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NEW FIXED POINT RESULTS IN FUZZY METRIC SPACE

SOUMEYEH KHALEGHIZADEH

Abstract. In this paper, we introduce certain new classes of contraction mapping and establish fixed point theorems for such kind of mapping in triangular fuzzy metric spaces.

Key Words: Triangular fuzzy metric space; Fixed points $\alpha - \eta - GF - contraction$; Generalized fuzzy metric space.

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1. Introduction

Banach contraction principle is a popular tool in solving existence problems in many branches of mathematics. This result has been extended in many directions. In 2008, in order to characterize the completeness of underlying metric spaces, Suzuki [9] introduced a weaker notion of contraction. Recently, Wardowski[10] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Abbas et al.[1] further generalized the concept of F-contraction and proved certain fixed and common fixed point results. In this paper, we introduce a fuzzy α-η-GF-contraction with respect to a more general family of functions G and obtain fixed point results in fuzzy metric space. As an application of our results we deduce Suzuki type results for GF-Contractions. In the last section, we introduce the new type of contractive maps and established

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∗Address correspondence to S. Khaleghizadeh; E-mail: skhaleghizadeh@yahoo.com.

a new fixed point theorem for such maps on the setting of generalized fuzzy metric spaces.

Definition 1.1. A 3-tuple $(X, M, *)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:
(ii) $M(x, y, t) > 0$;
(iii) $M(x, y, , t) = 1$ for all $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$;
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(vi) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left continuous;
(vii) $\lim_{t\to\infty} M(x, y, t) = 1$;

Definition 1.2. Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \qquad \Longrightarrow \qquad \alpha(Tx, Ty) \geq 1.$$

Definition 1.3. Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \qquad \alpha(x, y) \geq \eta(x, y) \qquad \Longrightarrow \qquad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.2. Also, if we take, $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

Definition 1.4. Let $(X, M, *)$ be an fuzzy metric space. Let $\alpha, \eta : X \times X \to [0, +\infty)$ and $T : X \to X$ be functions. We say T is an α-η-continuous mapping on $(X, M, *)$, if, for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ *as* $n \to \infty$,

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \textit{ for all } n \in \mathbb{N} \to Tx_n \to Tx.$$

Definition 1.5. Let $(X, M, *)$ be an intuitionistic fuzzy metric space. The fuzzy metric $(X, M, *)$ is called triangular whenever

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1,$$

for all $x, y, x \in X$ and all $t > 0$.

Definition 1.6. Let X be a nonempty set and let $\rho : X \times X \times (0,\infty) \to [0,\infty)$ be a function. We say ρ is a parametric metric on X if
(i) $\rho(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$;
(ii) $\rho(x,y,t) = \rho(y,x,t)$ for all $t > 0$;
(iii) $\rho(x,y,t) \leq \rho(x,z,t) + \rho(z,y,t)$ for all $x,y,z \in X$ and all $t > 0$
and one says the pair (X,ρ) is parametric metric space.

Let $(X,M,*)$ be an triangular intuitionistic fuzzy metric space. We consider the mapping $\rho : X\times X\times(0,\infty) \to [0,\infty)$ defined by $\rho(x,y,t) := \frac{1}{M(x,y,t)} - 1$ for all $x,y \in X$ and all $t > 0$. Then ρ is a parametric metric on X and hence (X,ρ) is a parametric metric space.

2. Fixed Point Result for fuzzy α-η-GF-Contractions

Consistent with Wordoresky we denote by Δ_F the set of all function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying following conditions:
(F_1) F is strictly increasing,
(F_2) for all sequence $\{\alpha_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if only if

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F_3) tere exists $0 < k < 1$ such that $\lim_{n\to 0^+} \alpha^k F(\alpha) = 0$.

Now, we introduce te following family of new function.

Let Δ_G denote the set of all functions $G : \mathbb{R}^{+4} \to \mathbb{R}^+$ satisfying:
(G) for all $t_1,t_2,t_3,t_4 \in \mathbb{R}^+$ with $t_1t_2t_3t_4 = 0$ there exists $\tau > 0$ such that $G(t_1,t_2,t_3,t_4) = \tau$.

Definition 2.1. Let $(X,M,*)$ be a triangular fuzzy metric space, and T be a self-mapping on X. Also suppose that $\alpha,\eta : X\times X \to [0,+\infty)$ be two functions. We say T is a fuzzy α-η-GF-contraction if for $x,y \in X$ with $\eta(x,Tx) \leq \alpha(x,y)$ and $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$ we have,

(2.1)
$$\begin{aligned} &G\left(1 - M(x,Tx,t), 1 - M(y,Ty,t), 1 - M(x,Ty,t), 1 - M(y,Tx,t)\right) \\ &\quad + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1). \end{aligned}$$

where $G \in \Delta_G$ and $F \in \Delta_F$

Now we state and prove our main result of this section.

