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ON PSEUDO-PROJECTIVE CURVATURE TENSOR OF SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION

ABHIJIT MANDAL AND ASHOKE DAS

Abstract. The purpose of the present paper is to study some properties of Sasakian manifolds with respect to Zamkovoy connection. Here, we study pseudo-projectively flat, quasi-pseudo-projectively flat and ϕ-pseudo-projectively flat Sasakian manifolds admitting Zamkovoy connection. Further, we study generalized pseudo-projective ϕ-recurrent Sasakian manifolds along with some more curvature properties of Sasakian manifolds with respect to Zamkovoy connection.

Key Words: Sasakian manifold, Zamkovoy connection, pseudo-projective curvature tensor.
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1. Introduction

A linear connection $\bar{\nabla}$ defined on a Riemannian manifold M is said to be symmetric if torsion $\bar{T}$ of $\bar{\nabla}$ defined by

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y],$$

is zero for any vector fields X, Y on M, otherwise, it is said to be non-symmetric. In 1932, Hayden [12] gave the idea of a metric connection on a Riemannian manifold and later named such connection a Hayden connection. A linear connection $\bar{\nabla}$ is called metric connection on a

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∗Address correspondence to A. Mandal; E-mail: abhijit4791@gmail.com.

Riemannian manifold M if covariant derivative of the metric of M is zero, i.e., $\bar{\nabla}g = 0$, otherwise non-metric.

In 2008, the notion of Zamkovoy canonical connection (briefly, Zamkovoy connection) was introduced by S. Zamkovoy [24] for a para-contact manifold. And this connection was defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold. Later, A. Biswas and K. K. Baishya studied this connection on generalized pseudo-Ricci symmetric Sasakian manifolds [2] and on almost pseudo-symmetric Sasakian manifolds [3]. This connection was further studied by A. M. Blaga [1] on para-Kenmotsu manifold. In 2020, A. Mandal and A. Das [5, 6, 7, 8, 9] studied in detail on various curvature tensors of Sasakian and LP-Sasakian manifolds admitting this new connection.

For an n-dimensional almost contact metric manifold M equipped with metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the relation between Zamkovoy connection (∇^*) and Levi-civita connection (∇) is given by [24]

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{1.1}$$

for all $X, Y \in \chi(M)$.

Sasakian manifold [18] with Riemannian metric was defined by Japanese mathematician S. Sasaki in the year 1960. Sasakian manifolds may be viewed as an odd dimensional analogous of Kâhler manifolds. This manifold was further studied by several authors, namely, B. P. Charles, J. Sparks, S. T. Yau, Z. Olszak, M. C. Chaki [11, 13, 16, 10] and many others.

The pseudo-projective curvature tensor on Riemannian manifold was introduced by B. Prasad [17] in 2002. In [14], H. G. Nagaraja and G. Somashekhara showed that every pseudo-projectively flat and pseudo-projective semi-symmetric Sasakian manifolds are locally isomorphic to unit sphere. The properties of this curvature tensor was further studied by many researcher. For details we refer [15, 22, 20] and the references therein. Pseudo-projective curvature tensor $\overline{P}$ of rank 3 is given by

$$\begin{aligned} \overline{P}(X,Y)Z &= aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] \\ &\quad + cr\left[g(Y,Z)X - g(X,Z)Y\right] \end{aligned} \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ denotes the set of all vector fields on the manifold M. Here, the non-zero constants a, b and c are related as

$$c=-\frac{1}{n}\left(\frac{a}{n-1}+b\right) \tag{1.3}$$

and r is scalar curvature, $R(X,Y)Z$ denotes the Riemannian curvature tensor of type $(1,3)$, S denotes the Ricci tensor of type $(0,2)$ and g is a Riemannian metric.

Definition 1.1. An n-dimensional Sasakian manifold M is said to be η-Einstein manifold if the Ricci tensor of type (0,2) is of the form:

$$S(X,Y)=k_1 g(X,Y)+k_2\eta(X)\eta(Y)$$

for all $X,Y\in\chi(M)$, where k_1, k_2 are scalars.

This paper has been organized as follows:

After introduction a short description of Sasakian manifold has been given in **Section-2**. In **Section-3**, we introduce Zamkovoy connection on Sasakian manifold. In **Section-4** we have obtained Riemannian curvature tensor R^*, Ricci curvature tensor S^*, Scalar curvature tensor r^*, Ricci operator Q^* with respect to Zamkovoy connection. **Section-5** contains discussion of pseudo-projectively flat Sasakian manifold with respect to the Zamkovoy connection. **Section-6** concerns with quasi-pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection. In **Section-7**, we have discussed ϕ-pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection. **Section-8** deals with the pseudo-projective ϕ-recurrent Sasakian manifold with respect to Zamkovoy connection. In **Section-9**, we have discussed Sasakian manifold satisfying $\overline{P}^*(\xi,X).R^*=0$, where $\overline{P}^*$ is the pseudo-projective curvature tensor with respect to Zamkovoy connection. Finally, **Section-10** contains an example of 5-dimensional Sasakian manifold admitting Zamkovoy connection.

2. Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then,

$$\phi^2V = -V+\eta(V)\xi,\eta(\xi)=1,\eta(\phi U)=0,\ \phi\xi=0, \tag{2.1}$$

$$g(U,V) = g(\phi U,\phi V)+\eta(U)\eta(V), \tag{2.2}$$

$$g(U,\phi V) = -g(\phi U,V),\eta(V)=g(V,\xi), \tag{2.3}$$

for all $U, V \in \chi(M)$, are all satisfied on M. An almost contact metric manifold M is said to be a contact metric manifold if

$$g(U, \phi V) = d\eta(U, V), \tag{2.4}$$

and a contact metric manifold M is called Sasakian if

$$(\nabla_U \phi)V = g(U, V)\xi - \eta(V)U, \tag{2.5}$$

In an n-dimensional Sasakian manifold the following relations also hold [4, 19, 23]:

$$\begin{aligned}
\nabla_U \xi &= -\phi U, && (2.6)\\
(\nabla_U \eta) V &= g(U, \phi V), && (2.7)\\
R(U, V)\xi &= \eta(V)U - \eta(U)V, && (2.8)\\
R(\xi, U)V &= g(U, V)\xi - \eta(V)U, &&\\
S(U, \xi) &= (n-1)\eta(U), &&\\
R(U, \xi)V &= \eta(V)U - g(U, V)\xi, &&\\
Q\xi &= (n-1)\xi, &&\\
S(\phi U, \phi V) &= S(U, V) - (n-1)\eta(U)\eta(V), && (2.9)
\end{aligned}$$

for all $U, V \in \chi(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator of M.

