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HYPERSURFACES IN THE GENERAL INNER PRODUCT SPACES

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ABSTRACT. Let A be a symmetric positive definite $(n+1)\times(n+1)$ real matrix for $n \geq 1$ and $S \in R^{n+1}$ be a hypersurface. We are supposed to determine the tangent space T_pS in an arbitrary point $p \in S$ in the case that the whole space R^{n+1} admits the inner product with matrix A. Among other things, some maximum and minimum properties for the vector fields perpendicular to tangent spaces of hypersurfaces, the compatibility of the image or inverse image of a hypersurface and its tangent space under an embedding, an isometry, and a submersion are also pointed out.

Key Words: Hypersurface, Integral curve, Vector field.
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1. Introduction and Preliminaries

The idea of the definition of a regular surface is to introduce a set S, that is, in a certain sense, two dimensional and that also is smooth so that the usual notions of calculus can be extended to it. For example, if $x : U \subseteq R^2 \to S$ be a parameterization of a regular surface S and $p \in U$, then the vector subspace $dx_pR^2 \subseteq R^3$ of dimension 2, coincides with the set of all tangent vectors of S at $x(p)$. In the case that $f : U \subseteq R^3 \to R$ is a smooth map and $c \in f(U)$ is a regular value of f, the set $S = f^{-1}(c)$ is a regular surface in R^3. As a result, in this case, the tangent space of S at p, coincides with $\nabla f(p)^{\perp}$, i.e., the set of all vectors at p which are perpendicular to $\nabla f(p)$ with respect to the usual inner product of

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R^3 [7, 18]. Recently, in last two dedicates, the problem of studying the tangent spaces to some zero sets at some remarkable points is considered by some authors. For example, a description of Zariski tangent spaces of Schubert varieties in some representation-theoretic terms is given in [13, 14, 17]. All the slopes of tangents to a real plane algebraic curve at a particular point also represented in [20]. Bouligand tangent of $S = \{x \in X | F(x) = 0\}$, where X is a Banach space and $F : X \to R^l$ is a sufficiently smooth map, is characterized in [1]. In this paper, we are supposed to characterize the tangent space T_pS for a hypersurface S, in the case that R^{n+1} admits the general inner product $\theta_A(u, v) = uAv^t$ for a symmetric positive definite $(n+1) \times (n+1)$ real matrix A [3]. This is mainly due to the applications to local differential geometry as well as to theory of vector fields.

Investigation the subject of hypersurfaces can be followed through the *hyperstructures*. Hyperstructures represent a suitable generalization of classical ones [8]. Informally, a hyperstructure H consists of a collection of bond sets at various levels $H = \{B_0, B_1, \cdots, B_n\}$ and the bond levels are connected by maps $\partial_i : B_{i+1} \to P(B_i)$ where taking a bond at the level $(i + 1)$ and assigning the collection of bonds at level i that it binds. Bonds may be viewed as a kind of general multirelations, but it should be noted that properties suitably defined at one level, play a role in the construction of the next level. Bonds are constructed or created in order to persist as functional units according to some presheaf property. For instance, let H_0 be a collection of hypersurfaces, and let Ω_0 assigned a vector field χ_0 for each hypersurface. Let B_1 be a bond of a collection of hypersurfaces with vector fields, i.e., $b_1\{b_0^{i_0}, \chi_0^{i_0}\}$ is a hypersurface b_1 where the $b_0^{i_0}$'s are embedded and b_1 has a vector field $\hat{\chi}_1$ extending the $\chi_0^{i_0}$'s. Let Ω_1 assign a vector field χ_1 to each b_1. Possibly $\hat{\chi}_1 = \chi_1$, but this is not required and depends on the situation. Let B_2 be a collection of hypersurfaces $b_2\{b_1^{i_1}, \chi_1^{i_1}\}$. A bond b_2 of this collection will be a hypersurface with the $b_1^{i_1}$'s embedded and with a vector field $\hat{\chi}_2$ extending the $\chi_1^{i_1}$'s. Then the procedure continues to obtain any number of levels [2]. Some of the numerous applications of hyperstructures, especially those that were found and studied in the last fifteen years are to hypergraphs, hypergroups, binary relations, lattices, automata, geometry, fuzzy sets and rough sets [4, 6, 9, 10, 15]. A history and new possible research directions of hyperstructures are also provided in [5]. Before advancing any further, let us bring this section to an end by some notions and definitions on the essential ingredients of this manuscript.

