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ON THE FORMAL POWER SERIES ALGEBRAS GENERATED BY A VECTOR SPACE AND A LINEAR FUNCTIONAL

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Abstract. Let $\mathscr{R}$ be a vector space (on $\mathbb{C}$) and φ be an element of $\mathscr{R}^*$ (the dual space of $\mathscr{R}$), the product $r \cdot s = \varphi(r)s$ converts $\mathscr{R}$ into an associative algebra that we denote it by $\mathscr{R}_\varphi$. We characterize the nilpotent, idempotent and the left and right zero divisor elements of $\mathscr{R}_\varphi[[x]]$. Also we show that the set of all nilpotent elements and also the set of all left zero divisor elements of $\mathscr{R}_\varphi[[x]]$ are ideals of $\mathscr{R}_\varphi[[x]]$.

Key Words: Vector space, Formal power series algebra, Nilpotent, Idempotent, Algebraic homomorphism.
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1. Introduction

Let A be an associative algebra (on $\mathbb{C}$) and

$$A[[x]] = \Big\{ \sum_{i=0}^{\infty} a_i x^i \quad | \quad a_i \in A \Big\},$$

be the set of all formal power series with coefficients in A. It is well known that the set $A[[x]]$ by the following operations of addition, multiplication and scalar multiplication is an associative algebra that is called

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the formal power series algebra over A.

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} (\sum_{k=0}^{i} a_k b_{i-k}) x^i,$$

$$\alpha(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} \alpha a_i x^i, \quad \alpha \in \mathbb{C} \quad \text{and} \quad \sum_{i=0}^{\infty} a_i x^i, \quad \sum_{i=0}^{\infty} b_i x^i \in A[[x]].$$

Similarly if R is a ring then $R[[x]]$ is the formal power series ring over R.

We recall some terminology. An element r of a ring R is called a right zero divisor, if there exists a nonzero y such that $yr = 0$. Similarly an element r is called a left zero divisor, if there exists a nonzero x such that $rx = 0$. An element r that is both a left and a right zero divisor is called a two-sided zero divisor. Also an element $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$. Finally $r \in R$ is idempotent if $r^2 = r$.

Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element in $\mathscr{R}^*$ (the dual space of $\mathscr{R}$). The product $r \cdot s = \varphi(r)s$, where $r, s \in \mathscr{R}$ converts $\mathscr{R}$ into an associative algebra that we denote it by $\mathscr{R}_\varphi$. Endomorphisms and also automorphisms on $\mathscr{R}_\varphi$ are investigated in [3]. And also in the case where $\mathscr{R}$ is a normed vector space and $\|\varphi\| \leq 1$,

- Arens regularity and also $n-$weak amenability of $\mathscr{R}_\varphi$ are investigated in [1].
- Strongly zero-product preserving maps, strongly Jordan zero-product preserving maps on $\mathscr{R}_\varphi$ and also polynomial equations with coefficients in $\mathscr{R}_\varphi$ are investigated in [2].
- Strongly Lie zero-product preserving maps on $\mathscr{R}_\varphi$ and $\mathscr{R}_\varphi^*$ are investigated in [4].

In the case where $\mathscr{R}$ is a vector space, we recall some properties of $\mathscr{R}_\varphi$ [1]. Let $Hom(\mathscr{R}_\varphi, \mathbb{C})$ be the set of all algebraic homomorphisms from $\mathscr{R}_\varphi$ into $\mathbb{C}$. Then $Hom(\mathscr{R}_\varphi, \mathbb{C}) = \{0, \varphi\}$. $\mathscr{R}_\varphi$ is commutative if and only if $\dim(\mathscr{R}) \leq 1$. Also in the case where $\dim \mathscr{R} > 1$ then $Z(\mathscr{R}_\varphi) = \{0\}$, where $Z(\mathscr{R}_\varphi)$ is the algebraic center of $\mathscr{R}_\varphi$.

The aim of the present paper is to show that although $\mathscr{R}_\varphi$ is not commutative and unital in general, the set of all nilpotent elements and also the set of all left zero divisor elements of $\mathscr{R}_\varphi[[x]]$ are ideals of $\mathscr{R}_\varphi[[x]]$. Also the set of all idempotent elements of $\mathscr{R}_\varphi[[x]]$ is multiplicative. These facts reveal that $\mathscr{R}_\varphi[[x]]$ is a source of example or counterexample in the field of algebraic theory.

2. Main Results

In this section we characterize the idempotent and also the nilpotent elements of $\mathscr{R}_\varphi[[x]]$.

