $\circ$-congruence on $H$. Hence $\beta$ is a strongly congruence on $H$. Now let $a\beta^*b$. Therefore there exist $z_1, z_2, ..., z_{n+1} \in H$ such that $\{z_i, z_{i+1}\} \subseteq u_i \in U$, $1 \leq i \leq n+1$ and $a = z_1\beta z_2\beta z_3...\beta z_n\beta z_{n+1} = b$. Since $\beta$ is a strongly congruence on $H$, we obtain $a*x = (z_1*x)\bar{\bar{\beta}}(z_2*x)\bar{\bar{\beta}}(z_3*x)...\bar{\bar{\beta}}(z_n*x)\bar{\bar{\beta}}(z_{n+1}*x) = b*x$ and so $(a*x)\bar{\bar{\beta}}(b*x)$. Hence $\beta^*$ is a right strongly $*$-congruence on $H$. Similarly, we can show that $\beta^*$ is a strongly left $*$-congruence and strongly left and right $\circ$-congruence on $H$. Therefore $\beta^*$ is a strongly congruence on $H$. □

In the sequel, for any element $a \in H$, we denote the congruence class of $a$ under $\beta^*$ by $\beta^*(a)$ .

In the following, we give an example of a hyper pseudo $BCK$-algebra in which $\beta^*$ is regular.

*Example* 3.2. Let $H = \{0, a, b, c\}$. Hyperoperations $\circ$ and $*$ on $H$ given by the following tables:

| $\circ$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0,a\}$ | $\{0,a\}$ | $\{0,a\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0,a\}$ | $\{0\}$ |
| $c$ | $\{c\}$ | $\{c\}$ | $\{c\}$ | $\{0\}$ |

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{a\}$ | $\{0,a\}$ | $\{0\}$ | $\{0\}$ |
| $b$ | $\{b\}$ | $\{b\}$ | $\{0\}$ | $\{0\}$ |
| $c$ | $\{c\}$ | $\{c\}$ | $\{c\}$ | $\{0\}$ |

Then $(H; *, \circ, 0)$ is a hyper pseudo $BCK$-algebra. We can check that $\beta^*(0) = \{0, a\}$. Since there is not $x, y \in H$ such that both $(x \circ y)\beta^* 0$ and $(y \circ x)\beta^* 0$, it follows that $\beta^*$ is regular.

We now give an example to show that $\beta^*$ is not necessarily regular, in general.

*Example* 3.3. Let $H = \{0, a, b, c, d\}$ be a hyper pseudo $BCK$-algebra as in Example 3.1. We can check that $\beta^*(0) = \{0, a, d\}$. It follows $(c \circ d)\beta^*\{0\}$ and $(d \circ c)\beta^*\{0\}$. But $\beta^*(d) = \{0, a, d\} \neq \{c\} = \beta^*(c)$. This implies that $\beta^*$ is not regular.

**Proposition 3.5.** *For any hyper pseudo $BCK$-algebra $H$, the following are equivalent:*

- *(i) $\beta^*$ is regular;*
- *(ii) For any $x, y \in H$, if $(x \circ y)\beta^*\{0\}$ and $(y \circ x)\beta^*\{0\}$ then there exist $v \in x * (x \circ y)$ and $v' \in y * (y \circ x)$ such that $v\beta^* v'$.*

*Proof.* $(i) \Rightarrow (ii)$ Let $(x \circ y)\beta^*\{0\}$ and $(y \circ x)\beta^*\{0\}$, then there exist $t \in (x \circ y)$ and $t' \in (y \circ x)$ such that $t\beta^* 0$ and $t'\beta^* 0$. Since $\beta^*$ is a strongly congruence on $H$, we get $(x * t)\bar{\bar{\beta}}^*\{x\}$ and $(y * t')\bar{\bar{\beta}}^*\{y\}$. Therefore there exist $v \in x * (x \circ y)$ and $v' \in y * (y \circ x)$ such that $v\beta^* x$ and $v'\beta^* y$. Since $\beta^*$ is regular, it follows from $(x \circ y)\beta^*\{0\}$ and $(y \circ x)\beta^*\{0\}$ that $x\beta^* y$ and so $v\beta^* v'$.