3. Zamkovoy connection on Sasakian manifold

Lemma 3.1. *Let M be an n-dimensional Sasakian manifold, then the relation between Zamkovoy connection and Levi-Civita connection on M is given by*

$$\nabla^*_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y, \tag{3.1}$$

where the torsion tensor of Zamkovoy connection is given by

$$T^*(X, Y) = 2g(X, \phi Y)\xi, \tag{3.2}$$

for all vector fields X, Y on M. Here, ∇^* denotes the Zamkovoy connection and ∇ denotes the Levi-Civita connection.

Proof. Suppose that the Zamkovoy connection and the Levi-Civita connection be related as

$$\nabla^*_X Y = \nabla_X Y + \mathcal{H}(X, Y), \tag{3.3}$$

for all $X, Y \in \chi(M)$, where $\mathcal{H}(X,Y)$ is a tensor field of type $(1,1)$. Then, by definition of torsion tensor we have

$$T^*(X,Y) = \mathcal{H}(X,Y) - \mathcal{H}(Y,X). \tag{3.4}$$

It follows that

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z). \tag{3.5}$$

In view of (1.1), (2.7) and (3.5), we have $\nabla^* g = 0$, i.e., Zamkovoy connection is a metric compatible connection and hence from (3.3), we get

$$0 = g(\mathcal{H}(X,Y), Z) + g(\mathcal{H}(X,Z), Y). \tag{3.6}$$

Now, in reference to (3.4) and (3.6), we have

$$\begin{aligned}
& g(T^*(X,Y), Z) + g(T^*(Z,X), Y) + g(T^*(Z,Y), X) \\
= \; & g(\mathcal{H}(X,Y), Z) - g(\mathcal{H}(Y,X), Z) + g(\mathcal{H}(Z,X), Y) \\
& - g(\mathcal{H}(X,Z), Y) + g(\mathcal{H}(Z,Y), X) + g(\mathcal{H}(Y,Z), X) \\
= \; & 2g(\mathcal{H}(X,Y), Z).
\end{aligned} \tag{3.7}$$

Setting

$$g(T^*(Z,X), Y) = g(\overline{T}(X,Y), Z), \tag{3.8}$$

$$g(T^*(Z,Y), X) = g(\overline{T}(Y,X), Z). \tag{3.9}$$

in the equation (3.7), we have

$$\begin{aligned}
& g(T^*(X,Y), Z) + g(\overline{T}(X,Y), Z) + g(\overline{T}(Y,X), Z) \\
= \; & 2g(\mathcal{H}(X,Y), Z),
\end{aligned} \tag{3.10}$$

which implies that

$$\mathcal{H}(X,Y) = \frac{1}{2}\left[T^*(X,Y) + \overline{T}(X,Y) + \overline{T}(Y,X)\right]. \tag{3.11}$$

From (3.2), (3.8) and (3.9), it follows that

$$\overline{T}(X,Y) = 2\eta(Y)\phi X, \; \overline{T}(Y,X) = 2\eta(X)\phi Y. \tag{3.12}$$

Using (3.2) and (3.12) in (3.11), we have

$$\mathcal{H}(X,Y) = g(X,\phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y. \tag{3.13}$$

By the help of (3.3) and (3.13), we can easily bring out the equation (3.1). □

Proposition 3.2. *Zamkovoy connection on Sasakian manifold is a metric compatible linear connection.*

Proposition 3.3. *The structure vector field ξ of a Sasakian manifold M is parallel with respect to Zamkovoy connection.*

Proof. From the equation (3.1), it is obvious that $\nabla^*_X \xi = 0$ for all $X \in \chi(M)$. □

Proposition 3.4. *The integral curve of ξ with respect to Zamkovoy connection is a geodesic in Sasakian manifold.*

4. Riemannian curvature tensor of Sasakian manifold with respect to Zamkovoy connection

By the help of (2.6), (2.5), (2.8) and (3.1), we get the following results:

$$\nabla^*_X \eta(Y) = \eta(\nabla_X Y) + g(X, \phi Y), \tag{4.1}$$

$$\begin{aligned} \nabla^*_X(\phi Y) &= \nabla_X(\phi Y) - g(\phi X, \phi Y)\xi \\ &\quad -\eta(X)Y + \eta(X)\eta(Y)\xi, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \nabla^*_X g(Y, \phi Z) &= g(\nabla_X Y, \phi Z) + \eta(X) g(\phi Y, \phi Z) + g(Y, \nabla_X(\phi Z)) \\ &\quad -\eta(X) g(Y, Z) + \eta(X)\eta(Y)\eta(Z). \end{aligned} \tag{4.3}$$

Let us denote the Riemannian curvature tensor with respect to Zamkovoy connection by R^* and it be defined as

$$R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z, \tag{4.4}$$

for all $X, Y, Z \in \chi(M)$.

