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COMPUTER ALGEBRA AND THURSTON GEOMETRIES

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Abstract. The article illustrates the graphical study of geodesic motion on $H^3, H^2 \times R, Nil_3$ and Sol_3 using the symbolic and graphical computation of MATLAB platform.

Key Words: Thurston geometries, Computer algebra, geodesic..
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1. Introduction

Let (M, g) be a Riemannian manifold. If for any $x, y \in M$ there does exist an isometry $\pi : M \to M$ such that $y = \pi(x)$, then the Riemannian manifold is called *homogeneous*. In [5] W. P. Thurston formulated a geometrization conjecture for three-manifolds which states that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian 3-geometries $E^3, S^3, H^3, S^2 \times R, H^2 \times R, \widetilde{SL_2 R}, Nil_3$ and Sol_3.
Obviously, the Poincar'e conjecture (a compact three-manifold with trivial fundamental group is necessarily homeomorphic to the 3-sphere) is a special case of the Thurston conjecture. In the past thirty years, many mathematicians have contributed to the understanding of this problem, maybe the most important attempts are due to R. Hamilton. In 2006 a scoop went round the world claiming that a Russian mathematician, G. I. Perelman could give a complete proof of the Thurston conjecture and

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so the Poincare conjecture, too. Followed by the complex and knotty proof (usingmodern differential geometry of Ricci flows) the interest has turned to homogeneous spaces.
In this paper we use MATLAB platform to calculate Christoffel symbols and geodesic equations then plot the geodesic of $H^3, H^2 \times R, Nil_3$ and Sol_3.
MATLAB files are included in the end of paper.
The Christoffel symbols of first kind are defined by:

$$\Gamma_{ij}^{m} = \frac{1}{2} g^{km} \left(-\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} \right)$$

Also we know that in local coordinates the curve

$$\gamma(t) = \big(x_1(t), \ldots, x_n(t)\big)$$

Is a geodesic if and only if it satisfies in geodesic equation:

$$\frac{d^2 x_k}{dt^2} + \Gamma_{ij}^{k} \frac{dx_i}{dt} \frac{dx_j}{dt} = 0$$

2. Thurston geometries

In this section, we investigate eight model of Thurston geometries and by MATLAB calculate some of geometric quantities of them.

2.1. E^3.

Euclidean 3-space, E^3 is the space R^3 with the metric

$$ds^2 = dx^2 + dy^2 + dz^2.$$

This space is flat (i.e curvature is zero). As in E^2 any isometry of E^3 can be written as $x \to Ax + b$ but now A is a real orthogonal 3×3 matrix and bis a translation vector in R^3.
The group of isometries (also called rigid motions of Euclidean space) is the semidirect product

$$\text{Isom}(E^3) = R^3 \times O(3)$$

where R^3 acts by translations and $O(3)$ by rotations.

2.2. S^3.

The spherical geometry is the three-sphere and its isometry group. S^3 can be embedded in R^4, in this case

$$S^3 = \big\{ x \in R^4 \mid \|x\| = 1 \big\}$$

and thus the metric on S^3 is the one induced from R^4, that is,

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2.$$

In this metric the geodesics of S^3 are exactly the great circles of S^3. We may also think of S^3 as ordered pairs of complex numbers. In this case,

$$S^3 = \{(z_1, z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

The isometry group of S^3 is $O(3)$, the group of orthogonal 3×3 matrices. the dimension of the group of isometries in S^3 is 6.

2.3. H^3. Hyperbolic three-space can be defined as the upper half of Euclidean three-space, $R^3_+ = \{(x, y, z) \in R^3 \mid Z > 0\}$ with the metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$.

The isometry group of H^3 is generated by reflections, which are reflections across planes perpendicular to the xy-plane, and inversions in a sphere with center on the xy-plane. The group of orientation preserving isometries of H^3 can be identified with a Moebius transformation of $C \cup \{\infty\}$.

Recall that a moebius transformation of $C \cup \{\infty\}$ is a map of the form

$$z \to \frac{az + b}{cz + d}$$

where $a, b, c, d \in C$ and $ad - bc \neq 0$.

