Journal of Hyperstructures 10 *(2) (2021), 137-149.*
ISSN: 2322-1666 print/2251-8436 online

A NOTE ON THE LOCATION OF POLES OF MEROMORPHIC FUNCTIONS

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Abstract. A meromorphic function on an open set D contained in the finite complex plane $\mathbb{C}$ is of the form of the ratio between two analytic functions defined on D with denominator not identically zero. Poles of meromorphic functions are those zeros of the denominator where numerator does not vanish. Finding all poles of a meromorphic function is too much difficult. So, it is desirable to know a region where these poles lie. In the paper we derive a region containing all the poles of some meromorphic functions. A few examples with related figures are given here to validate the results obtained.

Key Words: Meromorphic function, poles, order.
2010 Mathematics Subject Classification: Primary: 30D30; Secondary: 30A10, 30B10, 30C15.

1. Introduction.

Problems involving location of zeros of polynomials have a long history [12]. In 1829, Cauchy [12] proved the following classical result.

Theorem A. [12] If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients, then all the zeros of $P(z)$ lie in $|z| \leq 1 + max_{0\leq j\leq(n-1)}|\frac{a_j}{a_n}|$.

In a different manner, G. Enström and S. Kakeya [8] introduced following result known as Enström-Kakeya theorem.

Received: 29 April 2020, Accepted: 22 November 2021. Communicated by Nasrin Eghbali;
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Theorem B. [8] If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of $P(z)$ lie in $|z| \leq 1$.

There are so many improvements and generalizations of Theorem A for polynomials in the existing literature [6, 11]. Also, a lots of results on generalization of Theorem B for polynomials and analytic functions are found in [1,2,4,5,7–10].Though, such type of results for poles of a meromorphic function are not available in the literature.

Generally the poles of a meromorphic function $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ are the zeros of $\frac{1}{f}$ in D. A meromorphic function f in a domain $D \subseteq \mathbb{C}$ analytic in the annulus $R_1 <| z |< R_2$ in D can be represented by Laurent's series as $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ for any z in $R_1 <| z |< R_2$ where $a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta, n \in \mathbb{Z}$ with $C = \{\zeta :| \zeta |= r\}$ and $R_1 < r < R_2$.

The main aim of this paper is to establish some results about the region of the poles of meromorphic functions under various conditions on the above coefficients a_n's. We do not explain the standard theories, notations and definitions of entire and meromorphic functions as those are available in [13] & [14].

The following definition is well known:

Definition 1.1. The order ρ of a meromorphic function f is defined as

$$\rho = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

In this paper we first prove the following result:

Theorem 1.2. *Let $f(z)$ be a meromorphic function of finite order ρ in a domain $D \subseteq \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ be analytic in the annulus $R_1 \leq| z |\leq R_2$ in D. Also let $t_1(< R_1)$ & $t_2(> R_2)$ be any two positive real numbers such that $f(z)$ is analytic in $t_1 <| z |< t_2$ contained in D with*

$$0 < a_0 + \rho \geq t_2 a_1 \geq t_2^2 a_2 \geq ...$$

and

$$0 < a_{-1} \geq \frac{a_{-2}}{t_1} \geq \frac{a_{-3}}{t_1^2} \geq ....$$

Then the poles of $f(z)$ lie in the region $D_1 \cup D_2$ where

$$D_1 = \left\{ z \in D : \min\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right) \leq \mid z \mid \leq \max\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right) \right\}$$

and

$D_2 = \{z \in D : \mid z \mid \leq t_1\}$.

Remark 1.3. The following example with related figure ensures the validity of Theorem 1.2.

Example 1.4. Let $f(z) = \frac{1}{(z-1)(z-2)(z-3)}$.

Then $f(z)$ is meromorphic in $\mathbb{C}$ and the poles are at $z = 1,\ 2\ \&\ 3$.