Using (3.1), (4.1), (4.2) and (4.3) in (4.4), we obtain the followings:

$$\begin{aligned} &\nabla^*_X \nabla^*_Y Z \\ = {} &\nabla_X \nabla_Y Z + g(X, \phi \nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X + \eta(X)\phi \nabla_Y Z \\ &+ g(\nabla_X Y, \phi Z)\xi + \eta(X) g(\phi Y, \phi Z)\xi + g(Y, \nabla_X(\phi Z))\xi \\ &-\eta(X) g(Y, Z)\xi + \eta(X)\eta(Y)\eta(Z)\xi - g(Y, \phi Z)\phi X \\ &+ g(Y, \phi Z)\phi X + \eta(\nabla_X Z)\phi Y + g(X, \phi Z)\phi Y \\ &+ g(Z, \phi X)\phi Y + \eta(Z)\nabla_X(\phi Y) - \eta(Z) g(\phi X, \phi Y)\xi \\ &-\eta(Z)\eta(X)Y + \eta(Z)\eta(X)\eta(Y)\xi + \eta(\nabla_X Y)\phi Z \\ &+ g(X, \phi Y)\phi Z - g(Y, \phi X)\phi Z + g(Y, \phi X)\phi Z \\ &+ \eta(Y)\nabla_X(\phi Z) - \eta(Y) g(\phi X, \phi Z)\xi - g(Z, \phi X)\phi Y \\ &-\eta(Y)\eta(X)Z + \eta(Y)\eta(X)\eta(Z)\xi, \end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
& \nabla^*_{[X,Y]}Z \\
= \; & \nabla_{[X,Y]}Z + g(\nabla_X Y, \phi Z)\xi - g(\nabla_Y X, \phi Z)\xi + \eta(Z)\phi\nabla_X Y \\
& -\eta(Z)\phi\nabla_Y X + \eta(\nabla_X Y)\phi Z - \eta(\nabla_Y X)\phi Z. \qquad (4.6)
\end{aligned}
$$

Interchanging X and Y in (4.5) and using it along with (4.5), (4.6) in (4.4), we obtain the Riemannian curvature tensor in Sasakian manifold M with respect to Zamkovoy connection as

$$
\begin{aligned}
R^*(X,Y)Z &= R(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X \\
&\quad -2g(Y,\phi X)\phi Z + g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi \\
&\quad +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \qquad (4.7)
\end{aligned}
$$

Consequently, one can easily bring out the following results:

$$
\begin{aligned}
(4.8)\quad S^*(Y,Z) &= S(Y,Z) + 2g(Y,Z) - (1+n)\eta(Y)\eta(Z), \\
(4.9)\quad S^*(Y,\xi) &= 0, S^*(\xi,Z) = 0, \\
(4.10)\quad Q^*Y &= QY + 2Y - (1+n)\eta(Y)\xi, Q^*\xi = 0, \\
(4.11)\quad r^* &= r + n - 1, \\
(4.12)\quad R^*(X,Y)\xi &= 0, \\
(4.13)\quad R^*(\xi,Y)Z &= 0, \\
(4.14)\quad R^*(X,\xi)Z &= 0.
\end{aligned}
$$

Proposition 4.1. *Let M be an n-dimensional Sasakian manifold admitting Zamkovoy connection ∇^*, then*

(i) The Riemannian curvature tensor R^* with respect to ∇^* is given by (4.7),

(ii) The Ricci tensor S^* with respect to ∇^* is given by (4.8),

(iii) The scalar curvature r^* with respect to ∇^* is given by (4.11),

(iv) The Ricci tensor S^* with respect to ∇^* is symmetric.

Theorem 4.2. *If an n-dimensional Sasakian manifold M is Ricci flat with respect to Zamkovoy connection, then M is an η-Einstein manifold.*

Proof. Let us assume that the Sasakian manifold M be Ricci flat with respect to the Zamkovoy connection. Then, equation (4.8) gives

$$S(Y,Z) = -2g(Y,Z) + (1+n)\eta(Y)\eta(Z),$$

which shows that M is an η-Einstein manifold. □

The pseudo-projective curvature tensor $\left(\overline{\mathcal{P}}^{*}\right)$ with respect to Zamkovoy connection in an n-dimensional Sasakian manifold M is given by

$$\begin{aligned} \overline{\mathcal{P}}^{*}(X,Y)Z &= aR^{*}(X,Y)Z + b\left[S^{*}(Y,Z)X - S^{*}(X,Z)Y\right] \\ &\quad + cr^{*}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{4.15}$$

for all $X, Y, Z \in \chi(M)$, where the non-zero constants a, b and c are related as

$$c = -\frac{1}{n}\left(\frac{a}{n-1} + b\right), \tag{4.16}$$

and r^{*} is the scalar curvature with respect to Zamkovoy connection, $R^{*}(X,Y)Z$ denotes the Riemannian curvature tensor with respect to Zamkovoy connection of type $(1,3)$, S^{*} denotes the Ricci tensor with respect to Zamkovoy connection of type $(0,2)$ and g is a Riemannian metric.

5. Pseudo-projectively flat and ξ- pseudo-projectively flat Sasakian manifolds with respect to Zamkovoy connection

Definition 5.1. An n-dimensional Sasakian manifold M is said to be pseudo-projectively flat with respect to Zamkovoy connection if

$$\overline{\mathcal{P}}^{*}(X,Y)Z = 0,$$

for all $X, Y, Z \in \chi(M)$.

Definition 5.2. An n-dimensional Sasakian manifold M is said to be ξ-pseudo-projectively flat with respect to Zamkovoy connection if

$$\overline{\mathcal{P}}^{*}(X,Y)\xi = 0,$$

for all X, $Y \in \chi(M)$.

Theorem 5.3. *A pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection is an η-Einstein manifold.*

Proof. Let us consider a pseudo-projectively flat Sasakian manifold (M) with respect to Zamkovoy connection, i.e., $\overline{\mathcal{P}}^{*}(X,Y)Z = 0$. In view of (4.15), we have

$$\begin{aligned} 0 &= aR^{*}(X,Y)Z + b\left[S^{*}(Y,Z)X - S^{*}(X,Z)Y\right] \\ &\quad + cr^{*}\left[g(Y,Z)X - g(X,Z)Y\right]. \end{aligned} \tag{5.1}$$

Taking inner product of (5.1) with a vector field V, we have

$$\begin{aligned} 0 &= ag\left(R^{*}\left(X,Y\right)Z,V\right)+b\left[S^{*}\left(Y,Z\right)g\left(X,V\right)-S^{*}\left(X,Z\right)g\left(Y,V\right)\right] \\ &\quad +cr^{*}\left[g\left(Y,Z\right)g\left(X,V\right)-g\left(X,Z\right)g\left(Y,V\right)\right], \end{aligned} \tag{5.2}$$

for all vector fields X,Y,Z,V on M. Taking an orthonormal frame field of M and contracting (5.2) over X and V, we obtain

$$S^{*}\left(Y,Z\right)=-\frac{cr^{*}\left(n-1\right)}{\left[a+b\left(n-1\right)\right]}g\left(Y,Z\right). \tag{5.3}$$