In H^3 we design function file $H3\_1$ to compute Christoffel symbols. The function file $H3\_1$ has 3 arguments. The first two arguments are lower indexes in Γ^m_{ij} and the third argument is upper index in Γ^m_{ij} For example to compute Γ^3_{33} it is enough to call $H3\_1$ with $(3, 3, 3)$ or type $H3\_1(3, 3, 3)$ in MATLAB, so we have:

$\gg H3\_1(3, 3, 3)$

thechristoffel symbol $gama\_33$^3 is:

ans=

-1/z

and we will thave this equality:

$$\Gamma^m_{ij} = H3\_1(i, j, m)$$

By this process we have obtained Christoffel symbols. The nonzero components are the following:

$$\Gamma^3_{11} = \frac{1}{z}, \quad \Gamma^1_{13} = -\frac{1}{z}, \quad \Gamma^3_{22} = \frac{1}{z},$$
$$\Gamma^2_{23} = -\frac{1}{z}, \quad \Gamma^1_{31} = -\frac{1}{z}, \quad \Gamma^2_{32} = -\frac{1}{z},$$
$$\Gamma^3_{33} = -\frac{1}{z}.$$

So the geodesic equations of H^3 are:

$$\frac{d^2x(t)}{dt^2} + \frac{2}{z}\frac{dx(t)}{dt}\frac{dz(t)}{dt} = 0,$$
$$\frac{d^2y(t)}{dt^2} - \frac{2}{z}\frac{dy(t)}{dt}\frac{dz(t)}{dt} = 0,$$
$$\frac{d^2z(t)}{dt^2} + \frac{1}{z}\frac{dx(t)}{dt}\frac{dx(t)}{dt} + \frac{1}{z}\frac{dy(t)}{dt}\frac{dy(t)}{dt} - \frac{1}{z}\frac{dz(t)}{dt}\frac{dz(t)}{dt} = 0.$$

To solve the geodesic equations we use numerical solution of ODEs in MATLAB. First we build the function file $H3\_G\_1$ containing the geodesic equations and then we build the MATLAB script $H3\_G\_2$ with ODE45 to solve the geodesic equations and plot the result.

2.4. **$S^2 \times R$.** The space $S^2 \times R$ is precisely the product of the unit two-sphere and the real line with the product metric.
The isometry group of $S^2 \times R$ is isomorphic to $\mathrm{Isom}(S^2) \times \mathrm{Isom}(R)$.
We know $\mathrm{Isom}(S^2)$ is generated by the identity, antipodalmap, rotations, and reflections. $\mathrm{Isom}(R)$ consists only of identity, translations, and reflections.
There are only a few ways elements of these two groups can be paired to generatdiscret subgroup of $(S^2) \times \mathrm{Isom}(R)$.
Dimension of isometry group of $S^2 \times R$ is 4.

2.5. **$H^2 \times R$.** The space $H^2 \times R$ is the product of hyperbolic two-space $\big(H^2 = \{z = x + iy \in C \mid y > 0\}\big)$ and the real line. It has isometry group $\mathrm{Isom}(H^2 \times R) = \mathrm{Isom}(H^2) \times \mathrm{Isom}(R)$.
Let x_1, x_2 denote the coordinates in H^2 and x^3 the coordinate in R. The metric in $H^2 \times R$ is

$$ds^2 = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2$$

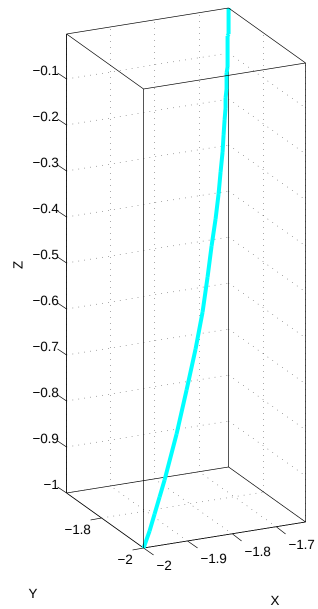

FIGURE 1. Geodesic of H^3

where

$$F = \left(\frac{1 - x_1^2 - x_2^2}{2}\right)^2 .$$

In $H^2 \times R$ we design function file H2Rchristo compute Christoffel symbols. The function fileH2Rchris has 3 arguments.
For example to compute Γ_{22}^2 it is enough to call H2Rchris with (2,2,2) or type H2Rchris (2,2,2) in MATLAB , so we have:
$\gg$ simplify(H2Rchris(2,2,2))
thechristoffel symbol *gama*_22^2 is:
ans=

```
(2*x2)/(x1^2 + x2^2-1)
```

