Journal of Hyperstructures 5 *(1)* (*2016*), *56-68.*
ISSN: 2322-1666 print/2251-8436 online

REPRODUCING KERNEL METHOD FOR SOLVING WIENER-HOPF EQUATIONS OF THE SECOND KIND

A. ALVANDI, T. LOTFI AND M. PARIPOUR

ABSTRACT. This paper proposed a reproducing kernel method for solving Wiener-Hopf equations of the second kind. In order to eliminate the singularity of the equation, a transform is used. The advantage of this numerical method is the representation of exact solution in reproducing kernel Hilbert space and accuracy in numerical computation is higher. On the other hand, by improving the traditional reproducing kernel method and the definition of the operator of W Hilbert space, the solutions of Wiener-Hopf equation of the second kind are obtained. The approximate solution converges uniformly and rapidly to the exact solution. Numerical examples indicate that this method is efficient for solving these equations. The validity of the method is illustrated with two examples.

Key Words: Reproducing kernel method, Wiener-Hopf equation, Singular integral equation..

2010 Mathematics Subject Classification: 13A15, 13F30, 13G05.

1. INTRODUCTION

In recent years, numerical methods for solving singular integral equations have attracted a lot of attention. These equations have many applications in mathematics and engineering, see for instance Hunter [1], Paget [2], Lu [3], Krenk [4], Pedas [5]. Recently, the reproducing kernel method for solving singular integral equations in reproducing kernel space is developed. The advantage of this method is that it converges

Received: 25 January 2016, Accepted: 23 May 2016. Communicated by Mohammad Zarebnia;
∗Address correspondence to Mahmoud Paripour; E-mail: paripour@hut.ac.ir

uniformly and rapidly to the exact solution. See Jin [6], Du [7], Chen [8], Shen [9]. The Wiener-Hopf equation of the second kind is of the form

$$y(t) + \int_0^\infty k(t-s)\, y(s)\, ds = g(t), \quad 0 \le t < \infty, \tag{1.1}$$

where $k(t) \in L_1(\mathbb{R})$ and $g(t) \in L_p[0,\infty)(1 \le p < \infty)$ are given functions. Many authors considered methods for solving equation (1.1) including the Clenshaw-Curtis quadrature method, Clenshaw-Curtis-Rational method and so on [10, 11, 12, 13, 14]. In this study, a new method of solving solution is proposed in a reproducing kernel Hilbert space(RKHS). It is called reproducing kernel method. The rest of the paper is organized as follows. In section next, the reproducing kernel Hilbert space for solving (1.1) is introduced. In section 3, we discuss reproducing kernel method for (1.1). We transform (1.1) into integral equation of finite interval by substituting the variables t and s by $t = \frac{\alpha(1-\tau)}{1+\tau}$, and $s = \frac{\alpha(1-z)}{1+z}$ respectively:

$$Y(t) + 2\alpha \int_{-1}^{1} \frac{K(\tau, z)}{(z+1)^2}\, Y(z)\, dz = G(\tau), \quad -1 < \tau < 1.$$

We will show that $K(\tau, z)$ has singularities along $\tau = z$ when τ tend to -1, to eliminate the singularities, we introduce a new function $X(z) \triangleq \dfrac{Y(z)}{(z+1)^2}$. We then proof the numerical method is stable and convergent. Section 4 illustrates two numerical examples. It is shown that the reproducing kernel method proposed in this paper is efficient. Finally, concluding remarks are given in Section 5.

2. Preliminaries

2.1. **A reproducing kernel Hilbert space $W^m[-1,1]$.** In the section, a RKHS $W^m[-1,1]$ is introduced for solving Eq. (1.1). The representation of reproducing kernel becomes simple by improving the definition of traditional inner product see [15, 16, 17, 18, 19], in $W^m[-1,1]$.