$(ii) \Rightarrow (i)$ Let $(x \circ y)\beta^*\{0\}$ and $(y \circ x)\beta^*\{0\}$. Since $\beta^*$ is a strongly congruence on $H$, we get $x * (x \circ y)\bar{\bar{\beta}}^*\{x\}$ and $y * (y \circ x)\bar{\bar{\beta}}^*\{y\}$. Now by the hypothesis, there exist $v \in x * (x \circ y)$ and $v' \in y * (y \circ x)$ such that $v\beta^* v'$. Since $\beta^*$ is a strongly congruence on $H$, we obtain $x\beta^* v\beta^* v'\beta^* y$ and so by transitivity of $\beta^*$ we have $x\beta^* y$. Therefore $\beta^*$ is regular. □

**Proposition 3.6.** *$[0]_{\beta^*}$ is a reflexive hyper pseudo $BCK$-ideal of type 1 of $H$.*

*Proof.* By Proposition 2.13, $[0]_{\beta^*}$ is a hyper pseudo $BCK$ ideal of type 1. Now let $x \in H$ and $t \in x * x$. Since $0 \in x * x$, we get $t\beta 0$. Hence $t\beta^* 0$

and so $t \in [0]_{\beta^*}$. Therefore $x * x \subseteq [0]_{\beta^*}$. By the same argument, we can show that $x \circ x \subseteq [0]_{\beta^*}$ for all $x \in H$. Therefore $[0]_{\beta^*}$ is a reflexive hyper pseudo $BCK$-ideal of type 1. □

**Theorem 3.7.** *$\beta^*$ is the smallest strongly congruence on $H$ such that:*

(3.1) *$\beta^*(x) * \beta^*(y)$ and $\beta^*(x) \circ \beta^*(y)$ are singleton for all $x, y \in H$.*

*Proof.* By Theorem 3.4, $\beta^*$ is a strongly congruence on $H$. Now let $\beta^*(z), \beta^*(z') \in \beta^*(a) * \beta^*(b)$, then by Definition 2.10, there exist $w, w' \in a * b$ such that $\beta^*(w) = \beta^*(z)$ and $\beta^*(w') = \beta^*(z')$. Since $w, w' \in a * b$, we get $w\beta^* w'$ and so $\beta^*(z) = \beta^*(w) = \beta^*(w') = \beta^*(z')$. Therefore $\beta^*(a) * \beta^*(b)$ is singleton for all $a, b \in H$. Similarly, we can show that $\beta^*(a) \circ \beta^*(b)$ is also singleton for all $a, b \in H$. Now assume that $\rho$ is a strongly congruence on $H$ satisfying the condition (3.1) and prove that $\beta \subseteq \rho$. Consider the natural homomorphism $\pi : H \to \frac{H}{\rho}$ with $\pi(x) = \rho(x)$ for all $x \in H$. Assume that $x\beta y$ for some $x, y \in H$. Then there exists $u \in U$ such that $\{x, y\} \subseteq u$. It follows that $\{\pi(x), \pi(y)\} \subseteq \pi(u) = \rho(u)$. Since $|\rho(u)| = 1$, we get $\pi(x) = \pi(y)$, that is, $\rho(x) = \rho(y)$. Hence $x\rho y$ and so $\beta \subseteq \rho$. Thus it follows from Theorem 3.2 that $\beta^* \subseteq \rho$, which completes the proof. □

**Corollary 3.8.** *If $\beta^*$ is regular on $H$, then $\frac{H}{\beta^*}$ is a pseudo $BCK$-algebra.*

*Proof.* It follows from Theorems 2.14 and 3.6. □

**Theorem 3.9.** *Let $\rho$ be an equivalence relation on $H$, then the following are equivalent.*
*(1) $\rho$ is a regular strongly congruence on $H$.*
*(2) $\frac{H}{\rho}$ is a pseudo $BCK$-algebra.*

*Proof.* (1) $\Rightarrow$ (2) It suffices to show that $\rho(x) * \rho(y)$ and $\rho(x) \circ \rho(y)$ are singleton for all $x, y \in H$. Since $x\rho x$ and $\rho$ is a strongly congruence on $H$, we get $(x * y)\bar{\bar{\rho}}(x * y)$ and $(x \circ y)\bar{\bar{\rho}}(x \circ y)$. This implies that $u\rho u'$ for all $u, u' \in x * y (u, u' \in x \circ y)$. Therefore $\rho(x) * \rho(y)$ and $\rho(x) \circ \rho(y)$ are singleton for all $x, y \in H$ and so the result holds.