Let $f : U \to R$ be a smooth map, where $U \subseteq R^{n+1}$ is an open set, a point $p \in U$ such that $\nabla f(p) \neq 0$ is called a *regular point* of f. If $c \in R$ be such that $S = f^{-1}(c)$ is non-empty, then S is called a *level set* of f or simply a level set. A level set $S = f^{-1}(c)$ is said a *hypersurface* if $\nabla f(p) \neq 0$ for all $p \in S$. A vector is said to be *tangent* to the level set S, if it is the velocity vector of a smooth parametric curve $\alpha : I \to R^{n+1}$ for some open interval I whose image is contained in S. A *vector field* χ on a hypersurface S, is a map which assigns to each point $p \in S$, a vector $\chi(p) \in T_pR^{n+1}$. Similarly, a *tangent vector field* on a hypersurface S, is a map which assigns to each point $p \in S$ an element of T_pS [7, 18]. A smooth parametric curve $\alpha : I \to R^{n+1}$, is said to be an *integral curve* of the vector field χ defined on the open set $U \subseteq R^{n+1}$, if $\alpha(t) \in U$ and $\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$ [12]. A map $g : S \to R^k$ for some $k \in N$, is called smooth if it is the restriction to S of some smooth function $\widetilde{g} : V \to R^k$ defined on some open set V [16]. A smooth vector field on S is defined similarly. If χ is a smooth tangent vector field on a hypersurface S, then $\alpha : I \to S$ satisfying previous conditions is called an integral curve of χ [11]. The smooth map $G : U \to V$ between two open subsets of R^{n+1} is called an *imbedding* if its derivative is nonsingular and it is a homeomorphism onto its image with the subspace topology [19]. A hypersurface, as a subset of the inner product space (R^{n+1}, θ_A), the norm and the orthogonality in (R^{n+1}, θ_A) are denoted by S_A, $\| . \|_A$ and $\perp_A$ respectively. An *isometry* of R^{n+1}, is a mapping $F : (R^{n+1}, \theta_A) \to (R^{n+1}, \theta_B)$ such that $d_A(p, q) = d_B(F(p), F(q))$ where $d_A(p, q) = \| p - q \|_A$ and $d_B(F(p), F(q)) = \| F(p) - F(q) \|_B$. Finally, if $m, n \in N, m \geq n$, and $U \subseteq R^{m+1}, V \subseteq R^{n+1}$ are two open subsets, then the smooth map $G : U \to V$ is called a *submersion* if the rank of its derivative at all points of U is equal to $n + 1$ [19].

2. Tangent Space of Hypersurfaces

In this section, we are going to find a representation for a hypersurface S_A and its tangent space T_pS_A at an arbitrary point $p \in S_A$.

Theorem 2.1. *Let S_A be a level set and $p \in S_A$, then the vector $\nabla f(p)A^{-1}$ is orthogonal to all vectors tangent to S_A at p.*

Proof. Each vector tangent to S_A at p is of the form $\dot{\alpha}(t_0)$ for some smooth parametric curve $\alpha : I \to R^{n+1}$ with $\alpha(t_0) = p$ and $Im\alpha \subseteq S_A$. But $Im\alpha \subseteq S_A$ implies that $f(\alpha(t)) = c$ for all $t \in I$, so the chain rule implies that

$$\theta_A(\nabla f(p)A^{-1}, \dot{\alpha}(t_0)) = \frac{d}{dt}(f \circ \alpha)(t)_{|t=t_0} = 0.$$

This completes the proof. □

Theorem 2.2. *Let S_A be a level set of f and $p \in S_A$ be a regular point of f, then the set of all vectors tangent to S_A at p is precisely $(\nabla f(p)A^{-1})^{\perp_A}$.*