Definition 1.2.1. $W^m[-1,1] = \{u(x) | u^{(m-1)}(x)$ is an absolutely continuous real value function, $u^{(m)}(x) \in L^2[-1,1]\}$.The inner product and norm in $W^m[-1,1]$ are given respectively by

$$\langle u, v\rangle = \sum_{i=0}^{m-1} u^{(i)}(-1) v^{(i)}(-1) + \int_{-1}^{1} u^{(m)}(x) v^{(m)}(x)\, \mathrm{d}x \tag{2.1}$$

and

$$\|u\|_m = \sqrt{\langle u, u\rangle_m}, \qquad u, v \in W^m[-1,1]. \tag{2.2}$$

$W^m[-1,1]$ is a reproducing kernel space and its reproducing kernel $R_x(y)$ can be obtained. Now let us find out the expression form of the reproducing kernel function $R_x(y)$ in $W^2[-1,1]$.

$$\begin{aligned}\langle u(y), R_x(y)\rangle &= u(-1)R_x(-1) + u'(-1)R'_x(-1) + \int_{-1}^{1} u''(x)R''_x(y)dy \\ &= u(-1)R_x(-1) + u'(-1)R'_x(-1) + u'(y)R''_x(y)|_{-1}^{1} - \\ u(y)R'''_x(y)|_{-1}^{1} &+ \int_{-1}^{1} u(y)R^{(4)}_x(y)dy.\end{aligned}$$

Not that the definition of the reproducing kernel $u(x) = \langle u(y), R_x(y)\rangle$ in $W^m[-1,1]$, the following equalities are necessary.

$$R^{(4)}_x(y) = \delta(y-x) \tag{2.3}$$

$$R_x(-1) + R'''_{(-1)} = 0, \tag{2.4}$$

$$R'_x(-1) - R''_x(-1) = 0, \tag{2.5}$$

$$R'''_x(1) = 0, \quad R''_x(1) = 0. \tag{2.6}$$

From (2.3), it has $R^{(4)}_x(y) = 0$ as $y \neq x$. $\lambda^4 = 0$ is its characteristic equation. Then the representation of the reproducing kernel is assumed by

$$R_x(y) = \begin{cases} \sum_{i=1}^{4} c_i y^{i-1}, & y \leq x, \\ \sum_{i=1}^{4} d_i y^{i-1}, & y > x, \end{cases} \tag{2.7}$$

where coefficients c_i, d_i, $\{i = 1,2,3,4\}$, could be obtained by solving the following equations

$$
\text{(2.8)} \qquad \begin{cases} R_x^{(m)}(x+0) = R_x^{(m)}(x-0), & (m=0,1,2), \\ R_x'''(x+0) - R_x'''(x-0) = 1, \\ R_x(-1) + R_x'''(-1) = 0, \\ R_x'(-1) - R_x''(-1) = 0, \\ R_x'''(1) = 0, \\ R_x''(1) = 0. \end{cases}
$$

3. Solving Eq. (1.1) in the Reproducing Kernel Space

3.1. An identical transformation of equation (1.1).

In this section, we proposed an identical transformation of equation (1.1):

$$
y(t) + \int_0^\infty k(t-s)\, y(s)\, ds = g(t), \qquad 0 \le t < \infty.
$$

We assume that $k(t) \in L_1(\mathbb{R})$ is semi-smooth, i.e., $k(t) \in C^r(0,\infty)$ and $k(t) \in C^r(-\infty,0)$ for certain positive integer r and $y(t) \in C^r(0,\infty)$ satisfying

$$
|y(t)| \le \frac{c}{t^2} \tag{3.1}
$$

for certain $c > 0$ for large t. Substituting the variables t and s in (1.1) $\frac{\alpha(1-\tau)}{1+\tau}$, and $\frac{\alpha(1-z)}{1+z}$ respectively, we get the following integral equation

$$
Y(\tau) + 2\alpha \int_{-1}^1 \frac{K(\tau,z)}{(z+1)^2}\, Y(z)\, dz = G(\tau), \quad -1 < \tau \le 1, \tag{3.2}
$$

where

$$
K(\tau,z) = k\left(\frac{\alpha(1-\tau)}{1+\tau} - \frac{\alpha(1-z)}{1+z}\right), \; Y(\tau) = y\left(\frac{\alpha(1-\tau)}{1+\tau}\right), \tag{3.3}
$$

$G(\tau) = g\left(\frac{\alpha(1-z)}{1+z}\right)$.