Theorem 2.2. *Let $(X,M,*)$ be a triangular intuitionstic fuzzy metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;
(ii) T is a fuzzy α-η-GF-contraction;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
(iv) T is a fuzzy α-η-continuous.
Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. Now since, T is a fuzzy α-admissible mapping with respect to η then, $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Also, let there exists $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $(\frac{1}{M(Tx_{n-1},Tx_n,t)} - 1) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since, T is a fuzzy α-η-GE-contraction, so we derive,

$$\begin{aligned}
&G(1 - M(x_{n-1}, Tx_{n-1}, t), 1 - M(x_n, Tx_n, t), 1 - M(x_{n-1}, Tx_n, t),\\
&\quad 1 - M(x_n, Tx_{n-1}, t)) + F(\frac{1}{M(Tx_{n-1}, Tx_n, t)} - 1)\\
&\quad \leq F(\frac{1}{M(x_{n-1}, x_n, t)} - 1),
\end{aligned}$$

which implies,

(2.2)
$$\begin{aligned}
&G(1 - M(x_{n-1}, x_n, t), 1 - M(x_n, x_{n+1}, t), 1 - M(x_{n-1}, x_{n+1}, t), 0)\\
&\quad + F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) \leq F(\frac{1}{M(x_{n-1}, x_n, t)} - 1).
\end{aligned}$$

Now since, $(1-M(x_{n-1}, x_n, t).1-M(x_n, x_{n+1}, t).1-M(x_{n-1}, x_{n+1}, t).0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$G(1 - M(x_{n-1}, x_n, t), 1 - M(x_n, x_{n+1}, t), 1 - M(x_{n-1}, x_{n+1}, t), 0) = \tau.$$

From (2.2) we deduce that,

$$F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) \leq F(\frac{1}{M(x_{n-1}, x_n, t)} - 1) - \tau.$$

Therefore:

(2.3)

$$F(\frac{1}{M(x_n,x_{n+1},t)}-1)\le F(\frac{1}{M(x_{n-1},x_n,t)}-1)-\tau$$
$$\le F(\frac{1}{M(x_{n-2},x_{n-1},t)}-1)-2\tau\le\cdots\le F(\frac{1}{M(x_0,x_1,t)}-1)-n\tau.$$

By taking limit as $n\to\infty$ in (2.3) we have, $\lim_{n\to\infty}F(\frac{1}{M(x_n,x_{n+1},t)}-1)=-\infty$, and since, $F\in\Delta_F$ we obtain,

$$\lim_{n\to\infty}(\frac{1}{M(x_n,x_{n+1},t)}-1)=0. \tag{2.4}$$

Now from $(F3)$, there exists $0<k<1$ such that,

$$\lim_{n\to\infty}[\frac{1}{M(x_n,x_{n+1},t)}-1]^kF(\frac{1}{M(x_n,x_{n+1},t)}-1)=0. \tag{2.5}$$

By (2.3) we have,

$$\begin{aligned}\lim_{n\to\infty}&[\frac{1}{M(x_n,x_{n+1},t)}-1]^kF(\frac{1}{M(x_n,x_{n+1},t)}-1)\\&-F(\frac{1}{M(x_0,x_1,t)}-1)\le-n\tau[\frac{1}{M(x_n,x_{n+1},t)}-1]^k\le0.\end{aligned} \tag{2.6}$$

By taking limit as $n\to\infty$ in (2.6) and applying (2.4) and (2.5) we have,

$$\lim_{n\to\infty}n[\frac{1}{M(x_n,x_{n+1},t)}-1]^k=0. \tag{2.7}$$

It follows from (2.7) that there exists, $n_1\in\mathbb{N}$ such that,

$$n[\frac{1}{M(x_n,x_{n+1},t)}-1]^k\le1,$$

for all $n>n_1$. This implies,

$$(\frac{1}{M(x_n,x_{n+1},t)}-1)\le\frac{1}{n^{\frac{1}{k}}}$$

for all $n>n_1$. Now for $m>n>n_1$ we have,

$$(\frac{1}{M(x_n,x_m,t)}-1)\le\Sigma_{i=n}^{m-1}(\frac{1}{M(x_i,x_{i+1},t)}-1)\le\Sigma_{i=n}^{m-1}\frac{1}{i^{\frac{1}{k}}}.$$

Since, $0<k<1$, then $\Sigma_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ converges. Therefore, $(\frac{1}{M(x_n,x_m,t)}-1)\to0$ as $m,n\to\infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^*\in X$ such that, $x_n\to x^*$ as $n\to\infty$. Now since, T is a fuzzy α-η-continuous and $\eta(x_{n-1},x_n)\le$

$\alpha(x_{n-1}, x_n)$ then, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Thus T has a fixed point.

Let $x, y \in Fix(T)$ where $x \neq y$. then from

$$G(1 - M(x, Tx, t), 1 - M(y, Ty, t), 1 - M(x, Ty, t), 1 - M(y, Tx, t)) + F(\frac{1}{M(Tx, Ty, t)} - 1) \leq F(\frac{1}{M(x, y, t)} - 1)$$

we get,

$$\tau + F(\frac{1}{M(x, y, t)} - 1) \leq F(\frac{1}{M(x, y, t)} - 1)$$

which is a contraction. Hence, $x = y$. Therefore, T has a unique fixed point. □

Theorem 2.3. *Let $(X, M, *)$ be a complete intuitionstic fuzzy metric space. Let $T : X \to X$ be a self-mapping satisfying the following asertions:*
(i) T is a α-admissible mapping with respect to η;
(ii) T is a fuzzy $\alpha - \eta - GF-$contraction;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x),$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Furthermore, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. As in proof of Theorem 2.2 we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x^* \text{ as } n \to \infty,$$

where, $x_{n+1} = Tx_n$. So from (iv), either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x^*) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x^*),$$

holds for all $n \in \mathbb{N}$. This implies,

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x) \text{ or } \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x),$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1} \leq \alpha(x_{n_k}, x^*),$$

and so from (2.1) we deduce that,

$$G(1-M(x_{n_k},Tx_{n_k},t),1-M(x^*,Tx^*,t),\\ 1-M(x_{n_k},Tx^*,t),1-M(x^*,Tx_{n_k},t))\\ +F(\frac{1}{M(Tx_{n_k},Tx^*,t)}-1)\leq F(\frac{1}{M(x_{n_k},x^*,t)}-1)$$

which implies,

$$F(\frac{1}{M(Tx_{n_k},Tx^*,t)}-1)\leq F(\frac{1}{M(x_{n_k},x^*,t)}-1).$$

From (F1) we have,

$$(\frac{1}{M(x_{n_k+1},Tx^*,t)}-1)<(\frac{1}{M(x_{n_k},x^*,t)}-1).$$

By taking limit as $k\to\infty$ in the above inequality we get, $(\frac{1}{M(x^*,Tx^*,t)}-1)=1$. i.e., $x^*=Tx^*$. Uniqueness follows similarly as in Theorem 2.2. □

Merging Theorem 2.3 and Theorem 2.2 we deduce following corollary.