Proof. It suffices to show that if $V = (p, v) \in (\nabla f(p)A^{-1})^{\perp_A}$, then $V = \dot{\alpha}(0)$ for some smooth parametric curve α with $Im\alpha \subseteq S$. Consider the constant vector field χ on U defined by $\chi(q) = (q, v)$. Let

$$Y(q) = \chi(q) - \frac{\theta_A(\nabla f(q)A^{-1}, \chi(q))}{\theta_A(\nabla f(q)A^{-1}, \nabla f(q))}\nabla f(q), \tag{2.1}$$

Y is defined on an open set U containing p, such that $\nabla f(q) \neq 0$ for all $q \in U$ and $Y(p) = \chi(p) = V \in (\nabla f(p)A^{-1})^{\perp_A}$. Moreover

$$\theta_A(\nabla f(q)A^{-1}, Y(q)) = 0, \tag{2.2}$$

for all $q \in U$. Thus $Y(q)\perp_A \nabla f(q)A^{-1}$ for all $q \in U$. If α is an integral curve of Y through p, then

$$\alpha(0) = p, \quad \dot{\alpha}(0) = Y(\alpha(0)) = Y(p) = \chi(p) = V, \tag{2.3}$$

and

$$\frac{d}{dt}(f \circ \alpha)(t) = \theta_A(\nabla f(\alpha(t))A^{-1}, Y(\alpha)(t))) = 0 \tag{2.4}$$

for all $t \in I$, so $f(\alpha(t)) = c$. Since $f(\alpha(0)) = f(p) = c$, so $Im\alpha \subseteq S_A$. This completes the proof. □

Theorem 2.3. *For any hypersurface S_A and $p \in S_A$, T_pS_A is a hypersurface.*

3. Existence and Uniqueness of Integral Curves of Tangent Vector Fields

The proof of the following theorems is an exploitation of the proof of Theorem 2.2, extends the fundamental theorem of local existence and uniqueness of integral curves (See [7, 18]) to a hypersurface. The fundamental theorem in two references is proved in the case that the vector field χ defined on an open set $U \subseteq R^{n+1}$ as a reformulation of the existence and uniqueness theorem for solutions of systems of first order differential equations [12].

Theorem 3.1. *Let S_A be a hypersurface, and χ be a smooth tangent vector field on S_A. There exists an open interval I containing 0 and a smooth parametric curve $\alpha : I \to S_A$ such that,*

(1) *$\alpha(0) = p$,*
(2) *$\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$,*
(3) *If $\beta : \tilde{I} \to S_A$ is any other parametric curve in S_A satisfying 1 and 2, then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.*

Proof. There exists an open set V containing S_A, and a smooth vector field $\tilde{\chi}$ on V such that $\tilde{\chi}(q) = \chi(q)$ for all $q \in S_A$. Let $f : U \to R$ and $c \in R$ be such that $S_A = f^{-1}(c)$ and $\nabla f(q) \neq 0$ for all $q \in S_A$. Let $W = \{q \in U \cap V | \nabla f(q) \neq 0\}$, then W is an open set containing S_A and both $\tilde{\chi}$ and f are defined on W. Let Y be defined as in (2.1) with $\tilde{\chi}$ instead of χ, then $Y(q) = \tilde{\chi}(q)$ for all $q \in S_A$. Let $\alpha : I \to W$ be the maximal integral curve of Y through p, then α maps I into S_A by (2.3) and (2.4). Conditions 1 and 2 are clearly satisfied, and condition 3 is also satisfied because any $\beta : I \to W$ satisfying 1 and 2 is also an integral curve of the vector field Y on W, so the fundamental theorem of local existence and uniqueness of integral curves of (2.1) applies [7]. This completes the proof. □

Theorem 3.2. *Let S_A be a hypersurface and χ be a smooth tangent vector field on S_A. If $\alpha : I \to U$ is any integral curve of χ such that $\alpha(t_0) \in S_A$ for some $t_0 \in I$, then $\alpha(t) \in S_A$ for all $t \in I$.*