and we will have this equatlity:

$$\Gamma_{ij}^m = \text{H2Rchris(i,j,m)}$$

By this process we have obtained Christoffelsymbols. The nonzero components are the following:

$$\Gamma^1_{11} = \frac{2x_1}{x_1^2 + x_2^2 - 1}, \quad \Gamma^1_{12} = \frac{2x_2}{x_1^2 + x_2^2 - 1}, \quad \Gamma^1_{21} = \frac{2x_2}{x_1^2 + x_2^2 - 1},$$
$$\Gamma^1_{22} = -\frac{2x_1}{x_1^2 + x_2^2 - 1}, \quad \Gamma^2_{11} = -\frac{2x_2}{x_1^2 + x_2^2 - 1}, \quad \Gamma^2_{12} = \frac{2x_1}{x_1^2 + x_2^2 - 1},$$
$$\Gamma^2_{21} = \frac{2x_1}{x_1^2 + x_2^2 - 1}, \quad \Gamma^2_{22} = \frac{2x_2}{x_1^2 + x_2^2 - 1}$$

so the geodesic equations of $H^2 \times R$ are:

$$\frac{d^2x_1(t)}{dt^2} + \frac{2x_1}{x_1^2 + x_2^2 - 1}\frac{dx_1(t)}{dt}\frac{dx_1(t)}{dt} + 2\left(\frac{2x_2}{x_1^2 + x_2^2 - 1}\right)\frac{dx_1(t)}{dt}\frac{dx_2(t)}{dt}$$
$$- \frac{2x_1}{x_1^2 + x_2^2 - 1}\frac{dx_2(t)}{dt}\frac{dx_2(t)}{dt} = 0$$
$$\frac{d^2x_2(t)}{dt^2} - \frac{2x_2}{x_1^2 + x_2^2 - 1}\frac{dx_1(t)}{dt}\frac{dx_1(t)}{dt} + 2\left(\frac{2x_1}{x_1^2 + x_2^2 - 1}\right)\frac{dx_1(t)}{dt}\frac{dx_2(t)}{dt}$$
$$+ \frac{2x_2}{x_1^2 + x_2^2 - 1}\frac{dx_2(t)}{dt}\frac{dx_2(t)}{dt} = 0$$
$$\frac{d^2x_3(t)}{dt^2} = 0.$$

To solve the geodesic equations we use numerical solution of ODEs in MATLAB. First we build the function file $H2R\_G\_1$ containing the geodesic equations and then we build the MATLAB script $H2R\_G\_2$ with ODE45 to solve the geodesic equations and plot the result.