We notice that the kernel function of (3.2) has singularities along $z = \tau$ as τ tend to -1 since the denominators $\tau+1, z+1$ and $(z+1)^2$ tend to infinity. On the other hand, under the assumption (3.1), the integral of (3.2) satisfies

$$
|\frac{K(\tau,z)}{(z+1)^2} Y(z)| = |\frac{K(\tau,z)}{(z+1)^2} y(\frac{\alpha(1-z)}{z+1})| \le |\frac{K(\tau,z)}{(z+1)^2} c(\frac{\alpha(1-z)}{(z+1)})^{-2}|
$$

$= |\frac{cK(\tau,z)}{\alpha^2(1-z)^2}|$,

i.e., $|\frac{K(\tau,z)}{(z+1)^2}Y(z)|$ is bounded. Now we proposed a way to eliminate the singularities. Since the factor $\dfrac{1}{(z+1)^2}$ in the kernel function of (3.2) is independent of τ, we define a new function $X(z) \triangleq \dfrac{Y(z)}{(z+1)^2}$ and then subtract the singularities by reformulating (3.2) as

(3.4)

$$(\tau+1)^2X(\tau)+2\alpha\int_{-1}^{1}K(\tau,z)\,X(\tau)\,dz+2\alpha\int_{-1}^{1}K(\tau,z)(X(z)-X(\tau))\,dz$$

$= G(\tau)$.

3.2. **Representation of Exact Solution for Wiener-Hopf Equations of the Second Kind.**

In this section, exact solution of Eq. (1.1) is obtained by defining operator $\mathbb{L}: W^2[-1,1] \longrightarrow L^2[-1,1]$, then Equation (3.4) can be converted into the form as follows :

$$(\mathbb{L}u)(\tau) = G(\tau), \qquad -1 < \tau \leq 1, \tag{3.5}$$

(3.6)

$$(\mathbb{L}u)(\tau) = ((\tau+1)^2+2\alpha\int_{-1}^{1}K(\tau,z)\,dz)u(\tau)+2\alpha\int_{-1}^{1}K(\tau,z)(u(z)-u(\tau))\,dz,$$

it is easy to prove $\mathbb{L}$ is a bounded linear operator, and let $\mathbb{L}^*$ is the conjugate operator of $\mathbb{L}$. In order to obtain the representation of the exact solution of Eq. (1.1), let

$\varphi_i(x) = R_{x_i}(x), \qquad \psi_i(x) = \mathbb{L}^*\varphi_i(x) = [\mathbb{L}_yR_x(y)](x_i), \quad (i = 1,2,\dots),$
where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[-1,1]$. Hence, one gets

$$\psi_i(x) = ((x_i+1)^2+2\alpha\int_{-1}^{1}K(x_i,y)\,dy)R(x_i,x) \tag{3.7}$$

$$+2\alpha\int_{-1}^{1}K(x_i,y)(R(x,y)-R(x_i,y)\,dy.$$

Theorem 3.1. If $\{x_i\}_{i=1}^{\infty}$ is dense in $[-1,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in $W^2[-1,1]$.