By the help of (4.8) and (5.3), we get

$$\begin{aligned} S\left(Y,Z\right) &= -\left[\frac{c\left(r+n-1\right)\left(n-1\right)}{a+b\left(n-1\right)}+2\right]g\left(Y,Z\right) \\ &\quad +\left(1+n\right)\eta\left(Y\right)\eta\left(Z\right). \end{aligned} \tag{5.4}$$

Using (4.16) in (5.4), we get

$$S\left(Y,Z\right)=\left[\frac{r-n-1}{n}\right]g\left(Y,Z\right)+\left(1+n\right)\eta\left(Y\right)\eta\left(Z\right),$$

which shows that M is an η-Einstein manifold. □

Theorem 5.4. *An n-dimensional Sasakian manifold M cannot be ξ-pseudo-projectively flat with respect to Zamkovoy connection if $r\neq-\left(n-1\right)$.*

Proof. If a Sasakian manifold M be ξ-pseudo-projectively flat with respect to Zamkovoy connection, i.e., $\overline{\mathcal{P}}^{*}\left(X,Y\right)\xi=0$, for all vector fields X,Y on M, then from (4.15), we get

$$\begin{aligned} 0 &= aR^{*}\left(X,Y\right)\xi+b\left[S^{*}\left(Y,\xi\right)X-S^{*}\left(X,\xi\right)Y\right] \\ &\quad +cr^{*}\left[\eta\left(Y\right)X-\eta\left(X\right)Y\right]. \end{aligned} \tag{5.5}$$

Using (2.8), (4.8), (4.9) and (4.12) in (5.5), we have

$$\left(r+n-1\right)R\left(X,Y\right)\xi=0.$$

Now, if $r\neq-\left(n-1\right)$, then $R\left(X,Y\right)\xi=0$, which is not admissible in M. □

6. Quasi-pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection

Definition 6.1. An n-dimensional Sasakian manifold M is said to be quasi-pseudo-projectively flat with respect to Zamkovoy connection if $g\left(\overline{P}^{*}\left(\phi X,Y\right)Z,\phi V\right)=0$, for all $X,Y,Z,V\in\chi(M)$.

Theorem 6.2. *If an n-dimensional Sasakian manifold M is quasi-pseudo-projectively flat with respect to Zamkovoy connection, then M is an η-Einstein manifold, provided that $a \neq b$.*

Proof. Let us consider a quasi-pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection, i.e.,

$$g\left(\overline{P}^{*}\left(\phi X, Y\right) Z, \phi V\right) = 0.$$

In view of (4.15), we have

$$\begin{aligned} 0 &= ag\left(R^{*}\left(\phi X, Y\right) Z, \phi V\right) \\ &\quad +b\left[S^{*}\left(Y, Z\right) g\left(\phi X, \phi V\right) - S^{*}\left(\phi X, Z\right) g\left(Y, \phi V\right)\right] \\ &\quad +cr^{*}\left[g\left(Y, Z\right) g\left(\phi X, \phi V\right) - g\left(\phi X, Z\right) g\left(Y, \phi V\right)\right], \end{aligned} \tag{6.1}$$

for all vector fields X, Y, Z, V on M. Taking an orthonormal frame field of M and contracting (6.1) over Y and Z, we get

$$\begin{aligned} 0 &= a\sum_{i=1}^{n} g\left(R^{*}\left(\phi X, e_i\right) e_i, \phi V\right) \\ &\quad +b\sum_{i=1}^{n}\left[S^{*}\left(e_i, e_i\right) g\left(\phi X, \phi V\right) - S^{*}\left(\phi X, e_i\right) g\left(e_i, \phi V\right)\right] \\ &\quad +cr^{*}\sum_{i=1}^{n}\left[g\left(e_i, e_i\right) g\left(\phi X, \phi V\right) - g\left(\phi X, e_i\right) g\left(e_i, \phi V\right)\right]. \end{aligned} \tag{6.2}$$

It is easily seen that

$$\sum_{i=1}^{n} S^{*}\left(\phi X, e_i\right) g\left(e_i, \phi V\right) = S^{*}\left(\phi X, \phi V\right), \tag{6.3}$$

$$\sum_{i=1}^{n} S^{*}\left(e_i, e_i\right) = r^{*}, \tag{6.4}$$

$$\sum_{i=1}^{n} g\left(e_i, e_i\right) = n. \tag{6.5}$$

Assuming $a \neq b$ and using (6.3), (6.4) and (6.5) in (6.2), we get

$$0 = nS^{*}\left(\phi X, \phi V\right) - r^{*} g\left(\phi X, \phi V\right). \tag{6.6}$$

Using (2.2), (2.9) and (4.11) in (6.6), we get

$$\begin{aligned} S(X,V) &= -\frac{1}{n}(n-r+1)\, g(X,V) \\ &\quad +\frac{1}{n}\left(n^2-2n+r-1\right)\eta(X)\,\eta(V). \end{aligned} \tag{6.7}$$

Therefore, M is an η-Einstein manifold. □

Corollary 6.3. *If a Sasakian manifold M of dimension n is quasi-pseudo-projectively flat with respect to Zamkovoy connection, then M is of constant scalar curvature, provided that $a \neq b$.*

Proof. Taking an orthonormal frame field of M and contracting (6.7) over X and V, we obtain $r = 3n + 1$. □

7. ϕ-pseudo-projectively flat Sasakian manifold with respect to Zamkovoy connection

Definition 7.1. An n-dimensional Sasakian manifold M is said to be ϕ-pseudo-projectively flat with respect to Zamkovoy connection if

$$g\left(\overline{\mathcal{P}}^*(\phi X, \phi Y)\,\phi Z, \phi V\right) = 0,$$

for all $X, Y, Z, V \in \chi(M)$.