(2)$\Rightarrow$(1) Let $\frac{H}{\rho}$ be a pseudo $BCK$-algebra and $x\rho y$, then $\rho(x) * \rho(t) = \rho(y) * \rho(t)$ is singleton for all $t \in H$. Hence for any $a \in x * t$ and $b \in y * t$, $\rho(a) = \rho(b)$. It follows that $(x * t)\bar{\bar{\rho}}(y * t)$. Similarly, we have $(t * x)\bar{\bar{\rho}}(t * y)$. Thus $\rho$ is a strongly $*$-congruence. By the similar argument, we can show that $\rho$ is a strongly $\circ$-congruence. Thus $\rho$ is strongly congruence on $H$. The regularity of $\rho$ follows from Theorem 2.11 and Definition 2.12. □

Note that in the proof of above theorem part (1) $\Rightarrow$ (2), we used only the condition strongly congruence of $\rho$. So we have the following result:

**Corollary 3.10.** *For every strongly congruence $\rho$ on $H$, $\rho(x) * \rho(y)$ and $\rho(x) \circ \rho(y)$ are singleton for all $x, y \in H$.*

**Corollary 3.11.** *For every regular strongly congruence $\rho$ on $H$, $[0]_\rho$ is a reflexive hyper pseudo $BCK$-ideal of type 1.*

*Proof.* It follows from Theorems 2.14 and 3.9. □

Note that if $\theta$ and $\rho$ are two relations on $H_1$ and $H_2$, respectively, then the relation $\theta \times \rho$ on $H_1 \times H_2$ is defined by

$$(\forall (a,b),(c,d) \in H_1 \times H_2)\ (a,b)\theta \times \rho(c,d) \Leftrightarrow a\theta c \text{ and } b\rho d.$$

It is well known that $\theta \times \rho$ is (reflexive, symmetric) transitive if and only if $\theta$ and $\rho$ are both (reflexive, symmetric) transitive.

**Lemma 3.12.** *Let $H_1, H_2$ be two hyper pseudo $BCK$-algebras, then*

(i) $\beta_{H_1 \times H_2} = \beta_{H_1} \times \beta_{H_2}$;
(ii) $\beta^*_{H_1 \times H_2} = \beta^*_{H_1} \times \beta^*_{H_2}$.

*Proof.* $(i)$ Let $U_1$ , $U_2$ and $U$ be the all finite combinations of elements of $H_1$, $H_2$ and $H_1 \times H_2$ respectively, then, by the composition of elements of $H_1 \times H_2$, we get $U = U_1 \times U_2$. Hence we have $(a,b)\beta_{H_1 \times H_2}(c,d) \Leftrightarrow \{(a,b),(c,d)\} \subseteq u$ for some $u \in U \Leftrightarrow \{a,c\} \subseteq u_1$ and $\{b,d\} \subseteq u_2$ for some $u_1 \in U_1$ and $u_2 \in U_2 \Leftrightarrow a\beta_1 c$ and $b\beta_2 d$. This completes the proof of $(i)$.

$(ii)$ By $(i)$, we have $(a,b)\beta_{H_1 \times H_2}(c,d) \Leftrightarrow a\beta_1 c$ and $b\beta_2 d$. Then using Definition 3.1, the proof is straightforward.

□

**Lemma 3.13.** *Let $H_1, H_2$ be two hyper pseudo $BCK$-algebras, then $\beta^*_{H_1 \times H_2}$ is regular if and only if $\beta^*_{H_1}$ and $\beta^*_{H_2}$ are regular.*