Proof. If for any $u(x) \in W^2[-1,1]$, it has $\langle u(x),\psi_i(x)\rangle = 0 \quad i = 1,2,\dots,$

namely

$$
\begin{aligned}
\langle u(x), \psi_i(x)\rangle &= \langle u(x), (\mathbb{L}y R_x(y)(x_i)\rangle \\
&= \mathbb{L}_y \langle u(x), R_x(y)\rangle (x_i) \\
&= [\mathbb{L}_y u(y)](x_i) = 0. \qquad (3.8)
\end{aligned}
$$

Note that $\{x_i\}_{i=1}^{\infty}$ is a dense set. It follows that $\mathbb{L}_y u(x) \equiv 0$. From the existence and uniqueness of the solution of Eq. (1.1), it follows that $u(x) \equiv 0$. So $\{\psi_i(x)\}_{i=1}^{\infty}$ is complete in $W^2[-1, 1]$. $\square$

By Gram-Schmidt process, we obtain an orthogonal basis $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^2[-1, 1]$, such that

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \tag{3.9}$$

where β_{ik} are orthogonal coefficients. In order to obtain β_{ik}, let

$$\psi_i(x) = \sum_{k=1}^{i} B_{ik} \bar{\psi}_k(x).$$

$$\langle \psi_i(x), \bar{\psi}_i(x)\rangle = \sum_{k=1}^{i-1} B_{ik}^2 + B_{ii}^2,$$

$$B_{ii} = \sqrt{\langle \psi_i(x), \psi_i(x)\rangle - \sum_{k=1}^{i-1} B_{ik}^2}.$$

$$\beta_{ii} = \frac{1}{\sqrt{\langle \psi_i(x), \psi_i(x)\rangle - \sum_{k=1}^{i-1} B_{ik}^2}}.$$

$$\beta_{ij} = \beta_{ii} \left(- \sum_{k=j}^{i-1} B_{ik} \beta_{kj} \right). \tag{3.10}$$

Theorem 3.2. If $u(x)$ is the solution of Eq. (1.1), then

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} G(x_k) \bar{\psi}_i(x), \tag{3.11}$$

Proof. $u(x)$ can be expanded to Fourier series in term of normal orthogonal basis $\bar{\psi}_i(x)$ in $W^2[-1,1]$,

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\langle u(x),\bar{\psi}_i(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),\psi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),\mathbb{L}^*\varphi_k(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \mathbb{L}u(x),\varphi_k(x)\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle G(x),\varphi_k(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}G(x_k)\bar{\psi}_i(x). \end{aligned} \tag{3.12}$$

The proof is complete.□

By truncating the series of the left-hand side of (3.11), we obtain the approximate solution of Eq. (1.1)

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}G(x_k)\bar{\psi}_i(x). \tag{3.13}$$

$u_n(x)$ in (3.13) is the n-term intercept of $u(x)$ in (3.11), so $u_n(x) \longrightarrow u(x)$ in $W^2[-1,1]$ as $n \longrightarrow \infty$.

Theorem 3.3. Suppose the following conditions are satisfied
(i) $\|u_n(x)\|_{W^2}$ is bounded;
(ii) $\{x_i\}_{i=1}^{\infty}$ is dense in $[-1,1]$. Then n-term approximate solution $u_n(x)$ converges to the exact solution $u(x)$ of Eq. (1.1) and the exact solution is expressed as

$$u(x) = \sum_{i=1}^{\infty}B_i\bar{\psi}_i(x), \tag{3.14}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}G(x_k)$.

Proof. (i) The convergence of $u_n(x)$ will be proved. From (3.13), one gets

$$u_n(x) = u_{n-1}(x) + B_n\bar{\psi}_n(x). \tag{3.15}$$

From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\|u_n(x)\|_{W^2}^2 = \|u_{n-1}(x)\|_{W^2}^2 + \|B_n\|^2.$$

The sequence $\|u_n(x)\|_{W^2}$ is monotone increasing. Due to $\|u_n(x)\|_{W^2}$ being bounded, $\|u_n(x)\|_{W^2}$ is convergent as soon as $n \longrightarrow \infty$. Then there exists a constant c such that

$$\sum_{i=1}^{\infty} B_i^2 = c. \tag{3.16}$$

let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} \|(u_m - u_n)\|^2_{W^2} &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|^2_{W^2} \\ &= \|u_m - u_{m-1}\|^2_{W^2} + \|u_{m-1} - u_{m-2}\|^2_{W^2} + \dots \\ &+ \|u_{n+1} - u_n\|^2_{W^2} = \sum_{i=n+1}^{m} (B_i)^2 \longrightarrow 0, (n \longrightarrow \infty). \end{aligned} \tag{3.17}$$