Theorem 7.2. *If an n-dimensional Sasakian manifold M is ϕ-pseudo-projectively flat with respect to Zamkovoy connection, then M is an η-Einstein manifold provided that $a + b(n-2) \neq 0$.*

Proof. Let a Sasakian manifold M be ϕ-pseudo-projectively flat with respect to Zamkovoy connection, i.e.,

$$g\left(\overline{\mathcal{P}}^*(\phi X, \phi Y)\,\phi Z, \phi V\right) = 0,$$

for all $X, Y, Z, V \in \chi(M)$. Then, in view of (4.15), we have

$$\begin{aligned} 0 &= ag\left(R^*(\phi X, \phi Y)\,\phi Z, \phi V\right) \\ &\quad +b\left[S^*(\phi Y, \phi Z)\, g(\phi X, \phi V) - S^*(\phi X, \phi Z)\, g(\phi Y, \phi V)\right] \\ &\quad +cr^*\left[g(\phi Y, \phi Z)\, g(\phi X, \phi V) - g(\phi X, \phi Z)\, g(\phi Y, \phi V)\right]. \end{aligned} \tag{7.1}$$

Let the tangent space at any point of M has a local orthonormal basis $\{e_i, \xi\}\,(1 \leq i \leq n-1)$. Using the fact that $\{\phi e_i, \xi\}\,(1 \leq i \leq n-1)$ is also a local orthonormal basis of the tangent space and setting $X =$

$V = e_i$ and taking summation over $i (1 \leq i \leq n-1)$, it follows from (7.1) that

$$\begin{aligned} & a \sum_{i=1}^{n-1} g\left(R^{*}\left(\phi e_i, \phi Y\right) \phi Z, \phi e_i\right) \\ = & -b\left[\sum_{i=1}^{n-1} S^{*}(\phi Y, \phi Z) g\left(\phi e_i, \phi e_i\right)-\sum_{i=1}^{n-1} S^{*}\left(\phi e_i, \phi Z\right) g\left(\phi Y, \phi e_i\right)\right] \\ & -c r^{*}\left[\sum_{i=1}^{n-1} g(\phi Y, \phi Z) g\left(\phi e_i, \phi e_i\right)-\sum_{i=1}^{n-1} g\left(\phi e_i, \phi Z\right) g\left(\phi Y, \phi e_i\right)\right] . \end{aligned} \tag{7.2}$$

It can be easily seen that

$$\sum_{i=1}^{n-1} S^{*}\left(\phi e_i, \phi e_i\right) = r^{*}, \tag{7.3}$$

$$\sum_{i=1}^{n-1} g\left(R^{*}\left(\phi e_i, \phi Y\right) \phi Z, \phi e_i\right) = S^{*}(\phi Y, \phi Z), \tag{7.4}$$

$$\sum_{i=1}^{n-1} g\left(\phi e_i, \phi Z\right) S^{*}\left(\phi Y, \phi e_i\right) = S^{*}(\phi Y, \phi Z), \tag{7.5}$$

$$\sum_{i=1}^{n-1} g\left(\phi e_i, \phi Y\right) g\left(\phi Z, \phi e_i\right) = g(\phi Y, \phi Z). \tag{7.6}$$

Using (2.9), (4.8), (4.11) and (7.3)-(7.6) in (7.2), we get

$$\begin{aligned} 0 = & [a+b(n-2)] S(Y, Z) \\ & +[2a+\{2b+c(r+n-1)\}(n-2)] g(Y, Z)-\gamma_1 \eta(Y) \eta(Z) . \end{aligned} \tag{7.7}$$

where $\gamma_1=[2a+\{2b+c(r+n-1)\}(n-2)+(n-1)\{a+b(n-2)\}]$.
If $a+b(n-2) \neq 0$, then from (7.7), it follows that

$$S(Y, Z)=\gamma_2 g(Y, Z)+\gamma_3 \eta(Y) \eta(Z), \tag{7.8}$$

where

$$\begin{aligned}
\gamma_2 &= -\left[\frac{2a+\{2b+c\,(r+n-1)\}\,(n-2)}{a+b\,(n-2)}\right], \\
\gamma_3 &= \left[\frac{2a+\{2b+c\,(r+n-1)\}\,(n-2)+(a+nb-2b)\,(n-1)}{a+b\,(n-2)}\right], \\
c &= -\left[\frac{a+b\,(n-1)}{n\,(n-1)}\right].
\end{aligned}$$

This gives the theorem. □

Corollary 7.3. *If an n-dimensional Sasakian manifold M is ϕ-pseudo-projectively flat with respect to Zamkovoy connection, then M is of constant scalar curvature, provided that $a+b\,(n-2)=0$.*

Proof. If $a+b\,(n-2)=0$, then (7.7) gives $r=-\,(n-1)$. □

8. Generalized pseudo-projective ϕ-recurrent Sasakian manifold with respect to Zamkovoy connection

Definition 8.1. An n-dimensional Sasakian manifold is said to be generalized pseudo-projective ϕ-recurrent with respect to Zamkovoy connection if

$$\begin{aligned}
\phi^2\left(\nabla^*_W\overline{P}^*\right)(X,Y)\,Z &= A\,(W)\,\overline{P}^*\,(X,Y)\,Z \\
&\quad +B\,(W)\,[g\,(Y,Z)\,X-g\,(X,Z)\,Y]\,,
\end{aligned}$$

for all $X,Y,Z,W\in\chi\,(M)$, where A and B are 1-forms and B is non vanishing such that $A\,(W)=g\,(W,\rho_1)\,,B\,(W)=g\,(W,\rho_2)$. Here, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

Theorem 8.2. *In a generalized pseudo-projective ϕ-recurrent Sasakian manifold admitting Zamkovoy connection, the associated 1-forms are related by $B=\left[\frac{(a-b)(r+n-1)}{n(n-1)}\right]A$.*