Now for $1 <\mid z \mid< \frac{3}{2}$, the Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{3} + \frac{7}{36}z + \frac{23}{216}z^2 + ... + \frac{1}{2z} + \frac{1}{2z^2} + \frac{1}{2z^3} + ... \,.$$

Here, $\rho = 0,\ t_1 = 1,\ t_2 = \frac{3}{2},\ a_0 = \frac{1}{3}$ and $a_1 = \frac{7}{36}$.

Now for $\rho = 0$ and $R_2 = \frac{7}{5}$,

$\min(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}) = \frac{3}{2}$ and $\max(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}) = 4.8$.

Hence by Theorem 1.2, the poles of $f(z)$ lie in

$$\{z \in \mathbb{C} :\mid z \mid\leq 1\} U \{z \in \mathbb{C} : \frac{3}{2} \leq\mid z \mid\leq 4.8\}\ .$$

Remark 1.5. Considering $\rho = (k-1)a_0$ where $k \geq 1$, the following result is an immediate consequence of Theorem 1.2.

Corollary 1.6. *Let $f(z)$ be a meromorphic function of finite order in a domain $D \subseteq \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ be analytic in the annulus $R_1 \leq\mid z \mid\leq R_2$ in D. Also let $t_1(< R_1)$ & $t_2(> R_2)$ be any two positive real numbers such that $f(z)$ is analytic in $t_1 <\mid z \mid< t_2$ contained in D with for some $k \geq 1$,*

$$0 < ka_0 \geq t_2a_1 \geq t_2^2a_2 \geq ...$$

and

$$0 < a_{-1} \geq \frac{a_{-2}}{t_1} \geq \frac{a_{-3}}{t_1^2} \geq ....$$

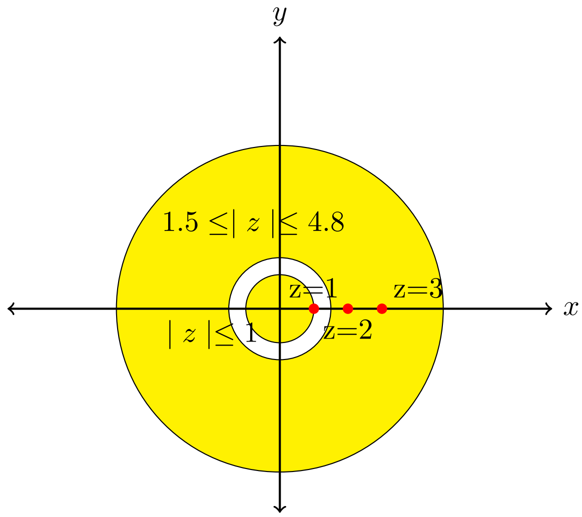

FIGURE 1. Distribution of poles of $f(z) = \frac{1}{3} + \frac{7}{36}z + \frac{23}{216}z^2 + ... + \frac{1}{2z} + \frac{1}{2z^2} + \frac{1}{2z^3} + ...$

Then the poles of $f(z)$ lie in the region $D_1' \cup D_2'$ where

$$D_1' = \left\{ z \in D : \min\left(t_2, \frac{ka_0R_2}{ka_0 - t_2a_1}\right) \leq\mid z \mid\leq \max\left(t_2, \frac{ka_0R_2}{ka_0 - t_2a_1}\right)\right\} \quad \textit{and}$$

$D_2' = \{z \in D :\mid z \mid\leq t_1\}$.

Remark 1.7. The following example with related figure justifies the validity of Corollary 1.6.

Example 1.8. Let $f(z) = \frac{1}{(z-1)(z-2)(3-z)}$.

Then $f(z)$ is meromorphic in $\mathbb{C}$ and the poles are at $z = 1,\ 2\ \&\ 3$.

Now for $2 <\mid z \mid< 3$, the Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{16} + \frac{1}{18}z + \frac{1}{54}z^2 + ... + \frac{1}{2z} + \frac{3}{2z^2} + ... \; .$$

Here, $t_1 = 2,\ t_2 = 3,\ a_0 = \frac{1}{16}$ and $a_1 = \frac{1}{18}$.

