Journal of Hyperstructures 10 *(2) (2021)129-136*
ISSN: 2322-1666 print/2251-8436 online

SOME GENERALIZED RESULTS OF ZEROS OF POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. In the present paper, we further generalize and extend various results on zeroes of polar derivatives of polynomials due to Gulzar, Zargar and Akhter (2019), who gave extension and generalization of various results on Enestrom-Kakeya theorem established by various researchers in the literature.

Key Words: Zeros, polynomial, Enestrom-Kakeya theorem, polar derivative, .
2010 Mathematics Subject Classification: 37K20 .

1. INTRODUCTION

If $f(z)$ be the k^{th} degree polynomial with real cofficients. Let $D_\beta f(z)$ be the polar derivative of $f(z)$ with respect to the point β and it is defined by $D_\beta f(z) = kf(z) + (\beta - z)f'(z)$. In this case the degree of $D_\beta f(z)$ at most $k-1$ and $\beta \to \infty$ then it generalized the ordinary derivative,

$$f'(\mathrm{z}) = \lim_{\beta\to\infty} \frac{D_\beta f(z)}{\beta}$$

Regarding the zeroes of $f(z)$, Enestrom-Kakeya proved the following result.

Theorem 1.1 Let $f(z) = \sum_{i=0}^{k} a_i z^i$ be the k^{th} degree polynomial with real coefficients such that $0 < a_0 \le a_1 \le \ldots \le a_{n-1} \le a_n$ then all the zeroes of $f(z)$ lies . Regarding the multiplicity of zeroes of $f(z)$, Aziz and Mahammad [1], proved the following result

Received: 12 May 2021, Accepted: 10 September 2021. Communicated by Mirela Stefanescu;
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Theorem 1.2 Let $f(z)=\sum_{i=0}^{k} a_i z^i$ be the k^{th} degree polynomial with real coefficients such that $0<a_0\leq a_1\leq\ldots\leq a_{n-1}\leq a_n$ then all the zeroes of $|f(z)|\geq\frac{k}{k+1'}$, are simple.

Gulzar, Zargar, and A khter [6] extended the above results to the polar derivatives, there exist some generalizations and extentions of Enestrom Kakeya theorem in [2, 3, 5, 8, 9, 10].

In the present paper, we generalize and extend various results on zeroes of polar derivatives of polynomials due to Gulzar, Zargar, and A khter [6]

2. Preparation of manuscript

Theorem 2.1 Let $f(z)=\sum_{i=0}^{k} a_i z^i$ be the k^{th} degree polynomial with real coefficients and β be a real number $\gamma>0, 0<t\leq 1$, such that for some

t$b_k\geq b_{k-1}\geq\cdots\geq b_3\geq b_2-\gamma$ then all zeroes of $D_\beta f(z)$ which does not lie in

$$|z|\leq\frac{|b_k|+\mathrm{t}\,(b_{k-}\,|b_k|-b_2+|b_2|+2\gamma}{|b_k|}$$

are simple, where $b_s=(s-1)\left[s\beta a_s+(k-(s-1))a_{s-1}\right]$, for, $s=1,2,3,\ldots,k$.

Proof- Let $f(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_1 z+a_0$ be the k^{th} degree polynomial with real coefficients, then by the definition of polar derivative, we know that

$$D_\beta f(z)=kf(z)+(\beta-z)f'(z),$$
$$D_\beta f(z)=kf(z)+\beta f'(z)-zf'(z),$$

Therefore,

$$D_\beta f(z)=k\left(a_k z^k+a_{k-1}z^{k-1}+\cdots+ka_1 z+a_0\right)+$$
$$\beta\left(a_k k z^{k-1}+a_{k-1}(k-1)z^{k-2}+\cdots+a_1\right)$$
$$-z\left(a_k k z^{k-1}+a_{k-1}(k-1)z^{k-2}+\cdots+a_1\right)$$
$$=ka_k z^k+ka_{k-1}z^{k-1}+\cdots+ka_1 z+ka_0+\beta a_k k z^{k-1}$$
$$\beta a_{k-1}(k-1)z^{k-2}+\cdots+\beta a_1$$
$$-a_k k z^k-a_{k-1}(k-1)z^{k-1}-\cdots-za_1$$

Thus,

$$D_\beta f(z)=\left[\beta k a_k+(k-(k-1)a_{k-1})\right]z^{k-1}+\left[\beta(k-1)a_{k-1}+\right.$$

$$(k-(k-2)a_{k-2})]\, z^{k-2}+\cdots+[2\beta a_2+(k-1)a_1]\, z+[\beta a_1+ka_0]$$