Considering the completeness of $W^2[-1, 1]$, it has

$$u_n(x) \stackrel{\|\cdot\|_{W^2}}{\longrightarrow} u(x), \quad n \longrightarrow \infty.$$

(ii) It is proved that $u(x)$ is the solution of Eq. (3.5).
From (3.14), it follows

$$\begin{aligned} (\mathbb{L}u)(x_j) &= \sum_{i=1}^{\infty} B_i \langle \mathbb{L}\bar{\psi}_i(x), \varphi_j(x)\rangle \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \mathbb{L}^* \varphi_j(x)\rangle \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \psi_j(x)\rangle, \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{i=1}^{n} \beta_{nj} (\mathbb{L}u)(x_j) &= \sum_{i=1}^{\infty} B_i \big\langle \bar{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj} \psi_j(x) \big\rangle_{W^2} \\ &= \sum_{i=1}^{\infty} B_i \langle \bar{\psi}_i(x), \bar{\psi}_n(x)\rangle_{W^2} = B_n. \end{aligned}$$

If $n = 1$, then $(\mathbb{L}u)(x_1) = G(x_1)$. If $n = 2$ then $\beta_{21}(\mathbb{L}u)(x_1) + \beta_{22}(\mathbb{L}u)(x_2) = \beta_{21}G(x_1) + \beta_{22}G(x_2)$. It is clear that $(\mathbb{L}u)(x_2) = G(x_2)$. Moreover, it is easy to see by induction that $(\mathbb{L}u)(x_j) = G(x_j)$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[-1, 1]$, for any $x \in [-1, 1]$

$$(\mathbb{L}u)(x) = G(x). \tag{3.18}$$

That is, $u(x)$ is the solution of Equation (3.5) and

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x). \tag{3.19}$$

The proof is complete. □

3.3. **The Stability of the Solution on the Eq. (3.5).**
Let $u(x)$ be a solution of (3.5). It is called that the approximate method on solution $u(x)$ from $u_n(x)$ with the right-hand side $G_n(x)$ is stable in $W^2[-1, 1]$, if $\lim_{n\to\infty} \|G - G_n\|_{W^2} = 0$, then $\lim_{n\to\infty} \|u - u_n\|_{W^2} = 0$.
Let $(\mathbb{L}u_n)(x) = G_n(x)$ and $G(x) = G_n(x) + \epsilon_n(x)$,
where $\epsilon_n(x)$ is a perturbation and $\epsilon_n(x) \to 0(n \to \infty)$. See [20, 21]. From the form (3.11), note that

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} G(x_k) \bar{\psi}_i(x),$$

and

$$u_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} G_n(x_k) \bar{\psi}_i(x),$$

it follows

$$u(x) - u_n(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \epsilon_n(x_k) \bar{\psi}_i(x) = \mathbb{L}^{-1} \epsilon_n(x).$$

From the continuity of $\mathbb{L}^{-1}$ and $\epsilon_n(x) \to 0(n \to \infty)$, it follows

$$\lim_{n\to\infty} \|u(x) - u_n(x)\| = \|\mathbb{L}^{-1}\| \lim_{n\to\infty} |\epsilon_n(x)| = 0.$$

Then, the method is stable.

4. Numerical examples

Example 4.1. Consider

$$u(t) + \int_0^{\infty} \frac{1}{1 + (t - s)^2} u(s)\, ds = g(t), \qquad 0 \le t < \infty,$$

where

$$g(t) = \frac{1}{1 + t^2} + \frac{1}{4 + t^2}(\pi + arctan(t)) + \frac{ln(1 + t^2)}{t(4 + t^2)}.$$

The exact solution of the equation is $u(t) = \dfrac{1}{(1+t^2)}$. Using the method presented in section 3, taking $n = 11$ and $n = 20$. The absolute errors $u_{11} - u$ and $u_{20} - u$ are given in Table 1 and Figure 1.