Now for $k = 4$ and $R_2 = \frac{14}{5}$,

$\min\left(t_2, \frac{ka_0R_2}{ka_0-t_2a_1}\right) = 3$ and $\max\left(t_2, \frac{ka_0R_2}{ka_0-t_2a_1}\right) = 8.4$.

Hence by Corollary 1.6, the poles of $f(z)$ lie in

$$\{z \in \mathbb{C} :| z |\leq 2\} U \{z \in \mathbb{C} : 3 \leq | z |\leq 8.4\} \ .$$

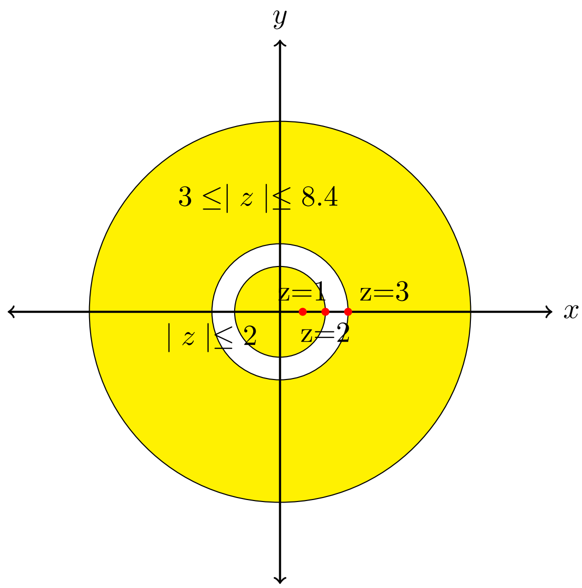

Figure 2. Distribution of poles of $f(z) = \frac{1}{16} + \frac{1}{18}z + \frac{1}{54}z^2 + ... + \frac{1}{2z} + \frac{3}{2z^2} + ...$

Finally, we establish following result without imposing any restrictions on the coefficients of the negative power of z.

Theorem 1.9. *Let $f(z)$ be a meromorphic function in a domain $D \subseteq \mathbb{C}$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be analytic in the annulus $R_1 \leq| z |\leq R_2$. Also let $t_1(< R_1)$ & $t_2(> R_2)$ be any two positive real numbers such that $f(z)$ is analytic in $t_1 <| z |< t_2$ contained in D with*

$$\underset{|z|=R_2}{Max} \mid \sum_{n=1}^{\infty}(a_{n-1} - t_2a_n)z^n \mid\leq M.$$

Then the poles of $f(z)$ lie in the region $D_3 \cup D_4$ where

$$D_3 = \left\{z \in D : \min\left(t_2, \frac{M}{\mid a_0 - t_2a_1 \mid}\right) \leq| z |\leq \max\left(t_2, \frac{M}{\mid a_0 - t_2a_1 \mid}\right)\right\} \quad and$$

$D_4 = \{z \in D :| z |\leq t_1\}$.

Remark 1.10. The following example with related figure ensures the validity of Theorem 1.9.

Example 1.11. Let $f(z) = \frac{1}{(z+i)(z-2)(z+3)}$.

Then $f(z)$ is meromorphic in $\mathbb{C}$ and the poles of $f(z)$ are at $z = -i,\ 2\ \&\ -3$.

Now for $2 <| z |< 3$, the Laurent's series expansion of $f(z)$ is

$$f(z) = \{\frac{1}{30} - \frac{1}{90}z + \frac{1}{270}z^2 - ...\} + \{-\frac{1}{10z} + (\frac{2}{15} + \frac{i}{6})\frac{1}{z^2} + ...\}$$

Here, $t_1 = 2,\ t_2 = 3$ and $a_n = (-1)^n \frac{1}{30.3^n},\ n = 0,\ 1,\ 2, ...$.