Corollary 2.4. *Let $(X,M,*)$ be a complete triangular intuitionistic fuzzy metric space. Let $T:X\to X$ be a self-mapping satisfying the following assertions:*
(i) T is a α-admissible mapping with respect to η;
(ii) If for $x,y\in X$ with $\eta(x,Tx)\leq\alpha(x,y)$ and $\frac{1}{M(Tx,Ty,t)}-1>0$ we have,

$$\tau+F(\frac{1}{M(Tx,Ty,t)}-1)\leq F(\frac{1}{M(x,y,t)}-1),$$

where $\tau>0$ and $F\in\Delta_F$.
(iii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$;
(iv) If $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ with $x_n\to x$ as $n\to\infty$, then either

$$\eta(Tx_n,T^2x_n)\leq\alpha(Tx_n,x)\ \text{or}\ \eta(T^2x_n,T^3x_n)\leq\alpha(T^2x_n,x),$$

holds for all $n\in\mathbb{N}$.

Then T has a fixed point. Furthermore, T has a unique fixed point when $\alpha(x,y)\geq\eta(x,x)$ for all $x,y\in Fix(T)$.

If in Corollary 2.4 we take $\alpha(x,y)=\eta(x,y)=1$ for all $x,y\in X$, then we deduce the following Corollary.

Corollary 2.5. *Let $(X, M, *)$ be a complete triangular intuitionistic fuzzy metric space and $T : X \to X$ be a self-mapping. If for $x, y \in X$ with $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$ we have,*

$$\tau + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1),$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a fixed point.

Recall that a self-mapping T is said to have the property P if $Fix(T^n) = F(T)$ for every $n \in \mathbb{N}$.

Theorem 2.6. *Let $(X, M, *)$ be a complete triangular intuitionistic fuzzy metric space and $T : X \to X$ be an α-continuous self-mapping. Assume that there exists $\tau > 0$ such that*

$$\tau + F(\frac{1}{M(Tx,T^2x,t)} - 1) \leq F(\frac{1}{M(x,Tx,t)} - 1) \tag{2.8}$$

holds for all $x \in X$ with $(\frac{1}{M(Tx,T^2x,t)} - 1) > 0$ where $F \in \Delta_F$. If T is an α-admissible mapping and there exists $x_0 \in X$ such that, $\alpha(x_0, Tx_0) \geq 1$, then T has the property P.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. Now since, T is an α-admissible mapping, so $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. By continuing this process we have,

$$\alpha(x_{n-1}, x_n) \geq 1,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $(\frac{1}{M(Tx_{n-1},T^2x_{n-1},t)}) - 1) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.8) we have,

$$\tau + F(\frac{1}{M(Tx_{n-1}, T^2x_{n-1}, t)} - 1) \leq F(\frac{1}{M(x_{n-1}, Tx_{n-1}, t)} - 1),$$

which implies,

$$\tau + F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) \leq F(\frac{1}{M(x_{n-1}, xn, t)} - 1)$$

and so,

$$F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) \leq F(\frac{1}{M(x_{n-1}, xn, t)} - 1) - \tau.$$

Therefore,

$$
\begin{aligned}
F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) &\leq F(\frac{1}{M(x_{n-1}, xn, t)} - 1) - \tau \\
&\leq F(\frac{1}{M(x_{n-2}, x_{n-1}, t)} - 1) - 2\tau \qquad (2.9)\\
&\leq \cdots \leq F(\frac{1}{M(x_0, x_1, t)} - 1) - n\tau.
\end{aligned}
$$

By taking limit as $n \to \infty$ in (2.9) we have, $\lim_{n\to\infty} F(\frac{1}{M(x_n,x_{n+1},t)} - 1) = -\infty$, and since, $F \in \Delta_F$ we obtain,

$$
\lim_{n\to\infty}(\frac{1}{M(x_n, x_{n+1}, t)} - 1) = 0. \qquad (2.10)
$$

Now from (F3), there exists $0 < k < 1$ such that,

$$
\lim_{n\to\infty}[(\frac{1}{M(x_n, x_{n+1}, t)} - 1)]^k F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) = 0. \qquad (2.11)
$$

By (2.9) we have,

$$
\begin{aligned}
&\lim_{n\to\infty}[(\frac{1}{M(x_n, x_{n+1}, t)} - 1)]^k [F(\frac{1}{M(x_n, x_{n+1}, t)} - 1) - \qquad (2.12)\\
&\quad F(\frac{1}{M(x_0, x_1, t)} - 1)] \leq -n\tau[(\frac{1}{M(x_n, x_{n+1}, t)} - 1)]^k \leq 0.
\end{aligned}
$$