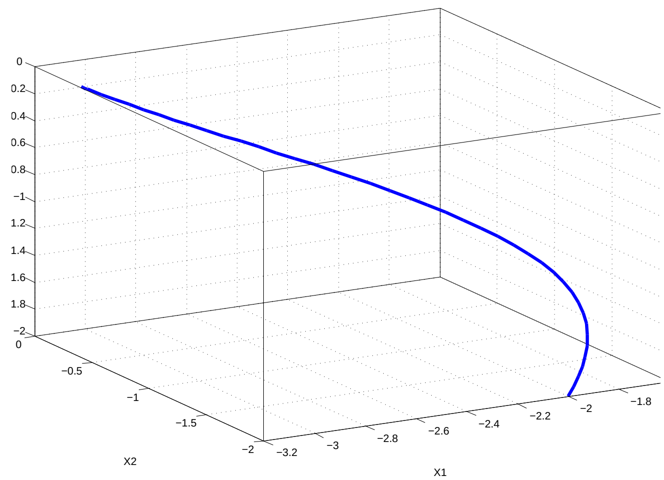

FIGURE 2. Geodesic of $H^2 \times R$

2.6. $\widetilde{SL_2R}$. The group SL_2R is the group of real matrices with determinant one, and is in fact a Lie group. The space $\widetilde{SL_2R}$ is the universal covering space of the Lie group SL2R.the space $\widetilde{SL_2R}$ is a Lie group.
The metric on $\widetilde{SL_2R}$ can be derived as follows:
The unit tangent bundle of H^2 can be identified with PSL_2R, which is covered by SL_2R. The metric on H^2 can then be pulled back to induce a metric on $\widetilde{SL_2R}$.

2.7. Nil_3. Nil3 is the three dimensional group of all real 3×3 uppertriangular matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

Nil3 Is defined as R^3 with the group operation

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{x\bar{y}}{2} - \frac{\bar{x}y}{2}\right)$$

The identity of the group is (0,0,0) and the inverse of (x, y, z) is given by $(-x, -y, -z)$. It is connected and nilpotent Lie group. This geometry is called Nil because the Lie group is nilpotent.

The following metric is left invariant

$$\bar{g} = dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy)\right)^2$$

The resulting Riemannian manifold $(Nil_3, \bar{g})$ is the model space of nil geometry in the sense of Thurston.
In Nil_3 we design function file $Nil3\_1$ to compute Christoffel symbols. The function file $Nil3\_1$ has 3 arguments.
For example to compute Γ^1_{23} it is enough to call $Nil3\_1$ with (2,3,1) or type $Nil3\_1(2, 3, 1)$ in MATLAB, so we have:

```
>>Nil3_1 (2,3,1)
thechristoffel symbol (gama_23^1) is:

ans

1/2
```

and we will have this equality:

$$\Gamma^m_{ij} = Nil3\_1(i, j, m)$$

By this process we have obtained Christoffelsymbols. The nonzero components are the following:

$$\Gamma^1_{23} = \frac{1}{2}, \quad \Gamma^1_{32} = \frac{1}{2}, \quad \Gamma^2_{13} = -\frac{1}{2},$$
$$\Gamma^2_{31} = -\frac{1}{2}, \quad \Gamma^3_{12} = \frac{1}{2}, \quad \Gamma^3_{21} = -\frac{1}{2}$$

So the geodesic equations of Nil_3 are:

$$\frac{d^2x(t)}{dt^2} + \frac{dy(t)}{dt}\frac{dz(t)}{dt} = 0,$$
$$\frac{d^2y(t)}{dt^2} - \frac{dx(t)}{dt}\frac{dz(t)}{dt} = 0$$
$$\frac{d^2z(t)}{dt^2} = 0.$$

To solve the geodesic equations we use numerical solution of ODEs in MATLAB. First we build the function file $Nil3\_G\_1$ containing the geodesic equations and then we build the MATLAB script $Nil3\_G\_2$ with ODE45 to solve the geodesic equations and plot the result.

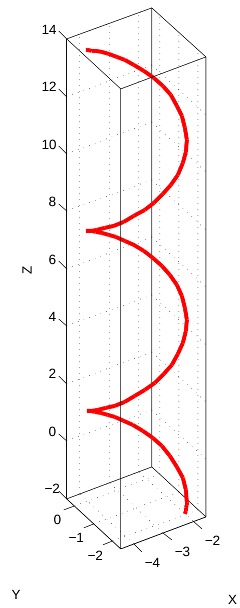

FIGURE 3. Geodesic of Nil_3

2.8. Sol_3**.** The space Sol3 is a connected 3-dimensional manifold whose one of the eight models of geometry of Thurston. This group is called Sol because it is a solvable group.
The space Sol3 can be viewed as R^3 with the metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

Where (x,y,z) are usual coordinates of R^3. The space sol_3 with the group operation

$$(x, y, z) * (\acute{x}, \acute{y}, \acute{z}) = (\acute{x} + e^{-z}\acute{x}, y + e^{z}\acute{y}, z + \acute{z})$$