Therefore,

$$D'_\beta f(z)=b_k z^{k-2}+b_{k-1}z^{k-3}+\cdots+b_4z^2+b_3z+b_2$$

where $b_s=(s-1)\,[s\beta a_s+(k-(s-1))a_{s-1}]$, for $s=2,3,4,\ldots\ldots\ldots,n$

Now consider $g(z)=(1-z)D'_\beta f(z)$, so that

$$g(z)=(1-z)\left[b_k z^{k-2}+b_{k-1}z^{k-3}+\cdots+b_4z^2+b_3z+b_2\right]$$

Then

$$|g(z)|\geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{\begin{array}{c} |b_k-b_{k-1}|+\frac{|b_{k-1}-b_{k-2}|}{|z|}+\frac{|b_{k-2}-b_{k-3}|}{|z|^2} \\ +\cdots+\frac{|b_3-b_2|}{|z|^{k-3}}+\frac{|b_2|}{|z|^{k-2}}\end{array}\right\}\right]$$

If $|z|<1$, then $\frac{1}{|z|}\leq 1$, therefore

$$\begin{aligned}|g(z)|&\geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{\begin{array}{c}|b_k-tb_k+tb_k-b_{k-1}|+\cdots\\ +|b_3-\gamma-(b_2-\gamma)+|\,b_2||\end{array}\right\}\right]\\ &\geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{\begin{array}{c}|b_k-tb_k|+|tb_k-b_{k-1}|+\cdots+\\ |b_3-(b_2-\gamma)|+|\gamma|+|b_2|\end{array}\right\}\right]\\ &\geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{(1-t)\,|b_k|+tb_k-b_2+2\gamma+|b_2|\right\}\right]\\ &\geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\}\right],\end{aligned}$$

Hence $g(z)>0$, if $|z|>\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\}$, this implies that $|g(z)|>1$ lie in

$$|z|\leq\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\}$$

Since the zeroes of $g(z)$ whose modulus is less than or equal to one are lie in

$$|z|\leq\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\}$$

It follows that all the zeroes of $g(z)$ lie in

$$|z|\leq\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\},$$

Since all the zeroes of $g(z)$ are also the zeroes of $D_\beta{}' f(z)$ lie in

$$|z|\leq\frac{1}{|b_k|}\left\{|b_k|+t\,(b_k-|b_k|)-b_2+2\gamma+|b_2|\right\}$$

Thus all the zeroes of $D_{\beta}'f(z)$ lie in

$$|z| \leq \frac{1}{|b_k|}\left\{|b_k| + t\left(b_k - |b_k|\right) - b_2 + 2\gamma + |b_2|\right\},$$

In other words all the zeroes of $D_{\beta}f(z)$ which does not lie in

$$|z| \leq \frac{1}{|b_k|}\left\{|b_k| + t\left(b_k - |b_k|\right) - b_2 + 2\gamma + |b_2|\right\}$$

are simple, where $b_s = (s-1)\left[s\beta a_s + (k-(s-1))a_{s-1}\right]$, for $s = 2, 3, 4, 5, \ldots, k$.

Theorem 2.2 Let $f(z) = \sum_{i=0}^{k} a_i z^i$ be the k^{th} degree polynomial with real coefficients and β be a real number $t \geq 1, 0 < \gamma < 1$, such that for some

$$\mathrm{t}b_k \geq b_{k-1} \geq \cdots \geq b_3 \geq b_2 - \gamma$$

then all zeroes of $D_{\beta}f(z)$ which does not lie in

$$|z + t - 1| \leq \frac{tb_{k-}b_2 + |b_2| + 2\gamma}{|b_k|}$$

are simple, where $b_s = (s-1)\left[s\beta a_s + (k-(s-1))a_{s-1}\right]$, for
$s = 1, 2, 3, \ldots, k$

Proof- Let $f(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0$ be the k^{th} degree polynomial with real coefficients, then by the definition of polar derivative,

$$D_{\beta}f(z) = kf(z) + (\beta - z)f'(z)$$
$$D_{\beta}f(z) = kf(z) + \beta f'(z) - zf'(z)$$

