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$(n-1,n)$-WEAKLY PRIME SUBMODULES IN DIRECT PRODUCT OF MODULES

MAHDIEH EBRAHIMPOUR

Abstract. Let $n \geq 2$ be a positive integer, R be a commutative ring with identity and M be a unitary R-module . In this paper we study the $(n-1,n)$-weakly prime submodules of direct product of modules. Also, we show that for some special cases, every proper submodule is $(n-1,n)$-weakly prime.

Key Words: Prime submodule, Weakly prime submodule, Quasi-local ring, $(n-1,n)$-weakly prime submodule.
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1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$ and all modules are unital. Let R be a ring, M be an $R-$module and N be a submodule of M. This is easy to show that $(N :_R M) = \{r \in R | rM \subseteq N\}$ is an ideal of R. M is called faithful if $Ann_R(M) = (0 :_R M) = 0$.

The concept of weakly prime submodule, i.e., a proper submodule P of M with the property that $r \in R$ and $x \in M$ together with $0 \neq rx \in P$ imply $x \in P$ or $r \in (P :_R M)$ has been introduced by Nekooei in [12]. In [1],[2] and [3], Ebrahimpour and Nekooei have introduced the concept of $(n-1,n)$-prime.

In [2], Ebrahimpour and Nekooei have said that a proper submodule P of M is $(n-1,n)$-prime if $a_1...a_{n-1}x \in P$ implies $a_1...a_{n-1} \in (P :_R M)$ or $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$, for some $i \in \{1,...,n-1\}$, where

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∗Address correspondence to M. Ebrahimpour; E-mail: m.ebrahimpour@vru.ac.ir.

$a_1,...,a_{n-1} \in R$ and $x \in M$. Note that a $(1,2)$-prime submodule is just a prime submodule. Also, a proper ideal P of R is said to be $(n-1,n)$-weakly prime if $a_1,\dots,a_n \in R$ together with $0 \neq a_1 \dots a_n \in P$ imply $a_1 \dots a_{i-1}a_{i+1} \dots a_n \in P$, for some $i \in \{1,\dots,n\}$, [3]. Other generalizations of prime submodules have been studied in [4],[5],...,[13] and [14].

In this paper we study $(n-1,n)$-weakly prime submodules. We say that a proper submodule P of M is $(n-1,n)$-weakly prime if $0 \neq a_1...a_{n-1}x \in P$ imply $a_1...a_{n-1} \in (P :_R M)$ or $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$, for some $i \in \{1,...,n-1\}$, where $a_1,...,a_{n-1} \in R$ and $x \in M$. So a $(1,2)$-weakly prime submodule is just a weakly prime submodule.

Let (R,Q) be a quasi-local ring and M be an $R-$module. If t is the smallest positive integer such that $Q^tM = 0$, then we say that the associated degree of M is t. If $Q^tM \neq 0$, for all $t \geq 1$, then we say that the associated degree of M is ∞.

Let $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times ... \times M_m$ be an $R-$module, where (R_i,Q_i) is a quasi-local ring and M_i is a non-zero R_i-module and the associated degree of M_i is t_i, for all $i \in \{1,\dots,m\}$.

In Theorem 1.4, we show that if every proper submodule of M is $(n-1,n)$-weakly prime, then $Q_i^{n-m}M_i = 0$, for all $i \in \{1,\dots,m\}$ and some m,n.

In Theorem 1.6, we show that if $\sum_{i=1}^{m} t_i \leq n-1$, then every proper submodule of M is $(n-1,n)$-weakly prime with $n \geq 2$ and $m \geq 1$.

2. MAIN RESULTES

Let $R = R_1 \times \cdots \times R_m$, $M = M_1 \times ... \times M_m$, where R_i is a ring and M_i is an R_i-module, for $i = 1,...,m$. Then every submodule of the R-module M is of the form $N_1 \times ... \times N_m$, where N_i is an R_i-submodule of M_i.