Table 1
Numerical results of Example 4.1

Node	$\lvert u_{11} - u\rvert$	$\lvert u_{20} - u\rvert$
-0.9	8.77083E-6	1.54070E-8
-0.7	5.11982E-6	9.70029E-9
-0.5	6.89540E-6	1.41000E-8
-0.3	2.08658E-6	5.96931E-8
-0.1	9.13911E-6	1.90117E-8
0.1	4.23227E-5	1.11963E-7
0.3	6.43560E-5	2.80711E-7
0.5	1.41120E-5	5.64622E-8
0.7	4.27360E-4	7.76620E-7
0.9	1.09958E-3	2.58520E-6

Example 4.2. Consider

$$u(t) + \int_0^\infty k(t-s)\,u(s)\,ds = (2 + t + \frac{t^2}{2} + \frac{t^3}{3})e^{-t}, \qquad 0 \le t < \infty,$$

where $k(t) = (1 + |t| + t^2)e^{-|t|}$. The exact solution of the above equation is $u(t) = e^{-t}$. Using the method presented in section 3, taking $n = 11$ and $n = 20$. The absolute errors $u_{11} - u$ and $u_{20} - u$ are given in Table 2 and Figure 2.

Table 2
Numerical results of Example 4.2

Node	$\lvert u_{11} - u\rvert$	$\lvert u_{20} - u\rvert$
-0.95	1.53861E-12	1.77174E-14
-0.75	2.72086E-6	1.16039E-9
-0.55	6.00161E-7	2.46969E-9
-0.35	6.74903E-6	5.62538E-9
-0.15	1.78160E-4	1.30169E-8
0.05	5.95211E-4	3.02134E-8
0.25	3.62010E-4	7.02237E-8
0.45	1.54430E-3	1.64547E-7
0.65	2.09719E-3	4.16000E-7
0.85	1.36754E-2	1.76546E-6

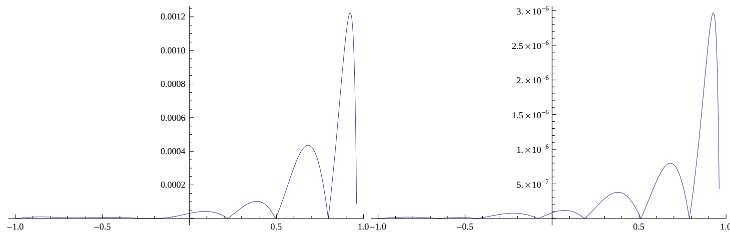

FIGURE 1. The absolute errors for n=11 and n=20, respectively.

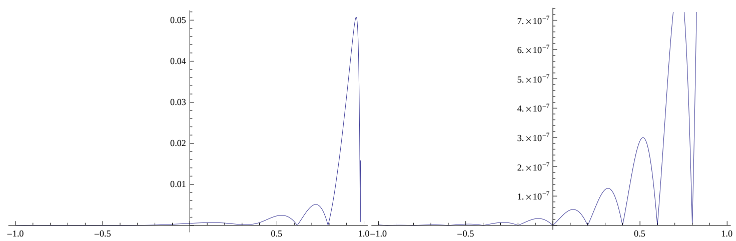

FIGURE 2. The absolute errors for n=11 and n=20, respectively.

5. Conclusion

In this paper, we use a new constructive method to find the approximate solution for Wiener-Hopf equations of the second kind in the reproducing kernel space. Using this method, we obtain the sequence which is proved to converge to the exact solution uniformly. The results from the numerical examples show that the present method is accurate and reliable for solving these equations.