Theorem 2.1. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Then an element $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ is nilpotent if and only if $a_i \in \ker(\varphi)$ for all $i \geq 0$.*

Proof. Let $P = \sum_{i=0}^{\infty} a_i x^i$ be nilpotent. Then there exists $n > 0$ such that $P^n = 0$. It follows that $a_0^n = 0$. So $\varphi(a_0^n) = (\varphi(a_0))^n = 0$, that implies $a_0 \in \ker(\varphi)$. As $a_0^2 = a_0 P = 0$, we can conclude that

$$\begin{aligned}(P - a_0)^2 &= P^2 - Pa_0 - a_0 P + a_0^2 \\ &= P^2 - Pa_0.\end{aligned}$$

So by induction we have

$$\begin{aligned}(P - a_0)^{n+1} &= P^{n+1} - P^n a_0 \\ &= 0.\end{aligned}$$

This shows that $Q = P - a_0 = \sum_{i=1}^{\infty} a_i x^i$ is nilpotent and $a_1^{n+1} = 0$, that implies $a_1 \in \ker(\varphi)$. Similarly by induction one can shows that

$$\begin{aligned}(Q - a_1 x)^{n+2} &= Q^{n+2} - Q^{n+1}(a_1 x) \\ &= 0.\end{aligned}$$

So $a_2^{n+2} = 0$, that implies $a_2 \in \ker(\varphi)$. Applying induction on i, we can conclude that $a_i^{n+i} = 0$, that implies $a_i \in \ker(\varphi)$ for all $i \geq 0$.
For the converse let $a_i \in \ker(\varphi)$ for all $i \geq 0$. So

$$\begin{aligned}P^2 &= \sum_{i=0}^{\infty}(\sum_{k=0}^{i} a_k a_{i-k})x^i \\ &= \sum_{i=0}^{\infty}(\sum_{k=0}^{i} \varphi(a_k) a_{i-k})x^i \\ &= \sum_{i=0}^{\infty}(\sum_{k=0}^{i} 0)x^i = 0.\end{aligned}$$

This shows that P is nilpotent. □

As the condition $a_i \in \ker(\varphi)$ is equivalent to $a_i^2 = 0$, by applying Theorem 2.1 we can present the following results.

Corollary 2.2. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Then an element $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ is nilpotent if and only if $a_i^2 = 0$ for all $i \geq 0$.*

It is well known that for a commutative ring R with an identity element, if $P = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is nilpotent, then a_i is nilpotent for all $i \geq 0$. But the converse is not the case in general. It is true whenever R is Noetherian. We recall that in the case where $\dim \mathscr{R} > 1$, $\mathscr{R}_\varphi$ is neither commutative nor unital. But Theorem 2.3 shows that the set of all nilpotent elements of $\mathscr{R}_\varphi[[x]]$ is an ideal that is worthy of consideration.

Theorem 2.3. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Also let $\mathscr{N}$ be the set of all nilpotent elements in $\mathscr{R}_\varphi[[x]]$. Then $\mathscr{N}$ is an ideal.*

Proof. Let $\sum_{i=0}^{\infty} a_i x^i, \quad \sum_{i=0}^{\infty} b_i x^i \in \mathscr{N}$ and $\sum_{i=0}^{\infty} c_i x^i \in \mathscr{R}_\varphi[[x]]$. So by Theorem 2.1 $a_i, b_i \in \ker(\varphi)$ for all $i \geq 0$. As

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{i=0}^{\infty} c_i x^i) = \sum_{i=0}^{\infty} (\sum_{k=0}^{i} a_k c_{i-k}) x^i,$$

$$(\sum_{i=0}^{\infty} c_i x^i)(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} (\sum_{k=0}^{i} c_k a_{i-k}) x^i,$$

and $\ker(\varphi)$ is an ideal, so

$$a_i + b_i, \quad \sum_{k=0}^{i} a_k c_{i-k}, \quad \sum_{k=0}^{i} c_k a_{i-k} \in \ker(\varphi),$$

for all $i \geq 0$. Hence by Theorem 2.1 $\mathscr{N}$ is an ideal. □

Theorem 2.4. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Then an element $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ is idempotent if and only if one of the following statements holds.*

1. *$P = 0$.*
2. *$\varphi(a_0) = 1$ and $a_i \in \ker(\varphi)$ for all $i \geq 1$.*

Proof. Let $P = \sum_{i=0}^{\infty} a_i x^i$ be idempotent. So $P^2 = P$. It follows that

$$a_i = \sum_{k=0}^{i} a_k a_{i-k}, \quad i \geq 0. \tag{2.1}$$

So $a_0 = a_0^2$, that implies $a_0 = \varphi(a_0)a_0$. Equivalently $(\varphi(a_0) - 1)a_0 = 0$. If $a_0 = 0$, then by (2.1) $a_i = 0$ inductively. So $P = 0$. In the case where $\varphi(a_0) = 1$ since $a_1 = a_0a_1 + a_1a_0$, we can conclude that

$$\begin{aligned} a_1 &= \varphi(a_0)a_1 + \varphi(a_1)a_0 \\ &= a_1 + \varphi(a_1)a_0. \end{aligned}$$