By taking limit as $n \to \infty$ in (2.12) and applying (2.10) and (2.11) we have,

$$
\lim_{n\to\infty} n[(\frac{1}{M(x_n, x_{n+1}, t)} - 1)]^k = 0. \qquad (2.13)
$$

consequently, there exists, $n_1 \in \mathbb{N}$ such that,

$$
n[(\frac{1}{M(x_n, x_{n+1}, t)} - 1)]^k \leq 1,
$$

for all $n > n_1$. This implies,

$$
(\frac{1}{M(x_n, x_{n+1}, t)} - 1) \leq \frac{1}{n^{\frac{1}{k}}},
$$

for all $n > n_1$. Now for $m > n > n_1$ we have,

$$
(\frac{1}{M(x_n, x_m, t)} - 1) \leq \Sigma_{i=n}^{m-1}(\frac{1}{M(x_i, x_{i+1}, t)} - 1) \leq \Sigma_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}}.
$$

Since, $0 < k < 1$, then $\Sigma_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges. Therefore, $(\frac{1}{M(x_n,x_m,t)} - 1) \to 0$ as $m, n \to \infty$.

Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $x^* \in X$ such that , $x_n \to x^*$ as $n \to \infty$. Now since, T is α-continuous and $\alpha(x_{n-1}, x_n) \geq 1$ then, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Thus T has a fixed point and $F(T^n) = F(T)$ for $n = 1$. Let $n > 1$. Assume contrarily that $w \in F(T^n)$ and $w \notin F(T)$. Then, $(\frac{1}{M(w,Tw,t)} - 1) > 0$. Now we have,

$$\begin{aligned}
F(\frac{1}{M(w,Tw,t)} - 1) &= F(\frac{1}{M(T(T^{n-1}w)), T^2(T^{n-1}w), t)} - 1)) \\
&\leq F(\frac{1}{M(T^{n-1}w), T^n w), t)} - 1)) - \tau \\
&\leq F(\frac{1}{M(T^{n-2}w), T^{n-1}w), t)} - 1)) - 2\tau \leq \cdots \\
&(\frac{1}{M(w,Tw,t)} - 1) - n\tau.
\end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality we have, $F(\frac{1}{M(w,Tw,t)} - 1) = -\infty$. Hence, By (F2) we get, $(\frac{1}{M(w,Tw,t)} - 1) = 0$ which is a contradictions. Therefore, $F(T^n) = F(T)$ for all $n \in \mathbb{N}$. □

Let $(X, M, *, \preceq)$ be a partially ordered fuzzy metric space. It should be considered that $T : X \to X$ is nondecreasing if $\forall x, y \in X, x \preceq y \Rightarrow T(x) \preceq T(y)$. Fixed point theorems for monotone operators in ordered fuzzy metric spaces are widely investigated and have found various applications in differential and integral equations. From Theorems 2.2-2.6, we derive following new result in partially ordered fuzzy metric spaces.

Theorem 2.7. *Let $(X, M, *, \preceq)$ be a complete partially ordered fuzzy metric space. Assume that the following assertions hold true:*
(i) T is nondecreasing and ordered GF-contraction;
(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(iii) eighter for a given $x \in X$ and sequence $\{x_n\}$ $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ we have $Tx_n \to Tx$ or if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either

$$Tx_n \preceq x, or T^2 x_n \preceq x,$$

holds for all $n \in \mathbb{N}$.
Then T has a fixed point.

Theorem 2.8. *Let $(X, M, *, \preceq)$ be a complete partially ordered fuzzy metric space. Assume that the following assertions hold true:*

(i) T is nondecreasing and satisfies (2.8) for all $x \in X$ with $(\frac{1}{M(Tx,T^2x,t)} - 1) > 0$ where $F \in \Delta_F$ and $\tau > 0$;
(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(iii) for a given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \text{ we have } Tx_n \to Tx.$$

Then T has a property P.

In this section, as an application of our results proved above, we deduce certain Suzuki-Wardowski type fixed point theorems.

Theorem 2.9. *Let $(X, M, *)$ be a complete fuzzy metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $(\frac{1}{M(x,Tx,t)} - 1) \leq (\frac{1}{M(x,y,t)} - 1)$ and $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$ we have,*

(2.14)
$$G(1 - M(x,Tx,t), 1 - M(y,Ty,t), 1 - M(x,Ty,t), 1 - M(y,Tx)) + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1)$$

where $G \in \Delta_G$ and $F \in \Delta_F$. Then T has a unique fixed point.

Proof. Define, $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = (\frac{1}{M(x,y,t)} - 1) \text{ and } \eta(x,y) = (\frac{1}{M(x,y,t)} - 1),$$

for all $x, y \in X$. Now, since, $(\frac{1}{M(x,y,t)} - 1) \leq (\frac{1}{M(x,y,t)} - 1)$ for all $x, y \in X$, so $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.2 hold true. Since T is continuous, so T is $\alpha - \eta$ -continuous. Let, $\eta(x,Tx) \leq \alpha(x,y)$ with $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$. Equivalently, if $(\frac{1}{M(x,Tx,t)} - 1) \leq (\frac{1}{M(x,y,t)} - 1)$ with $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$, then, from (2.14) we have,

$$G(1 - M(x,Tx,t), 1 - M(y,Ty,t), 1 - M(x,Ty,t), 1 - M(y,Tx)) + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1).$$

That is, T is a fuzzy α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.2 hold and T has a unique fixed point. □

Corollary 2.10. *Let $(X, M, *)$ be a complete fuzzy metric space and T be a continuous self-mapping on X. If for $x, y \in x$ with $(\frac{1}{M(x,Tx,t)} - 1) \leq$*