*Proof.* Let $\beta^*_{H_1}$ and $\beta^*_{H_2}$ be regular and let $(x_1,y_1) \circ (x_2,y_2)\beta^*_{H_1 \times H_2}\{(0,0)\}$ and $(x_2,y_2) \circ (x_1,y_1)\beta^*_{H_1 \times H_2}\{(0,0)\}$, then there exist $(a,b) \in (x_1,y_1) \circ (x_2,y_2)$ and $(a',b') \in (x_2,y_2) \circ (x_1,y_1)$ such that $(a,b)\beta^*_{H_1 \times H_2}(0,0)$ and $(a',b')\beta^*_{H_1 \times H_2}(0,0)$. Now by Lemma 3.12, we have $a\beta^*_{H_1}0$, $b\beta^*_{H_2}0$, $a'\beta^*_{H_1}0$ and $b'\beta^*_{H_2}0$. It follows that $(x_1 \circ x_2)\beta^*_{H_1}\{0\}$, $(x_2 \circ x_1)\beta^*_{H_1}\{0\}$, $(y_1 \circ y_2)\beta^*_{H_2}\{0\}$ and $(y_2 \circ y_1)\beta^*_{H_2}\{0\}$. Since $\beta^*_{H_1}$ and $\beta^*_{H_2}$ are regular, we get $x_1\beta^*_{H_1}x_2$ and $y_1\beta^*_{H_2}y_2$ and so by Lemma 3.12, we conclude $(x_1,y_1)\beta^*_{H_1 \times H_2}$ $(x_2,y_2)$. Therefore $\beta^*_{H_1 \times H_2}$ is regular.

Conversely, let $x, y \in H_1$ such that $x \circ y \beta^*_{H_1}\{0\}$ and $y \circ x \beta^*_{H_1}\{0\}$. Therefore there exist $a \in x \circ y$ and $b \in y \circ x$ such that $a\beta^*_{H_1}0$ and $b\beta^*_{H_1}0$. Hence $(a, 0)\beta^*_{H_1 \times H_2}(0, 0)$ and $(b, 0)\beta^*_{H_1 \times H_2}(0, 0)$. Since $(a, 0) \in (x, 0) \circ (y, 0)$ and $(b, 0) \in (y, 0) \circ (x, 0)$, we get $(x, 0) \circ (y, 0)\beta^*_{H_1 \times H_2}\{(0, 0)\}$ and $(y, 0) \circ (x, 0)\beta^*_{H_1 \times H_2}\{(0, 0)\}$. Thus by the regularity of $\beta^*_{H_1 \times H_2}$, we obtain $(x, 0)\beta^*_{H_1 \times H_2}(y, 0)$ and so by Lemma 3.12, we have $x\beta^*_{H_1}y$. Therefore $\beta^*_{H_1}$ is regular. Similarly, we can show that $\beta^*_{H_2}$ is also regular. □

**Theorem 3.14.** *Let $(X; *, \circ, 0)$ be a (hyper) pseudo $BCK$-algebra, then for any set $Y$ with $|X| = |Y|$, there are two (hyper) binary operations on $Y$, denoted by $*'$ and $\circ'$ such that $(Y; *', \circ', 0')$ is a (hyper) pseudo $BCK$-algebra and $(X; *, \circ, 0) \cong (Y; *', \circ', 0')$.*

*Proof.* Let $(X; *, \circ, 0)$ be a pseudo $BCK$-algebra. The case that $X$ is a hyper pseudo $BCK$-algebra can be proved in a similar way. By $|X| = |Y|$, there exists a bijection mapping $f : X \to Y$. We define the operations $*'$ and $\circ'$ on $Y$ as follows:

$$(3.2) \qquad (\forall x', y' \in Y) \qquad x' *' y' = f(f^{-1}(x') * f^{-1}(y')),$$

$$(3.3) \qquad x' \circ' y' = f(f^{-1}(x') \circ f^{-1}(y')),$$

$$(3.4) \qquad x' \preceq' y' \Leftrightarrow x' *' y' = 0' \Leftrightarrow x' \circ' y' = 0'.$$

Clearly, the operations $*'$ and $\circ'$ are well-defined. We take $0' = f(0)$. By (3.2) and (3.3), it is easy to see that

$$(3.5) \quad f^{-1}(x' *' y') = f^{-1}(x') * f^{-1}(y') \, ; \, f^{-1}(x' \circ' y') = f^{-1}(x') \circ f^{-1}(y').$$