Proof. Suppose $\alpha(t) \notin S_A$ for some $t \in I, t < t_0$. Let

$$t_1 = sup\{t \in I | t < t_0, \alpha(t) \notin S_A\}.$$

Then $f(\alpha(t)) = 0$ for $t_1 < t \leq t_0$, so by continuity $f(\alpha(t_1)) = c$; that is $\alpha(t_1) \in S_A$. Let $\beta : \tilde{I} \to S_A$ be an integral curve through $\alpha(t_1)$ of X. Thus $\beta(0) = \alpha(t_1)$, as the curve $\tilde{\alpha}$ defined by $\tilde{\alpha}(t) = \alpha(t + t_1)$. By Theorem 3.1, $\alpha(t) = \tilde{\alpha}(t - t_1) = \beta(t - t_1)$ for all t such that $t - t_1 \in Dom(\tilde{\alpha}) \cap Dom(\beta)$. But this contradicts the fact that $\alpha(t) \notin S_A$ for values of t arbitrary close to t_1. Hence $\alpha(t) \in S_A$ for all $t \in I$ with $t < t_0$. The proof for $t > t_0$ is similar. □

4. Maximum and Minimum Properties

Lagrange's Multiplier Theorem, is a consequence of the representation of tangent spaces as indicated in Theorem 2.2.

Theorem 4.1. *Let S_A be a hypersurface. Suppose V is an open subset of R^{n+1} containing S_A, $g : V \to R$ be a smooth map and $p \in S_A$ be an extreme point of g on S_A, then there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.*

Proof. The tangent space at p to S_A is $T_pS_A = (\nabla f(p)A^{-1})^{\perp_A}$ by Theorem 2.2. Hence $T_pS_A^{\perp_A}$ is the one dimensional subspace of T_pR^{n+1} spanned by $\nabla f(p)A^{-1}$. Moreover each $v \in T_pS_A$ is of the form $v = \dot{\alpha}(t_0)$ for some smooth parametric curve $\alpha : I \to S_A, t_0 \in I$ with $\alpha(t_0) = p$. Since p is an extreme point of g on S_A, so t_0 is an extreme point of $g \circ \alpha$ on I and

$$0 = \frac{d}{dt}(g \circ \alpha)(t_0) = \theta_A(\nabla g(p)A^{-1}, v)$$

for all $v \in T_pS_A$. Therefore $\nabla g(p)A^{-1} \in T_pS_A^{\perp_A}$ and there exists a real number λ such that $\nabla g(p)A^{-1} = \lambda \nabla f(p)A^{-1}$, i.e., $\nabla g(p) = \lambda \nabla f(p)$. This completes the proof. □

Theorem 4.2. *Let S_A be a hypersurface and $p_0 \in U - S_A$. Then the shortest line segment from p_0 to S_A, if one exists, is perpendicular to S_A, i.e., if $p \in S_A$ be such that $\theta_A(p - p_0, p - p_0) \leq \theta_A(q - p_0, q - p_0)$ for all $q \in S_A$ then $(p, p - p_0)\perp_A S_A$. The same conclusion also holds for the longest segment from p_0 to S_A.*

Proof. Let S_A be a hypersurface in U and $g : U \to R$ be defined by $g(u) = \theta_A(u - p_0, u - p_0)$. Then a computation using differentiation laws of smooth maps yields that $\nabla g(u) = 2(u - p_0)A$. Since g takes its minimum on S_A at p, Theorem 4.1 implies that $2(p - p_0)A = \lambda \nabla f(p)$ for some $\lambda \in R - \{0\}$ and so $(p, p - p_0)$ is parallel to $\nabla f(p)A^{-1}$. The proof of the longest segment is similar. □

Since a smooth map attains its extremums on a compact set, one can deduce the following Theorem.