Hence $\varphi(a_1) = 0$. Also

$$\begin{aligned} a_2 &= a_0a_2 + a_1a_1 + a_2a_0 \\ &= \varphi(a_0)a_2 + \varphi(a_1)a_1 + \varphi(a_2)a_0 \\ &= a_2 + 0 + \varphi(a_2)a_0. \end{aligned}$$

So $\varphi(a_2) = 0$. Applying (2.1) inductively, we can conclude that for all $i \geq 1, \varphi(a_i) = 0$.
For the converse if $P = 0$ then obviously P is idempotent. Let $\varphi(a_0) = 1$ and $\varphi(a_i) = 0$ for all $i \geq 1$. Then

$$\sum_{k=0}^{i} a_k a_{i-k} = \sum_{k=0}^{i} \varphi(a_k) a_{i-k} = a_i.$$

It follows that $P^2 = P$. □

Theorem 2.4 shows that in spite of $\mathscr{R}_\varphi[[x]]$ is not commutative, the set of all idempotent elements of $\mathscr{R}_\varphi[[x]]$ is multiplicative.

Theorem 2.5. *Let $\mathscr{R}$ be a vector space and $\dim \mathscr{R} > 1$. Also let φ be a non-zero element of $\mathscr{R}^*$. Then each element of $\mathscr{R}_\varphi[[x]]$ is a right zero divisor.*

Proof. Let $P = \sum_{i=0}^{\infty} a_i x^i$ be an arbitrary element of $\mathscr{R}_\varphi[[x]]$. As $\dim \mathscr{R} > 1$ so $\ker(\varphi) \neq \{0\}$. Let $0 \neq a \in \ker(\varphi)$. Obviously $aP = 0$. This shows that P is a right zero divisor. □

Note that in the case where $\dim \mathscr{R} = 1$, the only two-sided zero divisor in $\mathscr{R}_\varphi[[x]]$ is $P = 0$.

Theorem 2.6. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Then an element $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ is a left zero divisor if and only if $a_i \in \ker(\varphi)$ for all $i \geq 0$.*

Proof. Let $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ be a left zero divisor. Then there exists an element $0 \neq Q = \sum_{i=0}^{\infty} b_i x^i$ such that

$$\begin{aligned} PQ &= (\sum_{i=0}^{\infty} a_i x^i)(\sum_{i=0}^{\infty} b_i x^i) \\ &= \sum_{i=0}^{\infty} (\sum_{k=0}^{i} a_k b_{i-k}) x^i \\ &= 0. \end{aligned} \tag{2.2}$$

As $Q \neq 0$, let j be the smallest index such that $b_j \neq 0$. The equation (2.2) implies that $0 = \sum_{k=0}^{j} a_k b_{j-k} = a_0 b_j$. So $\varphi(a_0) b_j = 0$. This shows that $a_0 \in \ker(\varphi)$. Similarly

$$\begin{aligned} 0 &= \sum_{k=0}^{j+1} a_k b_{j+1-k} \\ &= a_0 b_{j+1} + a_1 b_j \\ &= \varphi(a_0) b_{j+1} + \varphi(a_1) b_j \\ &= \varphi(a_1) b_j. \end{aligned}$$

So $a_1 \in \ker(\varphi)$. Applying (2.2) inductively, we can conclude that $a_i \in \ker(\varphi)$ for all $i \geq 0$.
For the converse let $a_i \in \ker(\varphi)$ for all $i \geq 0$. Choose $0 \neq b \in \mathscr{R}_\varphi$. Clearly $Pb = 0$. This shows that P is a left zero divisor. □

Applying Theorems 2.5 and 2.6, we can conclude the following results.

Corollary 2.7. *Let $\mathscr{R}$ be a non-zero vector space and* $\dim \mathscr{R} > 1$. *Also let φ be a non-zero element of $\mathscr{R}^*$. Then an element $P = \sum_{i=0}^{\infty} a_i x^i \in \mathscr{R}_\varphi[[x]]$ is a two-sided zero divisor if and only if $a_i \in \ker(\varphi)$ for all $i \geq 0$.*

Corollary 2.8. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Then the set of all left zero divisor elements in $\mathscr{R}_\varphi[[x]]$ is an ideal.*

Proof. Let $\mathscr{L}$ be the set of all left zero divisor elements of $\mathscr{R}_\varphi[[x]]$. Because $\ker(\varphi)$ is an ideal, an argument similar to the proof of Theorem 2.3 can be applied to show that $\mathscr{L}$ is an ideal. □

In the sequel let $e \in \varphi^{-1}(\{1\})$ and $\mathscr{R}_\varphi[x]$ be the polynomial algebra over $\mathscr{R}_\varphi$. Also set $x^0 = e$.