For any $x', y' \in Y$, we have

$$\begin{aligned} x' \preceq' y' &\Leftrightarrow x' *' y' = 0' \quad \text{by (3.4)} \\ &\Leftrightarrow f(f^{-1}(x') * f^{-1}(y')) = 0' \quad \text{by (3.2)} \\ &\Leftrightarrow f^{-1}(x') * f^{-1}(y') = 0 \quad \text{by injectivity of } f \\ &\Leftrightarrow f^{-1}(x') \preceq f^{-1}(y') \quad \text{by (3.4).} \end{aligned}$$

Consequently,

$$(3.6) \qquad (\forall x', y' \in Y) \; x' \preceq' y' \Leftrightarrow f^{-1}(x') \preceq f^{-1}(y').$$

Now we show that $(Y; *', \circ', 0')$ satisfies the axioms of pseudo $BCK$-algebra. For any $x', y', z' \in Y$, by axiom $(a1)$ of $X$, we have

$$[f^{-1}(x') * f^{-1}(y')] \circ [f^{-1}(x') * f^{-1}(z')] \preceq f^{-1}(z') * f^{-1}(y').$$

Thus, by (3.5), we conclude

$$f^{-1}[(x^{'} *^{'} y^{'}) \circ^{'} (x^{'} *^{'} z^{'})] \preceq f^{-1}(z^{'} *^{'} y^{'}),$$

and so by (3.6), we get $(x^{'} *^{'} y^{'}) \circ^{'} (x^{'} *^{'} z^{'}) \preceq z^{'} *^{'} y^{'}$. Thus the axiom $(a1)$ of $Y$ holds. By the similar argument, we can prove the remainder axioms of pseudo $BCK$-algebra of $Y$. Therefore $(Y, *^{'}, \circ^{'}, 0^{'})$ is a pseudo $BCK$-algebra. □

**Theorem 3.15.** [6] *Let* $(G; ., \vee, \wedge, 0, 1)$ *be a lattice ordered group ($\ell$-group) and* $G^{+} = \{g \in G \mid g \geq 0\}$ *be its positive cone. If one defines* $x*y = x.y^{-1} \vee 0$ *and* $x \circ y = y^{-1} \cdot (x \vee 0)$ *($G$ is not necessarily commutative) then* $(G; *, \circ, 0)$ *is a pseudo $BCK$-algebra.*

*Example* 3.4. [9] Let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. We define binary operation $+$ on $G$ as follows:

$$(k_1, m_1, n_1)+(k_2, m_2, n_2) = \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2) & \text{if } n_2 \text{ is odd} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2) & \text{if } n_2 \text{ is even} \end{cases},$$

$0 = (0, 0, 0)$ is the neutral element, and

$$-(k, m, n) = \begin{cases} (-m, -k, -n) & \text{if } n \text{ is odd} \\ (-k, -m, -n) & \text{if } n \text{ is even} \end{cases},$$

then $G$ is a non abelian $\ell$-group with the positive cone $G^{+} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{Z} \times \{0\}$.

**Corollary 3.16.** *For any infinite countable set $X$, there exist the binary operations $*$ and $\circ$ on $X$ and a constant $0 \in X$ such that $(X; *, \circ, 0)$ is a pseudo $BCK$-algebra.*

*Proof.* It follows from Theorem 3.15 and Example 3.4. □

## 4. Fundamental pseudo $BCK$-algebra

In this section, applying the concept of fundamental algebra [12] on pseudo $BCK$-algebra, we prove that any pseudo $BCK$-algebra is fundamental.

**Definition 4.1.** *A pseudo $BCK$-algebra $H$ is called fundamental, if there exists a non-trivial hyper pseudo $BCK$-algebra $K$ such that* $\frac{K}{\beta^*} \cong H$.

**Theorem 4.2.** *Let $(H_1; *_1, \circ_1, 0_1)$ and $(H_2; *_2, \circ_2, 0_2)$ be two pseudo $BCK$-algebras. Define the hyperoperations $*$ and $\circ$ on $H = H_1 \times H_2$ as follows:*

$$(x_1, y_1) * (x_2, y_2) = \{(x_1 *_1 x_2, y_1), (x_1 *_1 x_2, y_1 *_2 y_2)\}; \tag{4.1}$$

$$(x_1, y_1) \circ (x_2, y_2) = \{(x_1 \circ_1 x_2, y_1), (x_1 \circ_1 x_2, y_1 \circ_2 y_2)\}.$$