Therefore,

$$\begin{aligned} D_{\beta}f(z) = k&\left(a_k z^k + a_{k-1}z^{k-1} + \cdots + ka_1 z + a_0\right) \\ +\beta&\left(a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2} + \cdots + a_1\right) \\ -z&\left(a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2} + \cdots + a_1\right) \\ &= ka_k z^k + ka_{k-1}z^{k-1} + \cdots + ka_1 z + ka_0 \\ &+\beta a_k k z^{k-1} + \beta a_{k-1}(k-1)z^{k-2} + \cdots + \beta a_1 \\ &-a_k k z^k - a_{k-1}(k-1)z^{k-1} - \cdots - za_1 \end{aligned}$$

Thus,

$$\begin{aligned} D_{\beta}f(z) &= \left[\beta k a_k + (k-(k-1)a_{k-1})\right] z^{k-1} \\ &+ \left[\beta(k-1)a_{k-1} + (k-(k-2)a_{k-2})\right] z^{k-2} \\ &+ \cdots + \left[2\beta a_2 + (k-1)a_1\right] z + \left[\beta a_1 + ka_0\right] \end{aligned}$$

Now, find $D'_\beta f(z)$ we get,

$$D'_\beta f(z) = b_k z^{k-2} + b_{k-1} z^{k-3} + \cdots + b_4 z^2 + b_3 z_+ b_2$$

where $b_s = (s-1)\left[s\beta a_s + (k-(s-1))a_{s-1}\right]$, for $s = 2, 3, 4, \ldots, n$ now consider $g(z) = (1-z)D'_\beta f(z)$, so that

then

$$g(z) = (1-z)\left[b_k z^{k-2} + b_{k-1} z^{k-3} + \cdots + b_4 z^2 + b_3 z_+ b_2\right]$$

$$|g(z)| \geq |b_k|\,|z|^{k-2}\left[|z+t-1| - \frac{1}{|b_k|}\left\{\begin{array}{c} |tb_k - b_{k-1}| + \frac{|b_{k-1}-b_{k-2}|}{|z|} + \frac{|b_{k-2}-b_{k-3}|}{|z|^2} \\ + \cdots + \frac{|b_3-b_2|}{|z|^{k-3}} + \frac{|b_2|}{|z|^{k-2}} \end{array}\right\}\right]$$

If $|z| < 1$, then $\dfrac{1}{|z|} \leq 1$

$$|g(z)| \geq |b_k|\,|z|^{k-2}\left[|z+t-1| - \tfrac{1}{|b_k|}\left\{|tb_k - b_{k-1}| + \cdots + |b_3 - \gamma - (b_2-\gamma) + |\,b_2||\right\}\right]$$
$$\geq |b_k|\,|z|^{k-2}\left[|z+t-1| - \tfrac{1}{|b_k|}\left\{|tb_k - b_{k-1}| + \cdots + |b_3 - (b_2-\gamma) + |\,\gamma| + |b_2||\right\}\right]$$
$$\geq |b_k|\,|z|^{k-2}\left[|z+t-1| - \tfrac{1}{|b_k|}\left\{(tb_k - b_{k-1}) + \cdots + b_3 - (b_2-\gamma) + |\gamma| + |b_2|\right\}\right]$$
$$\geq |b_k|\,|z|^{k-2}\left[|z+t-1| - \tfrac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}\right]$$

Hence $g(z) > 0$, if $|z+t-1| > \frac{1}{|b_k|}\left\{tb_k - b_{k-1} + 2\gamma + |b_2|\right\}$, this implies that $|g(z)| > 1$ are lie in

$$|z+t-1| \leq \frac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}$$

Since all the zeroes of $g(z)$ whose modulus is less than or equal to one already lie in

$$|z+t-1| \leq \frac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}$$

It follows that all the zeroes of $g(z)$ lie in

$$|z+t-1| \leq \frac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}$$

Since all the zeroes of $g(z)$ are also the zeroes of ${D_\beta}' f(z)$ therefore lie in

$$|z+t-1| \leq \frac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}$$

In other words all the zeroes of $D_\beta f(z)$ which does not lie in

$$|z+t-1| \leq \frac{1}{|b_k|}\left\{tb_k - b_2 + 2\gamma + |b_2|\right\}$$

are simple.