Theorem 2.9. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Also let $\psi : \mathscr{R}_\varphi[x] \longrightarrow \mathbb{C}$ be a linear mapping and $e \in \varphi^{-1}(\{1\})$. Then $\psi \in Hom(\mathscr{R}_\varphi[x], \mathbb{C})$ if and only if*

$$\psi(\ker(\varphi)[x]) = 0 \quad \textit{and} \quad \psi(ex^m) = (\psi(ex))^m$$

for all $m \geq 0$.

Proof. If $\psi = 0$, then the proof is clear. Let $0 \neq \psi \in Hom(\mathscr{R}_\varphi[x], \mathbb{C})$, $P \in \ker(\varphi)[x]$ and $Q \in \mathscr{R}_\varphi[x]$. As $PQ = 0$, so $\psi(P)\psi(Q) = \psi(PQ) = 0$. It follows that $\psi(P) = 0$. Also the equality $(ex)^m = ex^m$ implies,

$$\begin{aligned}\psi(ex^m) &= \psi((ex)^m) \\ &= (\psi(ex))^m, m \geq 0.\end{aligned}$$

For the converse let $\psi(\ker(\varphi)[x]) = 0$ and $\psi(ex^m) = (\psi(ex))^m$ for all $m \geq 0$. Clearly for all $a \in \mathscr{R}_\varphi$ we have

$$a = \varphi(a)e + K(a), \tag{2.3}$$

where $K(a) = a - \varphi(a)e \in \ker(\varphi)$. Let $P = \sum_{i=0}^n a_i x^i$ be an arbitrary element of $\mathscr{R}_\varphi[x]$. So by (2.3)

$$\begin{aligned}P &= \sum_{i=0}^n (\varphi(a_i)e + K(a_i))x^i \\ &= \sum_{i=0}^n \varphi(a_i)ex^i + \sum_{i=0}^n K(a_i)x^i.\end{aligned}$$

It follows that

$$\begin{aligned}\psi(P) &= \psi(\sum_{i=0}^n \varphi(a_i)ex^i + \sum_{i=0}^n K(a_i)x^i) \\ &= \psi(\sum_{i=0}^n \varphi(a_i)ex^i) + 0 \\ &= \sum_{i=0}^n \varphi(a_i)\psi(ex^i) \\ &= \sum_{i=0}^n \varphi(a_i)(\psi(ex))^i.\end{aligned}$$

Hence for $P = \sum_{i=1}^{n} a_i x^i$ and $Q = \sum_{i=1}^{m} b_i x^i$ we can conclude that

$$\begin{aligned}\psi(PQ) &= \psi(\sum_{i=0}^{m+n}(\sum_{k=0}^{i} a_k b_{i-k})x^i)\\ &= \sum_{i=0}^{m+n} \varphi(\sum_{k=0}^{i} a_k b_{i-k})(\psi(ex))^i\\ &= \sum_{i=0}^{m+n}(\sum_{k=0}^{i} \varphi(a_k)\varphi(b_{i-k}))(\psi(ex))^i\\ &= (\sum_{i=0}^{n} \varphi(a_i)(\psi(ex))^i)(\sum_{i=0}^{m} \varphi(b_i)(\psi((ex))^i)\\ &= \psi(P)\psi(Q).\end{aligned}$$

This shows that $\psi \in Hom(\mathscr{R}_\varphi[x], \mathbb{C})$. □

Applying Theorem 2.9, we can present the following result.

Corollary 2.10. *Let $\mathscr{R}$ be a non-zero vector space and φ be a non-zero element of $\mathscr{R}^*$. Also let $\psi : \mathscr{R}_\varphi[[x]] \longrightarrow \mathbb{C}$ be a linear mapping and $e \in \varphi^{-1}(\{1\})$. If $\psi \in Hom(\mathscr{R}_\varphi[[x]], \mathbb{C})$ then*

$$\psi(\ker(\varphi)[[x]]) = 0 \quad \text{and} \quad \psi(ex^m) = (\psi(ex))^m$$

for all $m \geq 0$.

Remark 2.11. It is clear that the map $\widehat{\varphi} : \mathscr{R}_\varphi[[x]] \longrightarrow \mathbb{C}$ defined by,

$$\widehat{\varphi}(\sum_{i=0}^{\infty} a_i x^i) = \varphi(a_0),$$

is an algebraic homomorphism.

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