Theorem 4.3. *Let S_A be a compact hypersurface and $p_0 \in U - S_A$. Then there exist points $p_i \in S_A, i = 1, 2$ such that $(p_i, p_i - p_0)\perp_A S_A$.*

In the following, we are going to present a general property of the integral curves of the vector fields $\pm(\nabla f)A^{-1}$.

Theorem 4.4. *If $f : U \to R$ be a smooth map, and $\alpha : I \to U$ be an integral curve of $(\nabla f)A^{-1}$ (res. $-(\nabla f)A^{-1}$), then for each $t_0 \in I$, the map f is increasing faster (res. slower) along α at $\alpha(t_0)$ than along any* *other curve passing through $\alpha(t_0)$ with the same speed, i.e., if $\beta : \tilde{I} \to U$ is such that $\beta(s_0) = \alpha(t_0)$ for some $s_0 \in \tilde{I}$ and $\|\dot{\beta}(s_0)\| = \|\dot{\alpha}(t_0)\|$, then $\frac{d}{dt}(f \circ \beta)(s_0) \leq \frac{d}{ds}(f \circ \alpha)(t_0)$ (res. $\frac{d}{dt}(f \circ \beta)(s_0) \geq \frac{d}{ds}(f \circ \alpha)(t_0)$).*

Proof. Let α be an integral curve of $(\nabla f)A^{-1}$. There exists a real number $k \in [-1, 1]$ such that

$$\begin{aligned} \frac{d}{dt}(f \circ \beta)(s_0) &= \theta_A(\nabla f(\beta(s_0))A^{-1}, \dot{\beta}(s_0)) \\ &= k\|\nabla f(\beta(s_0))A^{-1}\|\|\dot{\beta}(s_0)\| \\ &= k\|\nabla f(\alpha(t_0))A^{-1}\|\|\dot{\alpha}(t_0)\| \\ &= k\|\nabla f(\alpha(t_0))A^{-1}\|\|\nabla f(\alpha(t_0))A^{-1}\| \\ &= k\theta_A(\nabla f(\alpha(t_0))A^{-1}, \nabla f(\alpha(t_0))A^{-1}) \\ &\leq \theta_A(\nabla f(\alpha(t_0))A^{-1}, \nabla f(\alpha(t_0))A^{-1}) \\ &= \theta_A(\nabla f(\alpha(t_0))A^{-1}, \dot{\alpha}(t_0)) = \frac{d}{dt}(f \circ \alpha)(t_0). \end{aligned} \tag{4.1}$$

The proof of the theorem for $-(\nabla f)A^{-1}$ parallels that of $(\nabla f)A^{-1}$, as presented above. □

As a consequence of Theorems 3.2 and 4.4 we have the following Theorem.

Theorem 4.5. *Let S_A be a hypersurface with a smooth tangent vector field on it and $p \in S_A$. Then there exists an open interval I and a smooth parametric curve $\alpha : I \to S_A$ through p such that the map f is increasing faster along α at p than along any other curve passing through p (not essentially with image in S_A) with the same speed.*

The following example shows that the converse of Theorem 4.5 is not true.

Example 4.6. Let A be the $(n+1)\times(n+1)$ identity matrix, $p \in R^{n+1} - \{0\}$ and the smooth map $f : R^{n+1} - \{0\} \to R$ be defined by

$$f(x_1, \dots, x_{n+1}) = \frac{1}{2}\Sigma_{i=1}^{i=n+1} x_i^2.$$

Let $t_0 > 0$, I be an enough small interval about t_0, $\beta : I \to R^{n+1}$ be an arbitrary smooth parametric curve and the curves $\alpha, \gamma : I \to R^{n+1}$ are defined by

$$\alpha(t) = p \cdot exp(t - t_0), \quad \gamma(t) = p \cdot exp2(t - t_0).$$

Then α is an integral curve of $(\nabla f)A^{-1}$ through p at t_0, $\gamma(t_0) = p$ and

$$\frac{d}{dt}(f \circ \gamma)(t) = 4exp4(t - t_0)\theta_A(pA^{-1}, p) \tag{4.2}$$

$$> exp(t - t_0)\theta_A(pA^{-1}, p) \tag{4.3}$$

$$= \frac{d}{dt}(f \circ \alpha)(t) \geq \frac{d}{dt}(f \circ \beta)(t). \tag{4.4}$$

Therefore (4.1) is hold for γ instead of α by (4.2) and etc., not only for a single point but for all points of an interval, but γ is not an integral curve of ∇f.