$(\frac{1}{M(x,y,t)}-1)$ *and* $(\frac{1}{M(Tx,Ty,t)}-1)>0$ *we have*

$$\tau+F(\frac{1}{M(Tx,Ty,t)}-1)\leq F(\frac{1}{M(Tx,Ty,t)}-1),$$

where $\tau>0$ *and* $F\in\Delta_F$. *Then* T *has a unique fixed point.*

Corollary 2.11. *Let* $(X,M,*)$ *be a complete fuzzy metric space and* T *be a continuous self-mapping on* X. *If for* $x,y\in X$ *with* $(\frac{1}{M(x,Tx,t)}-1)\leq(\frac{1}{M(x,y,t)}-1)$ *and* $(\frac{1}{M(Tx,Ty,t)}-1)>0$ *we have,*

$$\tau e^{Lmin(\frac{1}{M(x,Tx,t)}-1),(\frac{1}{M(y,Ty,t)}-1),(\frac{1}{M(x,Ty,t)}-1),(\frac{1}{M(y,Tx,t)}-1)}$$
$$+F(\frac{1}{M(Tx,Ty,t)}-1)\leq F(\frac{1}{M(x,y,t)}-1),$$

where $\tau>0,L\geq 0$ *and* $F\in\Delta_F$. *Then* T *has a unique fixed point.*

Theorem 2.12. *Let* $(X,M,*)$ *be a complete fuzzy metric space and* T *be a self-mapping on* X. *Assume that there exists* $\tau>0$ *such that*

$$\frac{1}{2(1+\tau)}(\frac{1}{M(x,Tx,t)}-1)\leq(\frac{1}{M(x,y,t)}-1), \tag{2.15}$$

implies

$$\tau+F(\frac{1}{M(Tx,Ty,t)}-1)\leq F(\frac{1}{M(x,y,t)}-1),$$

for $x,y\in X$ *with* $(\frac{1}{M(Tx,Ty,t)}-1)>0$ *where* $F\in\Delta_F$. *Then* T *has a unique fixed point.*

Proof. Define, $\alpha,\eta:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=(\frac{1}{M(x,y,t)}-1)\text{ and }\eta(x,y)=\frac{1}{2(1+\tau)}(\frac{1}{M(x,y,t)}-1),$$

for all $x,y\in X$ where $\tau>0$. Now, since $\frac{1}{2(1+\tau)}(\frac{1}{M(x,y,t)}-1)\leq(\frac{1}{M(x,y,t)}-1)$ for all $x,y\in X$, so $\eta(x,y)\leq\alpha(x,y)$ for all $x,y\in X$. That is to say, condotions (i) and (iii) of Theorem 2.3 hold true. Let, $\{x_n\}$ be a sequence with $x_n\to x$ as $n\to\infty$. Assume that $(\frac{1}{M(Tx_n,T^2x_n,t)}-1)=0$ for some n. Then $Tx_n=T^2x_n$. That is Tx_n is a fixed point of T and we have nothing to prove. Hence we assume, $Tx_n\neq T^2x_n$ for all $n\in\mathbb{N}$. Since, $\frac{1}{2(1+\tau)}(\frac{1}{M(Tx_n,T^2x_n,t)}-1)\leq(\frac{1}{M(Tx_n,T^2x_n,t)}-1)$ for all $n\in\mathbb{N}$. Then from

(2.15) we get,

$$F(\frac{1}{M(T^2x_n, T^3x_n, t)} - 1) \leq \tau + F(\frac{1}{M(T^2x_n, T^3x_n, t)} - 1)$$
$$\leq (\frac{1}{M(Tx_n, T^2x_n, t)} - 1),$$

and so from (F1) we get,

$$(\frac{1}{M(T^2x_n, T^3x_n, t)} - 1) \leq (\frac{1}{M(Tx_n, T^2x_n, t)} - 1). \tag{2.16}$$

Assume there exists $n_0 \in \mathbb{N}$ such that,

$$\eta(Tx_{n_0}, T^2x_{n_0}) > \alpha(Tx_{n_0}, x) \text{ and } \eta(T^2x_{n_0}, T^3x_{n_0}) > \alpha(T^2x_{n_0}, x),$$

then,

$$\frac{1}{2(1+\tau)}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1) > (\frac{1}{M(Tx_{n_0}, x, t)} - 1),$$

and

$$\frac{1}{2(1+\tau)}(\frac{1}{M(T^2x_{n_0}, T^3x_{n_0}, t)} - 1) > (\frac{1}{M(T^2x_{n_0}, x, t)} - 1),$$

so by (2.16) we have,

$$\begin{aligned}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1) &\leq (\frac{1}{M(Tx_{n_0}, x, t)} - 1) + (\frac{1}{M(T^2x_{n_0}, x, t)} - 1)\\
&< \frac{1}{2(1+\tau)}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1)\\
&+ \frac{1}{2(1+\tau)}(\frac{1}{M(T^2x_{n_0}, T^3x_{n_0}, t)} - 1)\\
&\frac{1}{2(1+\tau)}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1)\\
&+ \frac{1}{2(1+\tau)}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1)\\
&= \frac{2}{2(1+\tau)}(\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1)\\
&\leq (\frac{1}{M(Tx_{n_0}, T^2x_{n_0}, t)} - 1),\end{aligned}$$

which is a contradiction. Hence, eighter

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x),$$

holds for all $n \in \mathbb{N}$. That is condition (iv) of Theorem 2.3 holds. Let, $\eta(x,Tx) \leq \alpha(x,y)$. So, $\frac{1}{2(1+\tau)}(\frac{1}{M(x,Tx,t)} - 1) \leq (\frac{1}{M(x,y,t)})$. Then from (**??**) we get, $\tau + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq (\frac{1}{M(x,y,t)} - 1)$. Hence, all conditions of Theorem 2.3 hold and T has a unique fixed point. □

Theorem 2.13. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*
(i) for $x, y \in O(\omega)$ with $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$ we have,

$$G(1 - M(x,Tx,t), 1 - M(y,Ty,t), 1 - M(x,Ty,t), 1 - M(y,Tx,t)) + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1)$$

where $G \in \Delta_G$ and $F \in \Delta_F$;
(ii) T is an orbitally continuous function.
Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(\omega)$.