*Then $H = (H_1 \times H_2; *, \circ, (0, 0))$ is a hyper pseudo $BCK$-algebra.*

*Proof.* We show that the hyperoperations $*$ and $\circ$ defined on $H$ satisfy the axioms of hyper pseudo $BCK$-algebras. Applying (4.1) and the axiom $(PHK1)$ of $H_1$ and $H_2$, we have, for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in H$,
$[(x_1, y_1) * (x_2, y_2)] * [(x_3, y_3) * (x_2, y_2)]=$
$\{(x_1 *_1 x_2, y_1), (x_1 *_1 x_2, y_1 *_2 y_2)\} * \{(x_3 *_1 x_2, y_3), (x_3 *_1 x_2, y_3 *_2 y_2)\}=$
$\{(x_1 *_1 x_2, y_1) * (x_3 *_1 x_2, y_3)\} \cup \{(x_1 *_1 x_2, y_1) * (x_3 *_1 x_2, y_3 *_2 y_2)\} \cup \{(x_1 *_1 x_2, y_1 *_2 y_2) * (x_3 *_1 x_2, y_3)\} \cup \{(x_1 *_1 x_2, y_1 *_2 y_2) * (x_3 *_1 x_2, y_3 *_2 y_2)\}=$
$\{((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1), ((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1 *_2 y_3),$
$((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1), ((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1 *_2 (y_3 *_2 y_2)),$
$((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1 *_2 y_2), ((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), (y_1 *_1 y_2) *_2 y_3),$
$((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), y_1 *_2 y_2), ((x_1 *_1 x_2) *_1 (x_3 *_1 x_2), (y_1 *_2 y_2) *_2 (y_3 *_2 y_2))\} \ll \{(x_1 * x_3, y_1), (x_1 * x_3, y_1 * y_3)\} = (x_1, y_1) * (x_3, y_3)$.

Similarly, we have $((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_2, y_2)) \ll (x_1, y_1) \circ (x_3, y_3)$. Hence $(PHK1)$ on $H$ holds. Similar to the proof of $(PHK1)$, we can show that the remainder axioms hold. Therefore $H = (H_1 \times H_2; *, \circ, (0, 0))$ is a hyper pseudo $BCK$algebra. □

**Lemma 4.3.** *If $H_1 \times H_2$ is the hyper pseudo $BCK$-algebra as in Theorem 4.2, then $\beta^*_{H_1 \times H_2}$ is regular.*

*Proof.* We first show that $(a, b)\beta(0, 0)$ implies $a = 0$ for any $(a, b) \in H_1 \times H_2$. By the assumption, there exists $u = (x_1, y_1) \otimes (x_2, y_2) ... \otimes (x_{n-1}, y_{n-1}) \otimes (x_n, y_n) \in U$ such that $\{(a, b), (0, 0)\} \subseteq u$. By the definition of hyperoperations $*$ and $\circ$ on $H_1 \times H_2$ as in Theorem 4.2, we have $u = \{(x_1 \otimes x_2 ... \otimes x_{n-1} \otimes x_n, t) \mid t \in T\}$, where
$T = \{y_1\} \cup \{y_1 \otimes y_{i1} \mid 1 < i1 \leq n\} \cup \{y_1 \otimes y_{i1} \otimes y_{i2} \mid 1 < i1 < i2 \leq n\} \cup ... \cup \{y_1 \otimes ... \otimes y_n\}$. By setting $x_0 = x_1 \otimes ... \otimes x_n$, it follows from $\{(a, b), (0, 0)\} \subseteq u$ that $x_0 = 0 = a$. For convenience, we use $\beta^*$ to denote the $\beta^*_{H_1 \times H_2}$. Now let $x \circ y\beta^*(0, 0)$ and $y \circ x\beta^*(0, 0)$, where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. Hence there exist $(a, b) \in x \circ y$ and $(a', b') \in y \circ x$ such that $(a, b)\beta^*(0, 0)$ and $(a', b')\beta^*(0, 0)$ and so by the previous argument $a = 0 = a'$. Thus $(0, b) \in x \circ y = \{(x_1 \circ$