5. Imbeddings, Isometries and Submersions

In this section we are going to investigate the properties of the image of a hypersurface under an imbedding, or an isometry and its inverse image under a submersion.

Let $G : U \to V$ be an imbedding. It follows from the inverse mapping theorem that G maps a neighborhood of a point in U diffeomorphically onto an open set in V. If $f : U \to R$ be a smooth map, and $J_{G^{-1}}((G(q))$ denotes the Jacobian of G^{-1} at $G(q)$, then a computation yields $\nabla(f \circ G^{-1})(G(q)) = \nabla f(q)J_{G^{-1}}(G(q))$ for all $q \in U$. As a result of above discussion and Theorem 2.2 we have the following corollaries.

Corollary 5.1. *Let A, B are two symmetric positive definite $(n+1) \times (n+1)$ real matrices, $G : (U, \theta_A) \to (V, \theta_B)$ be an imbedding, and $S_A = f^{-1}(c)$ be a hypersurface in U, then $G(S_A)$ is a hypersurface in V. Moreover, $T_{G(p)}G(S_A) = (\nabla(f \circ G^{-1})B^{-1})^{\perp_B}$.*

Corollary 5.2. *Let $U \subseteq R^{n+1}$ be an open set and $S_A \subseteq U$. Then S_A is a hypersurface if and only if it is the image of a hypersurface under an imbedding. Let $(p, v) \in T_pS_A$, then $(G(p), vG'(p)) \in T_{G(p)}G(S_A)$ if and only if*

$$\theta_B(\nabla f(p)J_{G^{-1}}(G(p))B^{-1}, vJ_G(p)) = 0.$$

Corollary 5.3. *If $(p, v) \in T_pS_A$ and $J_{G^{-1}}(G(p)(J_G(p))^t = I$ (In particular if $G'(p)$ is a symmetric linear map), then*

$$(G(p), vG'(p)) \in T_{G(p)}G(S_A).$$

Corollary 5.4. *Let A, B are two symmetric positive definite $(n+1) \times (n+1)$ real matrices. Let H_U (res. H_V) be the set of all hypersurfaces in U (res. V). Then, there is a one to one correspondence between H_U and H_V. Moreover, if D be any nonsingular symmetric matrix,*

$\hat{G} : (R^{n+1}, \theta_A) \to (R^{n+1}, \theta_B)$ be defined by $\hat{G}(x) = xD$ and $G = \hat{G}_{|U}$ be its restriction to U, then $(p, v) \in T_p S_A$ if and only if $(G(p), \hat{G}(v)) \in T_{G(p)} G(S_A)$.

Example 5.5. Let A, B are two symmetric positive definite 2×2 real matrices. Let $U = \{(x, y)|x > 0, y > 0\}$, $f : U \to R$ be defined by $f(x, y) = x^2 + y^2$ and $c = 1$. Then $S_A = \{(x, y)|x^2 + y^2 = 1, x > 0, y > 0\}$ and $G : U \to U$ which is defined by $G(x, y) = (2xy, x^2)$ is a nonlinear surjective imbedding with symmetric Jacobian matrix. Moreover, $T_{(a,b)} S_A = \{\lambda(-b, a)|\lambda \in R\}$ for $(a, b) \in S_A$, $G(S_A) = \{(x, y)|x^2 + (2y - 1)^2 = 1, x > 0, y > 0\}$, $T_{G(a,b)} G(S_A) = \{(x, y)|abx + (2a^2 - 1)y = 0\}$. On the other hand, $(x, y)G'(a, b) = (2bx + 2ay, 2ax)$. Thus, if $(x, y) \in T_{(a,b)} S_A$, then $x = -\lambda b, y = \lambda a$ for some λ, and $ab(2bx+2ay)+(2a^2-1)(2ax) = ab(-2\lambda b^2+2\lambda a^2)+(2a^2-1)(-2\lambda ab) = 0$, i.e., $(G(a, b), vG'(a, b)) \in T_{G(a,b)} G(S_A)$ as indicated in Corollary 5.3.