It is a solvable but not nilpotent Lie group and the metric ds^2 is left invariant.
The isometry group of sol_3 has dimension 3.
In [11] calculate some geometric quantities like Christoffel symbols, Riemannian curvature Tensor, Ricci tensor, Scalar curvature , Einstein tensor and geodesic in sol3 space by building MATLAB files.
By function file MZsol1 , MZsol7 and MZsol8 the Christoffel symbols , geodesic equations and geodesic obtained as follows:

Christoffel symbols:

$$\Gamma^1_{13} = 1, \quad \Gamma^1_{31} = 1, \quad \Gamma^2_{23} = -1,$$
$$\Gamma^2_{32} = -1, \quad \Gamma^3_{11} = -e^{2z}, \quad \Gamma^3_{22} = e^{-2z}.$$

Geodesic equations:

$$\frac{d^2x(t)}{dt} + 2\frac{dx(t)}{dt}\frac{dz(t)}{dt} = 0$$
$$\frac{d^2y(t)}{dt} - 2\frac{dy(t)}{dt}\frac{dz(t)}{dt} = 0$$
$$\frac{d^2z(t)}{dt} - e^{2z(t)}\frac{dx(t)}{dt}\frac{dx(t)}{dt} + e^{-2z(t)}\frac{dy(t)}{dt}\frac{dy(t)}{dt} = 0.$$

Geodesic:

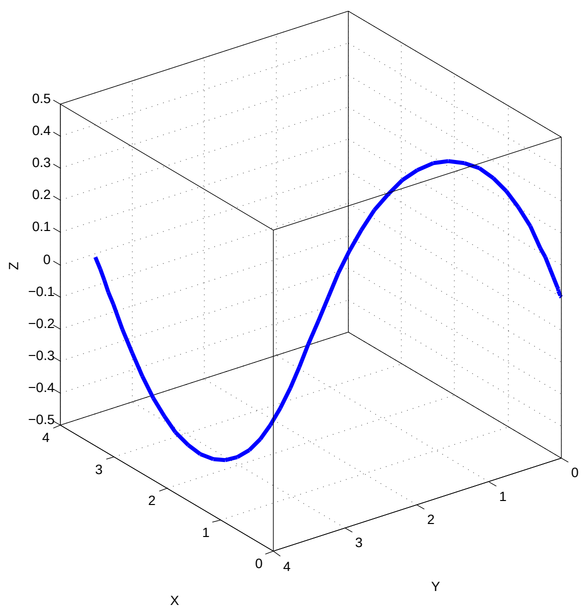

FIGURE 4. Geodesic of sol_3

3. APPENDIX (M-FILES)

*H*3_1**:**

```
functionGAMA_nm_k =H3_1(n,m,k)
%compute the christoffel symbol of H3 space
syms x y z;
g=[1/z^2 0 0;0 1/z^2 0;0 0 1/z^2];
G=inv(g);X=[x y z];
GAMA_nm_k=0;
for s=1:3
     ZR=1/2*G(s,k)*(diff(g(m,s),X(n))+diff(g(s,n),X(m))
     -diff(g(n,m),X(s)));
GAMA_nm_k=ZR+GAMA_nm_k;
end
 zr1=' the christoffel symbol';
 zr2='   gama _';zr3='   is :';
disp([zr1  zr2 num2str(n)  num2str(m) '^' num2str(k)  zr3]);
sum(GAMA_nm_k);
end
-------------------------------------------------------
```

$H3\_G\_1$:

```
functiondy =H3_G_1(t,y)
dy=zeros(6,1);
dy(1)=y(2);
dy(2)=-(-2/y(5))*(y(2))*(y(6));
dy(3)=y(4);
dy(4)=-(-2/y(5))*(y(4))*(y(6));
dy(5)=y(6);
dy(6)=-(1/y(5))*(y(2)^2)-(1/y(5))*(y(4)^2)+(1/y(5))*(y(6)^2);
end

-------------------------------------------------------
```