Proof. Let M be a generalized pseudo-projective ϕ-recurrent Sasakian manifold with respect to Zamkovoy connection, then

$$\begin{aligned}
&\phi^2\left(\nabla^*_W\overline{\mathcal{P}}^*\right)(X,Y)\,Z \\
&= A\,(W)\,\overline{P}^*\,(X,Y)\,Z+B\,(W)\,[g\,(Y,Z)\,X-g\,(X,Z)\,Y]\,, \qquad (8.1)
\end{aligned}$$

where the 1-forms are given by $A(W) = g(W, \rho_1), B(W) = g(W, \rho_2)$, $B(W) \neq 0$. Here, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

Using (2.1) in (8.1), we have

$$\begin{aligned} & -\left(\nabla^*_W \overline{\mathcal{P}}^*\right)(X, Y) Z \\ = & -\eta\left(\left(\nabla^*_W \overline{\mathcal{P}}^*\right)(X, Y) Z\right) \xi + A(W) \overline{\mathcal{P}}^*(X, Y) Z \\ & + B(W)[g(Y, Z) X - g(X, Z) Y]. \end{aligned} \tag{8.2}$$

The inner product of (8.2) with a vector field V gives

$$\begin{aligned} & g\left(\left(\nabla^*_W \overline{\mathcal{P}}^*\right)(X, Y) Z, V\right) \\ = & \ \eta\left(\left(\nabla^*_W \overline{\mathcal{P}}^*\right)(X, Y) Z\right) \eta(V) - A(W) g\left(\overline{\mathcal{P}}^*(X, Y) Z, V\right) \\ & - B(W)[g(Y, Z) g(X, V) - g(X, Z) g(Y, V)]. \end{aligned} \tag{8.3}$$

Contracting (8.3) over Y and Z, we have

$$\begin{aligned} & [b + c(n-1)] \nabla^*_W r^* g(X, V) \\ = & -(a-b)\left(\nabla^*_W S^*\right)(X, V) + [b + c(n-1)] \nabla^*_W r^* \eta(X) \eta(V) \\ & - A(W)\left[(a-b) S^*(X, V) + r^*\{b + c(n-1)\} g(X, V)\right] \\ & - B(W)(n-1) g(X, V). \end{aligned} \tag{8.4}$$

Setting $X = \xi$ and using (4.8), (4.11) in (8.4), we have

$$B(W) = \left[\frac{(a-b)(r+n-1)}{n(n-1)}\right] A(W).$$

□

9. Sasakian manifold admitting Zamkovoy connection and satisfying $\overline{\mathcal{P}}^*(\xi, X).R^* = 0$

Theorem 9.1. *A Sasakian manifold M admitting Zamkovoy connection and satisfying $\overline{\mathcal{P}}^*(\xi, X).R^* = 0$ is an η-Einstein manifold, where $\overline{P}^*$ and R^* are pseudo-projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection, respectively.*

Proof. Consider a Sasakian manifold M satisfying the condition

$$\left(\overline{\mathcal{P}}^*(\xi, X).R^*\right)(Y, Z) V = 0, \tag{9.1}$$

where $\overline{\mathcal{P}}^*$ and R^* are pseudo-projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection, respectively.

Then, we have

$$\begin{aligned} 0 &= \overline{\mathcal{P}}^*(\xi, X) R^*(Y, Z) V - R^*\left(\overline{\mathcal{P}}^*(\xi, X) Y, Z\right) V \\ &\quad -R^*\left(Y, \overline{\mathcal{P}}^*(\xi, X) Z\right) V - R^*(Y, Z) \overline{\mathcal{P}}^*(\xi, X) V. \end{aligned} \tag{9.2}$$

Replacing Y by ξ in (9.2), we get

$$\begin{aligned} &\overline{\mathcal{P}}^*(\xi, X) R^*(\xi, Z) V \\ &= R^*\left(\overline{\mathcal{P}}^*(\xi, X) \xi, Z\right) V + R^*\left(\xi, \overline{\mathcal{P}}^*(\xi, X) Z\right) V \\ &\quad +R^*(\xi, Z) \overline{\mathcal{P}}^*(\xi, X_0) V. \end{aligned} \tag{9.3}$$

Using (4.12), (4.13) and (4.14) in (9.3), we get

$$R^*\left(\overline{\mathcal{P}}^*(\xi, X) \xi, Z\right) V = 0. \tag{9.4}$$

In reference to (4.7), (4.15) and (9.4), we get

$$0 = R^*(X, Z) V. \tag{9.5}$$

Taking inner product of (9.5) with vector field W, we obtain

$$0 = g\left(R^*(X, Z) V, W\right). \tag{9.6}$$

Contracting (9.6) over X and W and using (4.8), we get

$$S(Z, V) = -2g(Z, V) + (1+n)\eta(Z)\eta(V). \tag{9.7}$$

This proves the theorem. □

Corollary 9.2. *A Sasakian manifold M of dimension n admitting Zamkovoy connection and satisfying $\overline{\mathcal{P}}^*(\xi, X).R^* = 0$, is of constant scalar curvature.*

Proof. Taking an orthonormal frame field and contracting (9.7) over Z and V, we obtain $r = -(n-1)$. □

10. Example of 5-dimensional Sasakian manifold admitting Zamkovoy Connection.

Consider 5-dimensional manifold $M^5 = \left\{(x, y, z, u, v) \in R^5\right\}$ where (x, y, z, u, v) are the std. co-ordinates in R^5

We choose the linearly independent vector fields

$$\epsilon_1 = e^z \frac{\partial}{\partial x}, \epsilon_2 = e^z \frac{\partial}{\partial y}, \epsilon_3 = \frac{\partial}{\partial z}, \epsilon_4 = e^z \frac{\partial}{\partial u}, \epsilon_5 = e^z \frac{\partial}{\partial v}$$

Let g be a Riemannian metric defined by

$$\begin{aligned} g(\epsilon_i, \epsilon_j) &= 0, \ if \ i \neq j \\ &= 1, \ if \ i = j \\ for \ i, j &= 1, 2, 3, 4, 5 \end{aligned}$$

Let η be the $1 - form$ defined by $\eta(X) = g(X, \epsilon_3)$ for any $X \in \chi(M^5)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi\epsilon_1 = \epsilon_1, \phi\epsilon_2 = \epsilon_2, \phi\epsilon_3 = 0, \phi\epsilon_4 = \epsilon_4, \phi\epsilon_5 = \epsilon_5 \tag{10.1}$$