Taking $R_2 = \frac{5}{2}$, we see that

$$Max_{|z|=\frac{5}{2}} \mid \sum_{n=1}^{\infty}(a_{n-1} - 3a_n)z^n \mid\leq 1\ .$$

Also $min(t_2, \frac{M}{|a_0 - t_2 a_1|}) = 3$ and $max(t_2, \frac{M}{|a_0 - t_2 a_1|}) = 15$.

Hence by Theorem 1.9, the poles of $f(z)$ lie in the region

$$\{z \in \mathbb{C} :| z |\leq 2\}U\{z \in \mathbb{C} : 3 \leq| z |\leq 15\}\ .$$

2. Lemmas.

In this section we present a lemma which will be needed in the sequel .

Lemma 2.1. *[3] If $f(z)$ is analytic in $| z |\leq R, f(0) = 0, f'(0) = b$ and $| f(z) |\leq M$ for $| z |= R$, then for $| z |\leq R$,*

$$| f(z) |\leq \frac{M \mid z \mid}{R^2}.\frac{M \mid z \mid + R^2 \mid b \mid}{M + \mid b \mid\mid z \mid}.$$

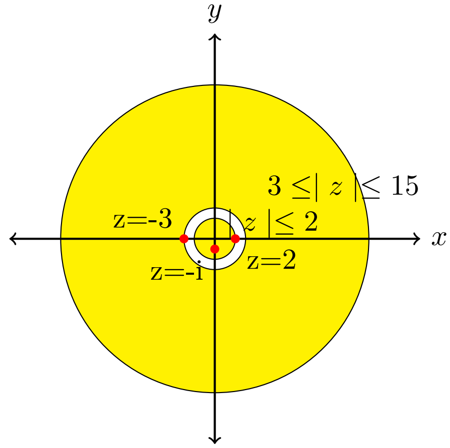

FIGURE 3. Distribution of poles of $f(z) = \frac{1}{30} - \frac{1}{90}z + \frac{1}{270}z^2 - ...\} + \{-\frac{1}{10z} + (\frac{2}{15} + \frac{i}{6})\frac{1}{z^2} + ...$

3. Proofs of the Theorems.

Proof of Theorem 1.2. For $R_1 \leq\mid z \mid\leq R_2$, it follows that

$$| f(z) |\leq| \sum_{n=0}^{\infty} a_n z^n | + | \sum_{n=-1}^{-\infty} a_n z^n | . \tag{3.1}$$

Clearly, $\lim_{n\to\infty} a_n R_2^n = 0$ and $\lim_{n\to-\infty} a_n R_1^n = 0$.

Now for $| z |\leq R_2 < t_2$, we get that

$$\begin{aligned} | (z - t_2)\sum_{n=0}^{\infty} a_n z^n | &= | \sum_{n=0}^{\infty} a_n z^{n+1} - t_2 \sum_{n=0}^{\infty} a_n z^n | \\ &= | -a_0 t_2 + (a_0 - t_2 a_1)z + \sum_{n=2}^{\infty}(a_{n-1} - t_2 a_n)z^n | \\ &= | -a_0 t_2 - \rho z + (a_0 + \rho - t_2 a_1)z + \sum_{n=2}^{\infty}(a_{n-1} - t_2 a_n)z^n | \\ &\leq | a_0 | t_2 + \rho | z | + | (a_0 + \rho - t_2 a_1)z + \sum_{n=2}^{\infty}(a_{n-1} - t_2 a_n)z^n | \\ &= | a_0 | t_2 + \rho | z | + | G(z) | . \end{aligned} \tag{3.2}$$