Due to investigating the image of a hypersurface under an isometry, at first, we characterize the structure of an isometry in a special case.

Theorem 5.6. *If F is an isometry of R^{n+1} such that $F(0) = 0$, then F is an orthogonal transformation.*

Proof. By hypothesis, F preserves norm, and $F(0) = 0$; hence $\| F(p) \|_B = d_B(0, F(p)) = d_B(F(0), F(p)) = d_A(0, p) = \| p \|_A$. Since F is an isometry, $d_B(F(p), F(q)) = d_A(p, q)$ for any pair of points. Hence $\theta_B(F(p) - F(q), F(p) - F(q)) = \theta_A(p - q, p - q)$. Since F preserves norms, we have $\theta_B(F(p), F(q)) = \theta_A(p, q)$, as required. It remains to prove that F is linear. Let $u_1, \cdots, u_{n+1}$ be some orthonormal vectors in (R^{n+1}, θ_A). Then $p = \sum p_i u_i$ and $\theta_A(u_i, u_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Since F preserves inner products, so $F(u_1), \cdots, F(u_{n+1})$ must also be orthonormal. Thus $F(p) = \sum \theta_B(F(p), F(u_i))F(u_i) = \sum \theta_A(p, u_i)F(u_i)$. Using this identity, it is a simple matter to check the linearity condition. □

We now give a concrete description of an arbitrary isometry.

Theorem 5.7. *Every isometry of R^{n+1} can be uniquely described as an orthogonal transformation followed by a translation.*

Proof. Let T be translation by $F(0)$. Then, $(T^{-1} \circ F)(0) = T^{-1}(F(0)) = F(0) - F(0) = 0$. Thus by Theorem 5.6, $T^{-1} \circ F$ is an orthogonal transformation, say $T^{-1} \circ F = C$, and so $F = T \circ C$. The proof of uniqueness is straightforward. □

As a result of Theorem 5.7 we have the following Theorem.

Theorem 5.8. *Every isometry of R^{n+1} is a smooth imbedding. Moreover, S_A is a hypersurface if and only if it is the image of a hypersurface under an isometry.*

An imbedding as defined above (res. an isometry) is a submersion. So by an argument using $\nabla(f \circ G)(q) = \nabla f(G(q))J_G(q)$ and $(rank\ G')(q) = n+1$ for all $q \in U$ one can generalize the Corollary 5.1 and Theorem 5.8 as the following Theorem.

Theorem 5.9. *Let A,(res. B) be a symmetric positive definite $(m+1) \times (m+1)$, (res. $(n+1) \times (n+1)$) real matrix. Let $U \subseteq R^{m+1}, V \subseteq R^{n+1}, G : U \to V$ be a submersion, and $f : V \to R$ be a smooth map. Let $c \in R$, and $S_B = f^{-1}(c)$ be a hypersurface in V, then $G^{-1}(S_B)$ is a hypersurface in U.*

Example 5.10. Let A, B are two symmetric positive definite 3×3, real matrices. The simplest construction of one-eighth of *Steiner* hypersurface is as the image of hypersurface $S_A = \{(x,y,z)|x^2 + y^2 + z^2 = 1, x > 0, y > 0, z > 0\}$ under the map $G(x,y,z) = (yz, xz, xy)$. Let $U = \{(x,y,z)|xyz \neq 0\}$, then $G : U \to U$ is an imbedding, and the image of S_A under G is the one-eighth of the Steiner hypersurface $S_B = \{(x,y,z)|x^2y^2 + y^2z^2 + z^2x^2 - xyz = 0, x > 0, y > 0, z > 0\}$. Moreover, $G : U \to U$ is also a submersion, and the inverse image of one-eighth of Steiner hypersurface under G is the hypersurface S_A.

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