$H3\_G\_2$:

```
clearall;clc;
tspan = [-1 1];
y0 = [-2;2;-2;2;-1;2];
[t, y] = ode45(@H3_G_1, tspan, y0)
plot3(y(:,1),y(:,3),y(:,5),'linewidth',3,'color','c');
axis equal;
xlabel('X');ylabel('Y');zlabel('Z');
gridon;box on;
```

H2Rchris:

```
functionL_nm_k =H2Rchris(n,m,k)
syms x1 x2 x3;
g=[((1-x1^2-x2^2)/2)^2 0 0;0 ((1-x1^2-x2^2)/2)^2 0;0 0 1];
G=inv(g);X=[x1 x2 x3];L_nm_k=0;
for s=1:3
     W=1/2*G(s,k)*(diff(g(m,s),X(n))+diff(g(s,n),X(m))
     -diff(g(n,m),X(s)));
L_nm_k=W+L_nm_k;
end
 M1=' the christoffel symbol';
 M2='gama _';M3=' is :';
disp([M1'('M2 num2str(n)num2str(m)'^'num2str(k)')' M3]);
sum(L_nm_k);
end
```

$H2R\_G\_1$:

```
functiondy =H2R_G_1(t,y)
dy=zeros(6,1);
dy(1)=y(2);
dy(2)=-((2*y(1))/(y(1)^2 + y(2)^2 - 1))*(y(2)^2) -
2*((2*y(2))/(y(1)^2+ y(2)^2 - 1))*(y(2))*(y(4))
- (-(2*y(1))/(y(1)^2+ y(2)^2 - 1))*(y(4)^2);
dy(3)=y(4);
dy(4)=-(-(2*y(2))/(y(1)^2 + y(2)^2 - 1))*(y(2)^2)
- 2*((2*y(1))/(y(1)^2+ y(2)^2 - 1))*(y(2))*(y(4))
- ((2*y(2))/(y(1)^2+ y(2)^2 - 1))*(y(4)^2);
dy(5)=y(6);
dy(6)=0 ;
end
```

H2R_G_2:

```
clearall;clc;
tspan = [-4 4];
y0 = [-2;2;-2;2;-2;2];
[t, y] = ode45(@Nil3_G_1, tspan, y0)
plot3(y(:,1),y(:,3),y(:,5),'linewidth',3,'color','r');
axis equal;
xlabel('X');ylabel('Y');zlabel('Z');
gridon;box on;
```

Nil3_1:

```
functiongama_nm_k =Nil3_1(ms1,ms2,ms3)
syms x y z ;
g=[1 (1/y)*x 0;0 1 0;2*(1/2)*y -2*(1/2)*x 1];
G=[1 0 0;0 1 0;1/(2*(1/2)*y) 1/(-2*(1/2)*x) 1];X=[x y z];
gama_nm_k=0;
for s=1:3
     zr=1/2*G(s,ms3)*(diff(g(ms2,s),X(ms1))
    +diff(g(s,ms1),X(ms2))-diff(g(ms1,ms2),X(s)));
gama_nm_k=zr+gama_nm_k;
end
 word1=' the christoffel symbol';
 word2='gama _';word3=' is :';
disp([word1 '(' word2  num2str(ms1)  num2str(ms2)
 '^' num2str(ms3) ')'  word3]);
sum(gama_nm_k);
end
```

Nil3_G_1:

```
functiondy =Nil3_G_1(t,y)
dy=zeros(6,1);
dy(1)=y(2);
dy(2)=-(y(4))*(y(6));
dy(3)=y(4);
dy(4)=-(-1)*(y(2))*(y(6));
dy(5)=y(6);
```

```
dy(6)=0 ;
end
```

Nil3_G_2:

```
clearall;clc;
tspan = [-4 4];
y0 = [-2;2;-2;2;-2;2];
[t, y] = ode45(@Nil3_G_1, tspan, y0)
plot3(y(:,1),y(:,3),y(:,5),'linewidth',3,'color','r');
axis equal;
xlabel('X');ylabel('Y');zlabel('Z');
gridon;box on;
```

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