For $| z |= R_2$, we have

$$\begin{aligned}
| G(z) | &= | (a_0 + \rho - t_2 a_1)z + \sum_{n=2}^{\infty}(a_{n-1} - t_2 a_n)z^n | \\
&\leq | a_0 + \rho - t_2 a_1 || z | + | \sum_{n=2}^{\infty} | (a_{n-1} - t_2 a_n) || z^n | \\
&= (a_0 + \rho - t_2 a_1)R_2 + \sum_{n=2}^{\infty}(a_{n-1} - t_2 a_n)R_2^n \\
&\leq (a_0 + \rho - R_2 a_1) + \sum_{n=2}^{\infty}(a_{n-1} - R_2 a_n)R_2^n \\
&= (a_0 + \rho)R_2.
\end{aligned}$$

Clearly $G(z)$ is analytic in $| z |\leq R_2, G(0) = 0, G'(0) = (a_0 + \rho - t_2 a_1)$ and $| G(z) |\leq (a_0 + \rho)R_2$ for $| z |= R_2$. Hence by Lemma 2.1, it follows that

$$\begin{aligned}
| G(z) | &\leq \frac{(a_0 + \rho)R_2 | z |}{R_2^2} . \frac{(a_0 + \rho)R_2 | z | + R_2^2 | a_0 + \rho - t_2 a_1 |}{(a_0 + \rho)R_2 + | a_0 + \rho - t_2 a_1 || z |} \\
&= \frac{(a_0 + \rho) | z | \{(a_0 + \rho) | z | + R_2 | a_0 + \rho - t_2 a_1 |\}}{(a_0 + \rho)R_2 + | a_0 + \rho - t_2 a_1 || z |} \\
&\leq \frac{(a_0 + \rho) | z | \{(a_0 + \rho) | z | + R_2 | a_0 + \rho - t_2 a_1 |\}}{(a_0 + \rho)R_2 - | a_0 + \rho - t_2 a_1 || z |} .
\end{aligned}$$

Therefore from (3.2), we obtain for $| z |\leq R_2 < t_2$ that

$$\begin{aligned}
| \sum_{n=0}^{\infty} a_n z^n | &\leq \frac{1}{| z - t_2 |} \left[| a_0 | t_2 + \rho | z | + \frac{(a_0 + \rho) | z | \{(a_0 + \rho) | z | + R_2 | a_0 + \rho - t_2 a_1 |\}}{(a_0 + \rho)R_2 - | a_0 + \rho - t_2 a_1 || z |} \right] \\
&\leq \frac{\begin{bmatrix} (| a_0 | t_2 + \rho | z |)\{(a_0 + \rho)R_2 - | a_0 + \rho - t_2 a_1 || z |\} \\ + (a_0 + \rho) | z | .\{(a_0 + \rho) | z | + R_2 | a_0 + \rho - t_2 a_1 |\} \end{bmatrix}}{(t_2 - | z |)\{(a_0 + \rho)R_2 - | a_0 + \rho - t_2 a_1 || z |\}} .
\end{aligned}$$

Now for $| z |\geq R_1 > t_1$, it follows that

$$
\begin{aligned}
&\mid \left(\frac{1}{z}-\frac{1}{t_1}\right)\sum_{n=-1}^{-\infty} a_n z^n \mid=\mid \sum_{n=-1}^{-\infty} a_n z^{n-1} - \frac{1}{t_1}\sum_{n=-1}^{-\infty} a_n z^n \mid \\
&=\mid -\frac{a_{-1}}{t_1 z} + \sum_{n=-1}^{-\infty}(a_n - \frac{a_{n-1}}{t_1})z^{n-1} \mid \\
&\leq \frac{a_{-1}}{t_1 \mid z \mid} + \sum_{n=-1}^{-\infty} \mid a_n - \frac{a_{n-1}}{t_1} \mid\mid z \mid^{n-1} \\
&\leq \frac{a_{-1}}{t_1 \mid z \mid} + \sum_{n=-1}^{-\infty}(a_n - \frac{a_{n-1}}{t_1})R_1^{n-1} \\
&\leq \frac{a_{-1}}{t_1 \mid z \mid} + \sum_{n=-1}^{-\infty}(a_n - \frac{a_{n-1}}{R_1})R_1^{n-1} \\
&= \frac{a_{-1}}{t_1 \mid z \mid} + \frac{a_{-1}}{R_1^2}.
\end{aligned}
$$