Theorem 2.1. *Let $R = R_1 \times \cdots \times R_m$, $M = M_1 \times ... \times M_m$, where R_i is a ring and M_i is a non-zero torsionfree R_i-module, for $i = 1,...,m$. Let $P = P_1 \times \cdots \times P_m$ be an $(n-1,n)$-weakly prime submodule of M together with $(P_i :_{R_i} M_i) \neq 0$, for all $i \in \{1,\dots,m\}$. Then either P is $(n-1,n)$-prime submodule of M or P_i is a $(n-2,n-1)$-prime submodule of M_i, for all $i \in \{1,\dots,m\}$ with $m \geq 2$ and $n \geq 3$.*

Proof. If there exists $j \in \{1,\dots,m\}$ such that $P_j = M_j$, then $(P : M)^{n-1}P \neq 0$. We claim that P is $(n-1,n)$-prime. Assume that P is not

$(n-1,n)$-prime. So there exist $a_1,\dots,a_{n-1}\in R$, $x\in M$ together with $a_1\dots a_{n-1}x=0$, $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x\notin P$, for all $i\in\{1,\dots,n-1\}$ and $a_1...a_{n-1}\notin (P:_R M)$ where $n\geq 2$. We show that $a_1\dots a_{n-k}(P:_R M)^{k-1}P=0$, for all $k\in\{1,2,...,n-1\}$. If $a_1\dots a_{n-k}(P:_R M)^{k-1}P\neq 0$, then $a_1\dots a_{n-k}p_1\dots p_{k-1}y\neq 0$, for some $p_1,\dots,p_{k-1}\in(P:_R M)$ and $y\in P$. Hence

$$a_1\dots a_{n-k}(a_{n-k+1}+p_1)(a_{n-k+2}+p_2)\dots(a_{n-1}+p_{k-1})(x+y)\in P\backslash\{0\}.$$

Since P is $(n-1,n)$-weakly prime, we have $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x\in P$, for some $i\in\{1,\dots,n-1\}$ or $a_1...a_{n-1}\in(P:_R M)$, which is a contradiction. So $a_1\dots a_{n-k}(P:_R M)^{k-1}P=0$.

By a same argument, we can prove that for all $\{i_1\dots i_{n-k}\}\subseteq\{1,\dots,n-1\}$, $a_{i_1}\dots a_{i_{n-k}}(P:_R M)^{k-1}P=0$.

Suppose $(P:_R M)^{n-1}P\neq 0$. According to the above discussion, we have $0\neq(a_1+p_1)\dots(a_{n-1}+p_{n-1})(x+y)\in P$. Since P is $(n-1,n)$-weakly prime, $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x\in P$, for some $i\in\{1,\dots,n-1\}$ or $a_1...a_{n-1}\in(P:_R M)$, which is a contradiction. Therefore, $(P:_R M)^{n-1}P=0$, which is a contradiction.

So we can assume that $P_j\neq M_j$, for all $j\in\{1,\dots,m\}$. We show that P_j is a $(n-2,n-1)$-prime submodule of M_j.

Let $a_1,\dots,a_{n-2}x\in P_j$ where $a_1,\dots,a_{n-2}\in R_j$, $x\in M_j$ and $i\in\{1,\dots,m\}$ such that $i\neq j$. Let $0\neq a\in(P_i:_{R_i} M_i)$ and $y\in M_i\backslash P_i$. Since M_i is torsionfree, $0\neq ay\in P_i$. Without loss of generality we can assume that $j<i$. We have

$$\begin{aligned}0 &\neq (0,\dots,0,a_1\dots a_{n-2}x,0,\dots,0,ay,0,\dots,0)\\ &= (a_{11},\dots,a_{1m})(a_{21},\dots,a_{2m})\dots(a_{(n-1)1},\dots,a_{(n-1)m})(x_1,...,x_m)\in P\end{aligned}$$

where $a_{kj}=a_k$ and $a_{ki}=a_{(n-1)j}=1$, for all $k\in\{1,\dots,n-2\}$. Let $a_{(n-1)i}=a$ and in other places $a_{hl}=0$. Also $x_i=y, x_j=x$ and $x_t=0$, for all $t\neq i,j$.

Since P is $(n-1,n)$-weakly prime, then $y\in P_i$, which is a contradiction or

$$a_1\dots a_{k-1}a_{k+1}\dots a_{n-2}x\in P_j,$$

for some $k\in\{1,\dots,n-2\}$ or $a_1...a_{n-2}\in(P_j:_{R_j} M_j)$. Therefore, P_j is $(n-2,n-1)$-prime. □

Theorem 2.2. *Let $n\geq 2$ be a positive integer, $R=R_1\times R_2$ and $M=M_1\times M_2$ where R_i is a ring and M_i is a non-zero R_i-module for $i=1,2$. suppose that P is a proper submodule of M_1. Then $P\times M_2$*

is an $(n-1,n)$-weakly prime ($(n-1,n)$-prime) submodule of M if and only if P is an $(n-1,n)$-prime submodule of M_1.