$x_2, y_1), (x_1 \circ x_2, y_1 \circ y_2)\}$ and so $x_1 \circ x_2 = 0$, that is, $x_1 \ll x_2$. Similarly, from $(0, b') \in \{(x_2 \circ x_1, y_2), (x_2 \circ x_1, y_2 \circ y_1)\}$, we get $x_2 \ll x_1$ and so $x_1 = x_2$. Hence $x = (x_1, y_1)$ and $y = (x_1, y_2)$, which imply $x \circ y = \{(0, y_1), (0, y_1 \circ y_2)\}$ and $y \circ x = \{(0, y_2), (0, y_2 \circ y_1)\}$. Since $(x \circ y)\beta^*(0,0)$ and $(y \circ x)\beta^*(0,0)$, it follows that $x \circ (x \circ y)\bar{\bar{\beta}}^* x$ and $y \circ (y \circ x)\bar{\bar{\beta}}^* y$. Now, since

$$(x_1, y_1) \circ ((x_1, y_1) \circ (x_1, y_2)) = \{(x_1, y_1), (x_1, 0), (x_1, y_1 \circ (y_1 \circ y_2))\}$$

and

$$(x_1, y_2) \circ ((x_1, y_2) \circ (x_1, y_1)) = \{(x_1, y_2), (x_1, 0), (x_1, y_2 \circ (y_2 \circ y_1))\},$$

we have $(x_1, 0) \in (x \circ (x \circ y)) \cap (y \circ (y \circ x))$. Hence $(x_1, 0)\beta^* x$ and $(x_1, 0)\beta^* y$ and so $x\beta^* y$. Therefore $\beta^*$ is regular. □

**Theorem 4.4.** *Every pseudo $BCK$-algebra is fundamental.*

*Proof.* Let $H$ be a pseudo $BCK$-algebra and consider the hyper pseudo $BCK$-algebra $H \times H$ as in Theorem 4.2. We show that $\beta^*((a, b)) = \{(a, x) | x \in H\}$ for all $(a, b) \in H \times H$. Let $(c, d)\beta(a, b)$, then $\{(a, b), (c, d)\} \subseteq u$ for some $u \in (x_1, y_1) \otimes ... \otimes (x_n, y_n) \in U$ in which $(x_i, y_i) \in H \times H$, $1 \leq i \leq n$. Thus similar to the proof of Lemma 4.3, it follows from $\{(a, b), (c, d)\} \subseteq u$ that $a = c = x_0$ and so $\beta^*((a, b)) \subseteq \{(a, x) \mid x \in H\}$. On the other hand, since $(a, x) * (0, x) = \{(a, x), (a, 0)\}$, we get $(a, x)\beta(a, 0)$ for all $x \in H$. Hence $(a, x)\beta(a, 0)\beta(a, b)$ and so $(a, x)\beta^*(a, b)$ for all $x \in H$. It follows that $\{(a, x)|x \in H\} \subseteq \beta^*(a, b)$ and so the equality holds. Now we define the function $\varphi : \frac{H \times H}{\beta^*_{H \times H}} \to H$ by $\varphi(\beta^*(a, b)) = a$. We note that $\beta^*(a, b) = \beta^*(c, d)$ if and only if $a = c$ if and only if $\varphi(\beta^*(a, b)) = \varphi(\beta^*(c, d))$. Thus $\varphi$ is well-defined and one to one. Clearly, $\varphi$ is onto. Since

$$\begin{aligned}\beta^*(a, b) * \beta^*(c, d) &= \{\beta^*(t, t') \mid (t, t') \in (a, b) * (c, d)\}\\ &= \{\beta^*(t, t') \mid (t, t') \in \{(a * c, b), (a * c, b * d)\}\}\\ &= \{\beta^*((a * c, t')) \mid t' = b \text{ } or \text{ } t' \in b * d\},\end{aligned}$$

it follows that $\varphi(\beta^*(a, b) * \beta^*(c, d)) = a * c = \varphi(\beta^*(a, b)) * \varphi(\beta^*(c, d))$. Similarly, we have $\varphi(\beta^*(a, b) \circ \beta^*(c, d)) = \varphi(\beta^*(a, b)) \circ \varphi(\beta^*(c, d))$. Thus $\varphi$ is an isomorphism and $\frac{H_1 \times H_2}{\beta^*_{H_1 \times H_2}} = \frac{H_1 \times H_2}{\beta^*} \cong H$. Therefore $H$ is fundamental. □

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