Therefore $\mid \sum_{n=-1}^{-\infty} a_n z^n \mid \leq \frac{1}{\mid \frac{1}{z}-\frac{1}{t_1} \mid}(\frac{a_{-1}}{t_1|z|}+\frac{a_{-1}}{R_1^2})$

$$
\begin{aligned}
&= \frac{a_{-1}(R_1^2+t_1|z|)}{|t_1-z|R_1^2} \\
&\leq \frac{a_{-1}(R_1^2+t_1|z|)}{(|z|-t_1)R_1^2}.
\end{aligned}
$$

Hence from (3.1), we get that

$$
|f(z)| \leq \frac{\begin{bmatrix}(\mid a_0 \mid t_2+\rho \mid z \mid)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\} \\ +(a_0+\rho)\mid z \mid .\{(a_0+\rho)\mid z \mid +R_2 \mid a_0+\rho-t_2a_1 \mid\}\end{bmatrix}}{(t_2- \mid z \mid)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\}} + \frac{a_{-1}(R_1^2+t_1 \mid z \mid)}{(\mid z \mid -t_1)R_1^2}
$$

$$
= \frac{\begin{bmatrix}(\mid z \mid -t_1)R_1^2[(\mid a_0 \mid t_2+\rho \mid z \mid)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\} \\ +(a_0+\rho)\mid z \mid .\{(a_0+\rho)\mid z \mid +R_2 \mid a_0+\rho-t_2a_1 \mid\}]+ \\ a_{-1}(R_1^2+t_1 \mid z \mid)[(t_2- \mid z \mid)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\}]\end{bmatrix}}{R_1^2(t_2- \mid z \mid)(\mid z \mid -t_1)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\}}
$$

Therefore $\frac{1}{|f(z)|} > 0$ if $(t_2- \mid z \mid)(\mid z \mid -t_1)\{(a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid\} > 0$.

Now for $\mid z \mid > t_2$, it follows that

$$\frac{1}{|f(z)|} > 0 \text{ if } (a_0+\rho)R_2- \mid a_0+\rho-t_2a_1 \mid\mid z \mid < 0$$

$$\text{i.e, } \frac{1}{|f(z)|} > 0 \text{ if } \mid z \mid > \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}.$$

Hence the zeros of $\frac{1}{f(z)}$ lie in the annular region

$$\min\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right) \leq\mid z\mid\leq \max\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right).$$

Consequently, the poles of $f(z)$ lie in

$$D_1=\left\{z\in D: \min\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right) \leq\mid z\mid\leq \max\left(t_2, \frac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}\right)\right\}.$$

Also for $\mid z\mid< t_1 < t_2$, we see that

$$\tfrac{1}{|f(z)|} > 0 \text{ if } \mid z\mid> \tfrac{(a_0+\rho)R_2}{a_0+\rho-t_2a_1}.$$

Hence the zeros of $\frac{1}{f(z)}$ lie in $\mid z\mid\leq t_1$.

Therefore the poles of $f(z)$ lie in $D_2=\{z\in D:\mid z\mid\leq t_1\}$.

Thus all the poles of $f(z)$ lie in the region $D_1\cup D_2$.

This proves the theorem. □

Proof of Theorem 1.9. For $R_1\leq\mid z\mid\leq R_2$,

$$\mid f(z)\mid\leq\mid \sum_{n=0}^{\infty} a_n z^n\mid+\mid\sum_{n=-1}^{-\infty} a_n z^n\mid, R_1\leq\mid z\mid\leq R_2. \tag{3.3}$$

Now for $\mid z\mid\leq R_2 < t_2$, it follows that

$$\begin{aligned}\mid (z-t_2)\sum_{n=0}^{\infty} a_n z^n\mid &=\mid -a_0t_2+\sum_{n=1}^{\infty}(a_{n-1}-t_2a_n)z^n\mid\\ &\leq\mid a_0t_2\mid+\mid\sum_{n=1}^{\infty}(a_{n-1}-t_2a_n)z^n\mid\\ &=\mid a_0t_2\mid+\mid G(z)\mid. \qquad (3.4)\end{aligned}$$