Proof. ($\Rightarrow$) Since $(P \times M_2 :_R M)^{n-1}(P \times M_2) \neq 0$, $P \times M_2$ is $(n-1,n)$-prime, similar to the proof of Theorem 1.2. It is easy to show that P is an $(n-1,n)$-prime submodule of M_1.

($\Leftarrow$) If P be an $(n-1,n)$-prime submodule of M_1, then it is easy to show that $P \times M_2$ is an $(n-1,n)$-prime submodule of M and thus $P \times M_2$ is an $(n-1,n)$-weakly prime submodule of M. □

Theorem 2.3. *Let $R = R_1 \times ... \times R_n$ be a ring and $M = M_1 \times ... \times M_n$ be an $R-$module, where M_i is an R_i-module, for all $i \in \{1, ..., n\}$. If every proper submodule of M is $(n-1,n)-$weakly prime, then M_i is a simple R_i-module, for all $i \in \{1, ..., n\}$ where $n \geq 2$.*

Proof. Let M_1 is not a simple R_1-module. So there exists a non-zero proper submodule P_1 of M_1. By hypothesis, the submodule $P = P_1 \times \{0\} \times ... \times \{0\}$ is an $(n-1,n)-$weakly prime submodule of M. Let $0 \neq x \in P_1$ and $0 \neq y_j \in M_j$ for all $j \in \{2, ..., n\}$. Then

$$(0, ..., 0) \neq (x, 0, ..., 0) = (1, a_{12}, ..., a_{1n})(1, a_{22}, ..., a_{2n})$$

$$...(1, a_{(n-1)2}, ..., a_{(n-1)n})(x, y_2, ..., y_n) \in P,$$

where $a_{i(i+1)} = 0$, for all $i \in \{1, ..., n-1\}$ and otherwise $a_{ij} = 1$.

Since P is $(n-1,n)-$weakly prime, $P_1 = M_1$ or $y_j = 0$, for some $j \in \{2, ..., n\}$, which are contradictions. So M_1 is a simple R_1-module. By a same argument, M_j is a simple R_j-module for all $j \in \{2, ..., n\}$. □

Theorem 2.4. *Let m, n be two positive integers such that $3 \leq m < n$, $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times ... \times M_m$ be an $R-$module, where (R_i, Q_i) is a quasi-local ring and M_i is a non-zero R_i-module, for all $i \in \{1, \ldots, m\}$. If every proper submodule of M is $(n-1,n)$-weakly prime, then $Q_i^{n-m} M_i = 0$, for all $i \in \{1, \ldots, m\}$.*

Proof. Let $Q_1^{n-m} M_1 \neq 0$. So there exist $a_1, \ldots, a_{n-m} \in Q_1$ and $x_1 \in M_1$ such that $a_1 \ldots a_{n-m} x_1 \neq 0$. By hypothesis, the submodule $P = (a_1 \ldots a_{n-m} x_1) \times \{0\} \times \cdots \times \{0\}$ is an $(n-1,n)$-weakly prime submodule of M. Let $0 \neq x_j \in M_j$ for all $j \in \{2, ..., m-1\}$. We have

$$(0, ..., 0) \neq (a_1 ... a_{n-m} x_1, 0, ..., 0) = (a_{11}, ..., a_{1m})$$

$$...(a_{(n-1)1}, ..., a_{(n-1)m})(x_{n1}, ..., x_{nm}) \in P,$$

for $a_{k1} = a_k$, where $1 \leq k \leq n - m$ and $a_{(n-m+t)(t+1)} = 0$, where $1 \leq t \leq (m-2)$. In other places, let $a_{ij} = 1$. $x_{nj} = x_j$, where $1 \leq j \leq m-1$ and $x_{nm} = 0$.