Proof. Define, $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & \textit{if } x, y \in O(\omega) \\ 0, & \textit{otherwise} \end{cases}$$

where $O(\omega)$ is an orbit of a point $\omega \in X$ and $\eta(x,y) = 1$. We know that T is a fuzzy α-η-continuous mapping. Let, $\alpha(x,y) \geq \eta(x,y)$, then $x, y \in O(\omega)$. So $Tx, Ty \in O(\omega)$. That is,$\alpha(Tx,Ty) \geq \eta(Tx,Ty)$. Therefore, T is an α-admissible mapping with respect to η. Since $\omega, T\omega \in O(\omega)$, then $\alpha(\omega, T\omega) \geq \eta(\omega, T\omega)$. Let, $\alpha(x,y) \geq \eta(x,Tx)$ and $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$. Then $x, y \in O(\omega)$ and $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$. Therefore from (i) we have,

$$G(1 - M(x,Tx,t), 1 - M(y,Ty,t), 1 - M(x,Ty,t), 1 - M(y,Tx,t)) + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1),$$

which implies, T is a fuzzy α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.2 hold true and T has a fixed point. If $Fix(T) \subseteq O(\omega)$, then, $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in Fix(T)$ and so from Theorem 2.2 T has a unique fixed point. □

Corollary 2.14. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions;*

(i) for $x, y \in O(\omega)$ with $(\frac{1}{M(Tx,Ty,t)} - 1) > 0$ we have,

$$\tau + F(\frac{1}{M(Tx,Ty,t)} - 1) \leq F(\frac{1}{M(x,y,t)} - 1),$$

where $\tau > 0$ and $F \in \Delta_F$;
(ii) T is orbitally continuous.
Then T has a fixed point. Furthermore, T has a unique fixed point when $Fix(T) \subseteq O(\omega)$.

Corollary 2.15. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*
(i) for $x, y \in O(\omega)$ with $(\frac{1}{M(Tx,Ty,t)} - 1)$ we have,

$$\tau e^{G(1-M(x,Tx,t),1-M(y,Ty,t),1-M(x,Ty,t),1-M(y,Tx,t))} + F(\frac{1}{M(Tx,Ty,t)} - 1)$$
$$\leq F(\frac{1}{M(x,y,t)} - 1),$$

where $\tau > 0$, $L \geq 0$ and $F \in \Delta_F$;
(ii) T is orbitally continuous.
Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(\omega)$.

Theorem 2.16. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*
(i) for $x \in X$ with $(\frac{1}{M(Tx,T^2x,t)} - 1) > 0$ we have,

$$\tau + F(\frac{1}{M(Tx,T^2x,t)} - 1) \leq F(\frac{1}{M(x,Tx,t)} - 1),$$

where $\tau > 0$ and $F \in \Delta_F$;
(ii) T is an orbitally continuous function.
Then T has the property P.

Proof. Define, $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in O(\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\omega \in X$. Let, $\alpha(x, y) \geq 1$, then $x, y \in O(\omega)$. So $Tx, Ty \in O(\omega)$. That is, $\alpha(Tx, Ty) \geq 1$. Therefore, T is α-admissible mapping. Since

$\omega, T\omega \in O(\omega)$, so $\alpha(\omega, T\omega) \geq 1$. We conclude that T is an α-continuous mapping. If, $x \in X$ with $(\frac{1}{M(Tx,T^2x,t)} - 1) > 0$, then, from (i) we have,

$$\tau + F(\frac{1}{M(Tx, T^2x, t)} - 1) \leq F(\frac{1}{M(x, Tx, t)} - 1).$$

Thus all conditions of Theorem 2.3 hold true and T has the property P. □

3. A fixed point result in generalized fuzzy metric space

We denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:
(Θ_1) θ is non-decreasing;
(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;
(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.
Before we prove the main results, we recall the following definitions introduced in [3].

Definition 3.1. Let $(X, M, *)$ be triangular fuzzy metric space and $M : X \times X \times [0, \infty) \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and $t \in (0, \infty)$ and for all distinct points $u, v \in X$, each of them different from x and y, one has
(i) $\frac{1}{M(x,y,t)} - 1 = 0 \iff x = y$;
(ii) $\frac{1}{M(x,y,t)} - 1 = \frac{1}{M(y,x,t)} - 1$;
(iii) $(\frac{1}{M(x,y,t)} - 1) \leq (\frac{1}{M(x,u,t)} - 1) + (\frac{1}{M(v,y,t)} - 1)$.
Then $(X, M, *)$ is called a generalized fuzzy metric space (or for short g.f.m.s.).

Definition 3.2. Let $(X, M, *)$ be a g.f.m.s. and $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is convergent to x if and only if $\frac{1}{M(x_n,x,t)} - 1 \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

Definition 3.3. Let $(X, M, *)$ be a g.f.m.s. and $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is Cauchy if and only if $\frac{1}{M(x_n,x_m,t)} - 1 \to 0$ as $n, m \to \infty$.