Let $X, Y, Z \in \chi(M^5)$ be given by

$$\begin{aligned} X &= x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + x_4\epsilon_4 + x_5\epsilon_5 \\ Y &= y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3 + y_4\epsilon_4 + y_5\epsilon_5 \\ Z &= z_1\epsilon_1 + z_2\epsilon_2 + z_3\epsilon_3 + z_4\epsilon_4 + z_5\epsilon_5 \end{aligned}$$

Then we have

$$\begin{aligned} g(X, Y) &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 \\ \eta(X) &= x_3 \\ g(\phi X, \phi Y) &= x_1y_1 + x_2y_2 + x_4y_4 + x_5y_5 \end{aligned}$$

Using the linearity of g and ϕ, $\eta(\epsilon_3) = 1, \phi^2 X = -X + \eta(X)\epsilon_3$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$.

We have

$$\begin{aligned} [\epsilon_1, \epsilon_3] &= -\epsilon_1, [\epsilon_2, \epsilon_3] = -\epsilon_2, [\epsilon_4, \epsilon_3] = -\epsilon_4, [\epsilon_5, \epsilon_3] = -\epsilon_5 \\ [\epsilon_i, \epsilon_j] &= 0 \text{ for all others } i \text{ and } j \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{aligned} \nabla_{\epsilon_1}\epsilon_1 &= \epsilon_3, \nabla_{\epsilon_1}\epsilon_2 = 0, \nabla_{\epsilon_1}\epsilon_3 = -\epsilon_1, \nabla_{\epsilon_1}\epsilon_4 = 0, \nabla_{\epsilon_1}\epsilon_5 = 0 \\ \nabla_{\epsilon_2}\epsilon_1 &= 0, \nabla_{\epsilon_2}\epsilon_2 = \epsilon_3, \nabla_{\epsilon_2}\epsilon_3 = -\epsilon_2, \ \nabla_{\epsilon_2}\epsilon_4 = 0, \nabla_{\epsilon_2}\epsilon_5 = 0 \\ \nabla_{\epsilon_3}\epsilon_1 &= 0, \ \nabla_{\epsilon_3}\epsilon_2 = 0 \ , \nabla_{\epsilon_3}\epsilon_3 = 0 \ , \nabla_{\epsilon_3}\epsilon_4 = 0 \ , \nabla_{\epsilon_3}\epsilon_5 = 0 \\ \nabla_{\epsilon_4}\epsilon_1 &= 0, \nabla_{\epsilon_4}\epsilon_2 = 0, \nabla_{\epsilon_4}\epsilon_3 = -\epsilon_4, \nabla_{\epsilon_4}\epsilon_4 = \epsilon_3, \nabla_{\epsilon_4}\epsilon_5 = 0 \\ \nabla_{\epsilon_5}\epsilon_1 &= 0, \nabla_{\epsilon_5}\epsilon_2 = 0, \nabla_{\epsilon_5}\epsilon_3 = -\epsilon_5, \nabla_{\epsilon_5}\epsilon_4 = 0, \ \nabla_{\epsilon_5}\epsilon_5 = \epsilon_3 \end{aligned}$$

From the above results we see that the structure (ϕ, ξ, η, g) satisfies

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \forall X, \ Y \in \chi(M^5). \tag{10.2}$$

where $\eta(\xi) = \eta(\epsilon_3) = 1$. Hence (ϕ, ξ, η, g) is a Sasakian structure and $M^5(\phi, \xi, \eta, g)$ is a 5-dimensional Sasakian manifold.

The components of Riemannian curvature with respect to Levi-Civita connectio ∇ are given by

$$
\begin{aligned}
R(\epsilon_1,\epsilon_2)\epsilon_1 &= \epsilon_2, R(\epsilon_1,\epsilon_2)\epsilon_2=-\epsilon_1, R(\epsilon_1,\epsilon_3)\epsilon_1=\epsilon_3, R(\epsilon_1,\epsilon_3)\epsilon_3=-\epsilon_1\\
R(\epsilon_1,\epsilon_4)\epsilon_1 &= \epsilon_4, R(\epsilon_1,\epsilon_4)\epsilon_4=-\epsilon_1, R(\epsilon_1,\epsilon_5)\epsilon_1=\epsilon_5, R(\epsilon_1,\epsilon_5)\epsilon_5=-\epsilon_1\\
R(\epsilon_2,\epsilon_1)\epsilon_2 &= \epsilon_1, R(\epsilon_2,\epsilon_1)\epsilon_1=-\epsilon_2, R(\epsilon_2,\epsilon_3)\epsilon_2=\epsilon_3, R(\epsilon_2,\epsilon_3)\epsilon_3=-\epsilon_2\\
R(\epsilon_2,\epsilon_4)\epsilon_2 &= \epsilon_4, R(\epsilon_2,\epsilon_4)\epsilon_4=-\epsilon_2, R(\epsilon_2,\epsilon_5)\epsilon_2=\epsilon_5, R(\epsilon_2,\epsilon_5)\epsilon_5=-\epsilon_2\\
R(\epsilon_3,\epsilon_1)\epsilon_3 &= \epsilon_1, R(\epsilon_3,\epsilon_1)\epsilon_1=-\epsilon_3, R(\epsilon_3,\epsilon_2)\epsilon_3=\epsilon_2, R(\epsilon_3,\epsilon_2)\epsilon_2=-\epsilon_3\\
R(\epsilon_3,\epsilon_4)\epsilon_3 &= \epsilon_4, R(\epsilon_3,\epsilon_4)\epsilon_4=-\epsilon_3, R(\epsilon_3,\epsilon_5)\epsilon_3=\epsilon_5, R(\epsilon_3,\epsilon_5)\epsilon_5=-\epsilon_3\\
R(\epsilon_4,\epsilon_1)\epsilon_4 &= \epsilon_1, R(\epsilon_4,\epsilon_1)\epsilon_1=-\epsilon_4, R(\epsilon_4,\epsilon_2)\epsilon_4=\epsilon_2, R(\epsilon_4,\epsilon_2)\epsilon_2=-\epsilon_4\\
R(\epsilon_4,\epsilon_3)\epsilon_4 &= \epsilon_3, R(\epsilon_4,\epsilon_3)\epsilon_3=-\epsilon_4, R(\epsilon_4,\epsilon_5)\epsilon_4=\epsilon_5, R(\epsilon_4,\epsilon_5)\epsilon_5=-\epsilon_4\\
R(\epsilon_5,\epsilon_1)\epsilon_5 &= \epsilon_1, R(\epsilon_5,\epsilon_1)\epsilon_1=-\epsilon_5, R(\epsilon_5,\epsilon_2)\epsilon_5=\epsilon_2, R(\epsilon_5,\epsilon_2)\epsilon_2=-\epsilon_5\\
R(\epsilon_5,\epsilon_3)\epsilon_5 &= \epsilon_3, R(\epsilon_5,\epsilon_3)\epsilon_3=-\epsilon_5, R(\epsilon_5,\epsilon_4)\epsilon_5=\epsilon_4, R(\epsilon_5,\epsilon_4)\epsilon_4=-\epsilon_5
\end{aligned}
$$