Since P is $(n-1, n)$-weakly prime, $M_m = 0$ or $x_{nj} = 0$, for some $2 \leq j \leq m-1$, which are contradictions, or $a_1 \dots a_{j-1}a_{j+1} \dots a_{n-m}x_1 \in (a_1 \dots a_{n-m}x_1)$, for some $j \in \{1, \dots, n-m\}$. So there exists an $r \in R$ such that

$$a_1 \dots a_{j-1}a_{j+1} \dots a_{n-m}x_1(1 - ra_j) = 0.$$

Since $a_j \in Q_1$, $1 - ra_j$ is a unit in R_1. So $a_1 \dots a_{j-1}a_{j+1} \dots a_{n-m}x_1 = 0$ which is a contradiction. Therefore $Q_1^{n-m}M_1 = 0$. By a same argument, $Q_i^{n-m}M_i = 0$, for all $i \in \{2, \dots, m\}$. □

Let (R, Q) be a quasi-local ring and M be an $R-$module. If t is the smallest positive integer such that $Q^tM = 0$, then we say that the associated degree of M is t. If $Q^tM \neq 0$, for all $t \geq 1$, then we say that the associated degree of M is ∞.

Corollary 2.5. *Let m, n be two positive integers such that $3 \leq m < n$ and $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times ... \times M_m$ be an $R-$module, where (R_i, Q_i) is a quasi-local ring and M_i is a non-zero R_i-module, where the associated degree of M_i is t_i, for all $i \in \{1, \dots, m\}$. If every proper submodule of M is $(n-1, n)$-weakly prime, then $t_i \leq n - m$, for all $i \in \{1, \dots, m\}$.*

Theorem 2.6. *Let $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times ... \times M_m$ be an $R-$module, where (R_i, Q_i) is a quasi-local ring, M_i is an R_i-module and the associated degree of M_i is t_i, for all $i \in \{1, \dots, m\}$. If $\sum_{i=1}^{m} t_i \leq n-1$, then every proper submodule of M is $(n-1, n)$-weakly prime with $n \geq 2$ and $m \geq 1$.*

Proof. Let $P = P_1 \times \cdots \times P_m$ be a proper submodule of M and
$(0, \dots, 0) \neq$
$(a_{11}, \dots, a_{1m})(a_{21}, \dots, a_{2m}) \dots (a_{(n-1)1}, \dots, a_{(n-1)m})(x_1, ..., x_m) \in P.$

So there exists a $j \in \{1, \dots, n-1\}$ such that $(\prod_{k=1}^{n-1} a_{kj})x_j \neq 0$. Since $Q_j^{t_j}M_j = 0$, there exist at most $t_j - 1$ elements of $\{a_{1j}, \dots, a_{(n-1)j}\}$ that are nonunits in R_j. So we need at most $t_j - 1$ parentheses such that the product of their $j'th$ elements product in x_j is in P_j.

For $i \neq j$ we have $Q_i^{t_i} M_i = 0$. If there exist t_i elements of $\{a_{1i}, \ldots, a_{(n-1)i}\}$ that are nonunits in R_i, then the product of these t_i elements is zero and we need t_i parentheses such that the product of their $i'th$ elements product in x_i is in P_i.

If there exist less than t_i elements of $\{a_{1i}, \ldots, a_{(n-1)i}\}$ that are nonunits in R_i, then we need less than t_i parentheses that the product of their $i'th$ elements product in x_i is in P_i.

So we need at most $(t_j - 1 + \sum_{i \neq j, i=1}^{m} t_i) + 1 = \sum_{i=1}^{m} t_i$ parentheses such that their product is in P. So P is $(n-1,n)$-weakly prime. □

Corollary 2.7. *Let $m < n$ be two positive integers, $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times ... \times M_m$ be an $R-$module, where R_i is a field and M_i is an R_i-vector space, for $i \in \{1, \ldots, m\}$. Then every proper submodule of M is $(n-1,n)$-weakly prime with $n \geq 2$.*

Proof. Let t_i be the associated degree of M_i. So $t_i = 1$, for all $i \in \{1, ..., m\}$. Thus $\Sigma_{i=1}^{m} t_i = m \leq n-1$. Therefore, every proper submodule of M is $(n-1,n)$-weakly prime, by Theorem 1.7. □

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M. Ebrahimpour
Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O.Box 518, Rafsanjan, Iran
`Email: m.ebrahimpour@vru.ac.ir`