Definition 3.4. Let $(X, M, *)$ be a g.f.m.s. We say that $(X, M, *)$ is complete if and only if every Cauchy sequence in X converges to some element in X.

Lemma 3.5. *Let $(X, M, *)$ be a g.f.m.s., $\{x_n\}$ be a Cauchy sequence in $(X, M, *)$, and $x, y \in X$. Suppose that there exists a positive integer N such that*
(i) $x_n \neq x_m$, for all $n, m > N$;
(ii) x_n and x are distinct points in X, for all $n > N$;
(iii) x_n and y are distinct points in X, for all $n > N$;
(iv) $\lim_{n\to\infty} \frac{1}{M(x_n,x,t)} - 1 = \lim_{n\to\infty} \frac{1}{M(x_n,y,t)} - 1$.
Then we have $x = y$.

We observe easily that if one of the conditions (ii) or (iii) is not satisfied, then the result of Lemma 3.5 is still valid.

Now, we are ready to state and prove our main result.

Theorem 3.6. *Let $(X, M, *)$ be a complete g.f.m.s. and $T : X \to X$ for all $x, y \in X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$\begin{aligned} &(\frac{1}{M(Tx,Ty,t)} - 1) \neq 0 \implies \theta(\frac{1}{M(Tx,Ty,t)} - M(y,Tx,t)) \\ &\leq [\theta(\frac{1}{M(x,y,t)} - M(y,Tx,t)]^k. \end{aligned} \tag{3.1}$$

Then T has a unique fixed point.

Proof. Let $x \in X$ be an arbitrary point in X. If for some $p \in \mathbb{N}$, we have $T^p x = T^{P+1}x$, then $T^p x$ will be a fixed point of T. So, without restriction of the generality, we can suppose that $(\frac{1}{M(T^n x,T^{n+1}x,t)} - 1) > 0$ for all $n \in \mathbb{N}$. Now, from (3.1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} &\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - M(T^n x, T^n x, t) \leq \\ &[\theta(\frac{1}{M(T^{n-1}x, T^n x, t)} - M(T^{n-1}x, T^{n-1}x, t))]^k \\ &\leq [\theta(\frac{1}{M(T^{n-2}x, T^{n-1}x, t)} - M(T^{n-2}x, T^{n-2}x, t))]^{k^2} \\ &\leq \cdots \leq [\theta(\frac{1}{M(x, Tx, t)} - M(x, x, t))]^{k^n}. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have

$$1 \leq \theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) \leq [\theta(\frac{1}{M(x, Tx, t)} - 1)]^{k^n}. \tag{3.2}$$

Letting $n \to \infty$ in (3.2), we obtain

$$\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) \to 1 \ as\ n \to \infty,$$

which implies from (Θ_2) that

$$\lim_{n\to\infty}(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) = 0. \tag{3.3}$$

From condition (Θ_3), there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1}{[\frac{1}{M(T^n x, T^{n+1}x, t)} - 1]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = l/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1}{[\frac{1}{M(T^n x, T^{n+1}x, t)} - 1]^r} - l\right| \leq B,\ for\ all\ n \geq n_0.$$

This implies that

$$\frac{\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1}{[\frac{1}{M(T^n x, T^{n+1}x, t) - 1}]^r} \geq l - B = B, for\ all\ n \geq n_0.$$

Then for all $n \geq n_0$

$$n[\frac{1}{M(T^n x, T^{n+1}, t)} - 1]^r \leq An[\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1],$$

where $A = 1/B$. Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1}{[\frac{1}{M(T^n x, T^{n+1}x, t)} - 1]^r} \geq B,\ for\ all\ n \geq n_o.$$

This implies that for all $n \geq n_0$

$$n[\frac{1}{M(T^n x, T^{n+1}, t)} - 1]^r \leq An[\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1],$$

where $A = 1/B$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$n[\frac{1}{M(T^n x, T^{n+1}, t)} - 1]^r \leq An[\theta(\frac{1}{M(T^n x, T^{n+1}x, t)} - 1) - 1].$$

Using (3.2), for all $n \geq n_0$ we obtain

$$n[\frac{1}{M(T^n x, T^{n+1}, t)} - 1]^r \leq An([\theta(\frac{1}{M(x, Tx, t)} - 1)]^{k^n} - 1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} n[\frac{1}{M(T^n x, T^{n+1} x, t)} - 1]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$(\frac{1}{M(T^n x, T^{n+1} x, t)} - 1) \leq \frac{1}{n^{\frac{1}{r}}}, \ \textit{for all } n \geq n_1. \tag{3.4}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case, then $T^n x \neq T^m x$ for every $n, m \in \mathbb{N}$ such that $n \neq m$, Using (3.1), we obtain

$$\begin{aligned} \theta(\frac{1}{M(T^n x, T^{n+2} x, t)} - 1) &\leq [\theta(\frac{1}{M(T^{n-1} x, T^{n+1} x, t)} - 1)]^k \\ &\leq [\theta(\frac{1}{M(T^{n-2} x, T^n x, t)} - 1)]^{k^2} \\ &\leq \cdots \leq [\theta(\frac{1}{M(x, T^2 x, t)} - 1)]^{k^n}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (Θ_2), we obtain

$$\lim_{n \to \infty} \frac{1}{M(T^n x, T^{n+2} x, t) - 1} = 0. \tag{3.5}$$

Similarly, from condition (Θ_3), there exists $n_2 \in \mathbb{N}$ such that

$$(\frac{1}{M(T^n x, T^{n+2} x, t)} - 1) \leq \frac{1}{n^{\frac{1}{r}}}, \ \textit{for all } n \geq n_2. \tag{3.6}$$

Let $N = \max\{n_0, n_1\}$. We consider two cases.