Also for $\mid z\mid= R_2$,

$$\mid G(z)\mid=\mid\sum_{n=1}^{\infty}(a_{n-1}-t_2a_n)z^n\mid\leq \underset{|z|=R_2}{Max}\mid\sum_{n=1}^{\infty}(a_{n-1}-t_2a_n)z^n\mid\leq M$$

and $G(z)$ being analytic in $\mid z \mid\leq R_2,\ G(0) = 0,\ G'(0) = (a_0 - t_2a_1)$, applying Lemma 2.1 we obtain that

$$\begin{aligned} \mid G(z) \mid &\leq \frac{M \mid z \mid}{R_2^2} \frac{M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid}{M + \mid a_0 - t_2a_1 \mid\mid z \mid} \\ &\leq \frac{M \mid z \mid (M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid)}{R_2^2(M - \mid a_0 - t_2a_1 \mid\mid z \mid)} \text{ for } \mid z \mid\leq R_2. \end{aligned}$$

Therefore for $\mid z \mid\leq R_2 < t_2$, it follows from (3.4) that

$$\begin{aligned} \mid \sum_{n=0}^{\infty} a_n z^n \mid &\leq \frac{1}{\mid z - t_2 \mid} \frac{R_2^2 \mid a_0 \mid t_2(M - \mid a_0 - t_2a_1 \mid\mid z \mid) + M \mid z \mid (M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid)}{R_2^2(M - \mid a_0 - t_2a_1 \mid\mid z \mid)} \\ &\leq \frac{\mid a_0 \mid t_2(M - \mid a_0 - t_2a_1 \mid\mid z \mid) + M \mid z \mid (M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid)}{R_2^2(t_2 - \mid z \mid)(M - \mid a_0 - t_2a_1 \mid\mid z \mid)}. \end{aligned}$$

Now for $\mid z \mid\geq R_1 > t_1$,

$$\begin{aligned} \mid (\frac{1}{z} - \frac{1}{t_1}) \sum_{n=-1}^{-\infty} a_n z^n \mid &= \mid -\frac{a_{-1}}{t_1 z} + \sum_{n=-1}^{-\infty} (a_n - \frac{1}{t_1} a_{n-1}) z^{n-1} \mid \\ &\leq \frac{\mid a_{-1} \mid}{t_1 \mid z \mid} + \mid \sum_{n=-1}^{-\infty} (a_n - \frac{1}{t_1} a_{n-1}) z^{n-1} \mid \\ &\leq \frac{\mid a_{-1} \mid}{t_1 \mid z \mid} + M_1 \end{aligned}$$

where $M_1 = \underset{|z|=R_1}{Max} \mid \sum_{n=-1}^{-\infty} (a_n - \frac{1}{t_1} a_{n-1}) z^{n-1} \mid$.

Therefore

$$\begin{aligned} \mid \sum_{n=-1}^{-\infty} a_n z^n \mid &\leq \frac{1}{\mid t_1 - z \mid} \{\mid a_{-1} \mid + t_1 M_1 \mid z \mid\} \\ &\leq \frac{1}{\mid z \mid - t_1} \{\mid a_{-1} \mid + t_1 M_1 \mid z \mid\} \text{ for } \mid z \mid\geq R_1 > t_1. \end{aligned}$$