References

[1] D.B. Hunter, *Some Gauss-type formulae for the evaluation of Cauchy principle values of integrals*, Numer. Math. **19** (1972), 419–424.

[2] D.F. Paget, D. Elliott, *An algorithm for the numerical evaluation of certain Cauchy principle values of integrals*, Numer. Math. **19** (1972), 373–385.

[3] C. K. Lu, *The approximate of Cauchy type integral by some kinds of interpolatory splines*, J. Approx. Theory. **36** (1982), 197–212.

[4] S. Krenk, *Numerical quadrature of periodic singular integral equations*, J. Inst. Math. Appl. **21** (1978), 181–187.

[5] A. Pedas, E. Tamme, *Discrete Galerkin method for Fredholm integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math. **213** (2008), 111–126.

[6] X. Jin, L. M. Keer, Q. Wang, *A practical method for singular integral equations of the second kind*, Eng. Fracture Mech. **206** (2007), 189–195.

[7] J. Du, *On the numerical solution for singular integral equations with Hilbert kernel*, Chin. J. Numer. Math. Appl. **11 (2)** (1989), 9–27.

[8] Z. Chen, Y.F. Zhou, *A new method for solving Hilbert type singular integral equations*, Appl. Math. Comput. **218** (2011), 406–412.

[9] H. Du, J.H. Shen, *Reproducing kernel method of solving singular integral equation with cosecant kernel*, J. Math. Anal. Appl. **348** (2008), 308–314.

[10] S. Y. Kang, I. Koltracht, G. Rawitscher,*Nystrom-Clenshaw-Curtis quadrature for integral equations with discontinuous kernels*, Math. Comput. **72 (242)** (2003), 729–756.

[11] Yan Xuan, Fu-Rong Lin, *Numerical methods based on rational variable substitution for Wiener-Hopf equation of the second kind*, Appl. Math. Comput. **236** (2012), 3528–3539.

[12] G.A. Chandler, I.G. Graham, *The convergence of Nystrom methods for Wiener-Hopf equations*, Numer. Math. **2 (52)** (1988), 345–364.

[13] I.G. Graham, W.R. Mendes, *Nystrom-product integration for Wiener-Hopf equations with applications to radiative transfer*, IMA J. Numer. Anal. **9** (1989), 261–284.

[14] G. Mastroianni, G. Monegato, *Nystrom interpolants based on zeros of Laguerre polynomials for some Wiener-Hopf equations*, IMA J. Numer. Anal. **17** (1997), 621–642.

[15] M. G. Cui, Y. Z. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science PubInc. Hauppauge, (2009).

[16] F. Z. Geng, M. G. Cui, *Solving a nonlinear system of second order boundary value problems*, J. Math. Appl. **327**(2007), 1167–1181.

[17] X.Y. Li, B.Y. Wu, *A continuous method for nonlocal functional differential equations with delayed or advenced arguments*, Mathematical Analysis and Applications, **409** (2014), 485–493.

[18] X.Y. Li, B.Y. Wu, *Error estimation for the reproducing kernel method to solve linear boundary value problems*, Computational and Applied Mathematics, **243** (2013), 10–15.

[19] F.Z. Geng, S.P. Qian, S. Li,*A numerical method for singularly perturbed turning point problems with an interior layer*, **255**(2014), 97–105.

[20] Hong Du, Minggen Cui, *Representation of the exact solution and a stability analysis on the Fredholm integral equation of the first kind in reproducing kernel space*, Appl. Math. Comput. **182 (2)** (2006), 1608–1614.

[21] M. Cui, Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science, (2008).

Azizallah Alvandi
Department of Mathematics, Hamedan Branch, Islamic Azad University , Iran.
`Email: alvandya@gmail.com`

Taher Lotfi
Department of Mathematics, Hamedan Branch, Islamic Azad University , Iran.
Email: lotfitaher@yahoo.com

M. Paripour
Department of Mathematics, Hamedan University of Technology, Hamedan, 65156-579, Iran.
Email: paripour@hut.ac.ir