Case 1. If $m > 2$ is odd, then writing $m = 2L + 1$, $L \geq 1$, and using (3.4), for all $n \geq N$, we obtain

$$\begin{aligned}(\frac{1}{M(T^n x, T^{n+m} x, t)} - 1) &\leq (\frac{1}{M(T^n x, T^{n+1} x, t)} - 1) \\ &+ (\frac{1}{M(T^{n+1} x, T^{n+2} x, t)} - 1) \cdots \\ &+ (\frac{1}{M(T^{n+2L} x, T^{n+2L+1} x, t)} - 1) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2L)^{\frac{1}{r}}} \\ &\leq \Sigma_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Case 2. If $m > 2$ is even, then writing $m = 2L$, $L \geq 2$, and using (3.4) and (3.6), for all $n \geq N$, we obtain

$$\begin{aligned}(\frac{1}{M(T^n x, T^{n+m} x, t)} - 1) &\leq (\frac{1}{M(T^n x, T^{n+2} x, t)} - 1) \\ &+ (\frac{1}{M(T^{n+2} x, T^{n+3} x, t)} - 1) + \\ &\cdots + (\frac{1}{M(T^{n+2L-1} x, T^{n+2L} x, t)} - 1) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2L-1)^{\frac{1}{r}}} \\ &\leq \Sigma_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Thus, combining all the cases we have

$$(\frac{1}{M(T^n x, T^{n+m} x, t)} - 1) \leq \Sigma_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}, \ \textit{for all } n \geq N, m \in \mathbb{N}.$$

From the convergence of the series $\Sigma_i \frac{1}{i^{\frac{1}{r}}}$ (since $\frac{1}{r} > 1$), we deduce that $\{T^n x\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete triangular fuzzy metric space, there is $z \in X$ such that $T^n x \to z$. On the other hand, observe that T to is continuous, indeed, if $Tx \neq Ty$, then we have from (3.1)

$$\ln[\theta(\frac{1}{M(Tx, Ty, t)} - 1)] \leq k \ln[\theta(\frac{1}{M(x, y, t)} - 1)] \leq \ln[\theta(\frac{1}{M(x, y, t)} - 1)],$$

Which implies from (Θ_1) that

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq \frac{1}{M(x,y,t)} - 1, \ \ \textit{for all } x,y \in X.$$

From this observation, for all $n \in \mathbb{N}$, we have

$$\frac{1}{M(T^{n+1}x,Tz,t)-1} \leq \frac{1}{M(T^n x,z,t)-1}.$$

Letting $n \to \infty$ in the above inequality, we get $T^{n+1}x \to Tz$. From Lemma 3.5, we obtain $z = Tz$, which is a contradiction with the assumption: T dose not have a periodic point. Thus T has a periodic point, say z, of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \ \ \textit{and} \ (\frac{1}{M(z,Tz,t)} - 1) > 0.$$

Using (3.1), we obtain

$$\begin{aligned}\theta(\frac{1}{M(z,Tz,t)} - 1) = \theta(\frac{1}{M(T^n z,T^{n+1}z,t)} - 1) &\leq [\theta(\frac{1}{M(z,Tz,t)} - 1)]^{k^n} \\ &< \theta(\frac{1}{M(z,Tz,t)} - 1),\end{aligned}$$

which is a contradiction. Thus, the set of fixed points of T is non-empty, that is, T has at least one fixed point. Now, suppose that $z,u \in X$ are two fixed points of T such that $\frac{1}{M(z,u,t)} - 1 = \frac{1}{M(Tz,Tu,t)} - 1 > 0$. Using (3.1), we obtain

$$\begin{aligned}&\theta(\frac{1}{M(z,u,t)} - 1) = \theta(\frac{1}{M(Tz,Tu,t)} - 1) \leq [\theta(\frac{1}{M(z,u,t)} - 1)]^{k} \\ &\quad < \theta(\frac{1}{M(z,u,t)} - 1),\end{aligned}$$

which is a contradiction. Then we have one and only one fixed point. □

Since a fuzzy metric space is a g.f.m.s., from Theoram 3.6, we deduce immediately the following result.

Corollary 3.7. *Let $(X,M,*)$ be a complete fuzzy metric space and $T : X \to X$ be a given map. suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$*

such that for all $x, y \in X$

$$(\frac{1}{M(Tx,Ty,t)} - 1) \neq 0 \implies \theta(\frac{1}{M(Tx,Ty,t)} - 1) \leq [\theta(\frac{1}{M(x,y,t)} - 1)]^k.$$

Then T has a unique fixed point.

Observe that the Banach contraction principle follows immediately from Corollary 3.7. Indeed, if T is a Banach contraction, i.e., there exists $\lambda \in (0,1)$ such that

$$(\frac{1}{M(Tx,Ty,t)} - 1) \leq \lambda(\frac{1}{M(x,y,t)} - 1), \text{ for all } x, y \in X,$$

then we have

$$e^{(\frac{1}{M(Tx,Ty,t)} - 1)} \leq [e^{(\frac{1}{M(x,y,t)}) - 1}]^k, \text{ for all } x, y \in X.$$

Clearly the function $\theta : (0,\infty) \to (1,\infty)$ defined by $\theta(t) := e^{\sqrt{t}}$ belongs to Θ. So, the existence and uniqueness of the fixed point follows from Corollary 3.7.

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Soumeyeh Khaleghizadeh
Department of Mathematics, Payame Noor University, Tehran, Iran
Email: `skhaleghizadeh@yahoo.com`