From (3.1), we obtain

$$
\begin{aligned}
\nabla^*_{\epsilon_1}\epsilon_1 &= 2\epsilon_3,\ \nabla^*_{\epsilon_1}\epsilon_2=0, \nabla^*_{\epsilon_1}\epsilon_3=0, \nabla^*_{\epsilon_1}\epsilon_4=0, \nabla^*_{\epsilon_1}\epsilon_5=0\\
\nabla^*_{\epsilon_2}\epsilon_1 &= 0, \nabla^*_{\epsilon_2}\epsilon_2=2\epsilon_3, \nabla^*_{\epsilon_2}\epsilon_3=0,\ \nabla^*_{\epsilon_2}\epsilon_4=0, \nabla^*_{\epsilon_2}\epsilon_5=0\\
\nabla^*_{\epsilon_3}\epsilon_1 &= \epsilon_1, \nabla^*_{\epsilon_3}\epsilon_2=\epsilon_2, \nabla^*_{\epsilon_3}\epsilon_3=0, \nabla^*_{\epsilon_3}\epsilon_4=\epsilon_4, \nabla^*_{\epsilon_3}\epsilon_5=\epsilon_5\\
\nabla^*_{\epsilon_4}\epsilon_1 &= 0,\ \nabla^*_{\epsilon_4}\epsilon_2=0, \nabla^*_{\epsilon_4}\epsilon_3=0, \nabla^*_{\epsilon_4}\epsilon_4=2\epsilon_3, \nabla^*_{\epsilon_4}\epsilon_5=0\\
\nabla^*_{\epsilon_5}\epsilon_1 &= 0, \nabla^*_{\epsilon_5}\epsilon_2=0, \nabla^*_{\epsilon_5}\epsilon_3=0, \nabla^*_{\epsilon_5}\epsilon_4=0,\ \nabla^*_{\epsilon_5}\epsilon_5=2\epsilon_3
\end{aligned}
$$

The non zero components of Riemannian curvature tensor with respect to Zamkovoy connection are given by

$$
\begin{aligned}
R^*(\epsilon_1,\epsilon_3)\epsilon_1 &= 4\epsilon_3, R^*(\epsilon_2,\epsilon_3)\epsilon_2=4\epsilon_3\\
R^*(\epsilon_4,\epsilon_3)\epsilon_4 &= 4\epsilon_3, R^*(\epsilon_5,\epsilon_3)\epsilon_5=4\epsilon_3\\
R^*(\epsilon_3,\epsilon_1)\epsilon_1 &= -4\epsilon_3, R^*(\epsilon_3,\epsilon_2)\epsilon_2=-4\epsilon_3\\
R^*(\epsilon_3,\epsilon_4)\epsilon_4 &= -4\epsilon_3, R^*(\epsilon_3,\epsilon_5)\epsilon_5=-4\epsilon_3
\end{aligned}
$$

Using the above curvature tensors the Ricci curvature tensors with respect to ∇ and ∇^* are:

$$
\begin{aligned}
S(\epsilon_1,\epsilon_1) &= S(\epsilon_2,\epsilon_2)=S(\epsilon_3,\epsilon_3)=S(\epsilon_4,\epsilon_4)=S(\epsilon_5,\epsilon_5)=-4\\
S^*(\epsilon_1,\epsilon_1) &= S^*(\epsilon_2,\epsilon_2)=S^*(\epsilon_4,\epsilon_4)=S^*(\epsilon_5,\epsilon_5)=-4, S^*(\epsilon_3,\epsilon_3)=0
\end{aligned}
$$

The relation between two scalar curvatures r^* and r in M^5 is obtained as follows

$$
\begin{aligned}
r^* &= \sum_{i=1}^{5} S^*\left(\epsilon_i, \epsilon_i\right) \\
&= -16 \\
&= \sum_{i=1}^{5} S\left(\epsilon_i, \epsilon_i\right) + 5 - 1 \\
&= r + n - 1 \qquad (10.3)
\end{aligned}
$$

which implies the relation (4.11). Similarly, we can verify the others results.

11. Conclusion

In this paper we have investigated that Ricci flat, pseudo-projectively flat, quasi-pseudo-projectively flat and ϕ-pseudo-projectively flat Sasakian manifolds with respect to Zamkovoy conection are η-Einstein manifolds. Besides these, we have investigated that Sasakian manifolds satisfying $\overline{P}^*(\xi, X) \circ R^* = 0$ are η-Einstein manifolds, where $\overline{P}^*$ and R^* are pseudo-projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection, respectively.

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Abhijit Mandal
Department of Mathematics, Raiganj Surendranath Mahavidyalaya, P.O.Box 733134, Raiganj, West Bengal, India
Email: abhijit4791@gmail.com

Ashoke Das
Department of Mathematics, Raiganj University P.O.Box 733134, Raiganj, West Bengal, India
Email: ashoke.avik@gmail.com