Hence for $R_1 \leq\mid z \mid\leq R_2$, we get from (3.3) that

$$\mid f(z) \mid\leq \frac{R_2^2 \mid a_0 \mid t_2(M- \mid a_0 - t_2a_1 \mid\mid z \mid) + M \mid z \mid (M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid)}{R_2^2(t_2- \mid z \mid)(M- \mid a_0 - t_2a_1 \mid\mid z \mid)} + \frac{\mid a_{-1} \mid + t_1M_1 \mid z \mid}{\mid z \mid - t_1}$$

$$= \frac{\left[\begin{array}{r}(\mid z \mid - t_1)\{R_2^2 \mid a_0 \mid t_2(M- \mid a_0 - t_2a_1 \mid\mid z \mid) + M \mid z \mid (M \mid z \mid + R_2^2 \mid a_0 - t_2a_1 \mid)\} \\ + (t_2- \mid z \mid)(M- \mid a_0 - t_2a_1 \mid\mid z \mid)(\mid a_{-1} \mid + t_1M_1 \mid z \mid)\end{array}\right]}{R_2^2(\mid z \mid - t_1)(t_2- \mid z \mid)(M- \mid a_0 - t_2a_1 \mid\mid z \mid)}.$$

Therefore $\frac{1}{|f(z)|} > 0$ if $(\mid z \mid - t_1)(t_2- \mid z \mid)(M- \mid a_0 - t_2a_1 \mid\mid z \mid) > 0$.

In a like manner as in the proof of Theorem 1.2, the poles of $f(z)$ lie in the region $D_3 \cup D_4$ where

$$D_3 = \left\{z \in D : \min\left(t_2, \frac{M}{\mid a_0 - t_2a_1 \mid}\right) \leq\mid z \mid\leq \max\left(t_2, \frac{M}{\mid a_0 - t_2a_1 \mid}\right)\right\} \quad \text{and}$$

$$D_4 = \{z \in D :\mid z \mid\leq t_1\}.$$

Thus the theorem is established. □

Future prospect. In the line of the works as carried out in the paper one may think of proving the results in case of meromorphic functions of infinite order.

References

[1] A. Aziz and Q.G. Mohammad, *On the zeros of a certain class of polynomials and related analytic functions*, J. Math. Anal. Appl., 1980; **75** (1980), 495502.

[2] A. Aziz and Q.G. Mohammad, *Zero-free regions for polynomials and some generalizations of Eneström-Kakeya theorem*, Can. Math. Bull., **27**(3) (1984), 265272.

[3] A. Aziz and W.M. Shah, *On the zeros of polynomials and related analytic functions*, Glasnik Matematicki, **33**(53) (1998), 173-184.

[4] A. Aziz and B.A. Zargar, *Bounds for the zeros of a polynomial with restricted coefficients*, Applied Mathematics, **3**(2012), 30-33.

[5] A. Chattopadhyay, S. Das, V.K. Jain and H. Konwar, *Certain generalizations of Eneströn-Kakeya theorem*, Journal of Indian Mathematical Society, **72**(1-4) (2005), 147-156.

[6] Y. Choo, *Contributions to the zeros of a polynomial*, Int. J. Math. Anal., **9** (2015), 15711577.

[7] N.K. Govil and Q.I. Rahaman, *On the Eneström-Kakeya theorem*, Tohoku Math. J., **20**(1968), 126-136.

[8] R. Gardner and N.K. Govil, *Enestrom- Kakeya theorem and some of its generalization*, Birkhauser, New Delhi, 2014.

[9] A. Joyal, G. Labelle and Q.I. Rahaman, *On the location of zeros of polynomials*, Canad. Math.Bull., **10** (1967), 53-63.
[10] V.K. Jain, *On the location of zeros of polynomials*, Ann. Univ. Mariae Curie-Sklodowska, Lublin-Polonia Sect. A, **30** (1976), 43-48.
[11] V.K. Jain, *On Cauchy's bound for zeros of a polynomial*, Turk. J. Math., **30**(1) (2006), 95-100.
[12] M. Marden, *Geometry of polynomials*, Math. Surveys. no. 3. Amer. Math. Society., Providence, 1966.
[13] W.K. Hayman, *Meromorphic functions*, The Clarendon Press, Oxford, 1964.
[14] V. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publishing Company, 1949.

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