Theorem 2.3 Let $f(z)=\sum_{i=0}^{k} a_i z^i$ be the k^{th} degree polynomial with real coefficients and β be a real number $\gamma>0, 0<t\leq 1$, such that for some

$$\text{t}b_k \geq b_{k-1} \geq \cdots \geq b_3 \geq b_2+\gamma$$

then all zeroes of $D_\beta f(z)$ which does not lie in

$$|z| \leq \frac{|b_k| - t\left(b_k+|b_k|\right)+b_2+|b_2|+2\gamma}{|b_k|}$$

are simple, where $b_s=(s-1)\left[s\beta a_s+(k-(s-1))a_{s-1}\right]$, for $s=1,2,3,\ldots,k$

Proof. Let $f(z)=a_k z^k+a_{k-1}z^{k-1}+\cdots+a_1 z+a_0$ be the k^{th} degree polynomial with real coefficients, then by the definition of polar derivative, we know that

$$D_\beta f(z)=kf(z)+(\beta-z)f'(z)$$
$$D_\beta f(z)=kf(z)+\beta f'(z)-zf'(z),$$

Therefore,

$$\begin{aligned} D_\beta f(z) &= k\left(a_k z^k+a_{k-1}z^{k-1}+\cdots+ka_1 z+a_0\right)\\ &+\beta\left(a_k k z^{k-1}+a_{k-1}(k-1)z^{k-2}+\cdots+a_1\right)\\ &-z\left(a_k k z^{k-1}+a_{k-1}(k-1)z^{k-2}+\cdots+a_1\right)\\ &= ka_k z^k+ka_{k-1}z^{k-1}+\cdots+ka_1 z+ka_0\\ &+\beta a_k k z^{k-1}+\beta a_{k-1}(k-1)z^{k-2}+\cdots+\beta a_1\\ &-a_k k z^k-a_{k-1}(k-1)z^{k-1}-\cdots-za_1 \end{aligned}$$

$$D_\beta f(z)=\left[\beta k a_k+(k-(k-1)a_{k-1})\right]z^{k-1}+\left[\beta(k-1)a_{k-1}+(k-(k-2)a_{k-2})\right]z^{k-2}$$
$$+\cdots+\left[2\beta a_2+(k-1)a_1\right]z+\left[\beta a_1+ka_0\right],$$

Therefore,

$$D_\beta{}' f(z)=b_k z^{k-2}+b_{k-1}z^{k-3}+\cdots+b_4 z^2+b_3 z+b_2$$

where $b_s=(s-1)\left[s\beta a_s+(k-(s-1))a_{s-1}\right]$, for $s=2,3,4,\ldots,n$

Now consider $g(z)=(1-z)D_\beta{}' f(z)$, so that

$$g(z)=(1-z)\left[b_k z^{k-2}+b_{k-1}z^{k-3}+\cdots+b_4 z^2+b_3 z+b_2\right]$$

Then

$$|g(z)| \geq |b_k|\,|z|^{k-2}\left[|z|-\frac{1}{|b_k|}\left\{\begin{array}{c} |b_k-b_{k-1}|+\frac{|b_{k-1}-b_{k-2}|}{|z|}+\frac{|b_{k-2}-b_{k-3}|}{|z|^2}\\ +\cdots+\frac{|b_3-b_2|}{|z|^{k-3}}+\frac{|b_2|}{|z|^{k-2}} \end{array}\right\}\right]$$

$$\text{If } |z|<1, \text{ then } \frac{1}{|z|}\le 1$$

$$|g(z)| \ge |b_k|\,|z|^{k-2}\left[|z| - \frac{1}{|b_k|}\left\{\begin{array}{c} |b_k - tb_k + tb_k - b_{k-1}| + |b_{k-1} - b_{k-2}| + \\ \cdots + |b_4 - b_3| + |b_3 + \gamma - (b_2+\gamma) + | b_2|| \end{array}\right\}\right]$$

$$|g(z)| \ge |b_k|\,|z|^{k-2}\left[|z| - \frac{1}{|b_k|}\left\{\begin{array}{c} |b_k - tb_k| + |tb_k - b_{k-1}| + |b_{k-1} - b_{k-2}| + \\ \cdots + |b_4 - b_3| + |b_3 - (b_2+\gamma) + | \gamma| + |b_2|| \end{array}\right\}\right]$$

$$\ge |b_k|\,|z|^{k-2}\left[|z| - \frac{1}{|b_k|}\left\{\begin{array}{c} |(1-t)b_k| + (b_{k-1} - tb_k) + \cdots + \\ (b_2 + \gamma - b_3) + |\gamma| + |b_2| \end{array}\right\}\right]$$

$$\ge |b_k|\,|z|^{k-2}\left[|z| - \frac{1}{|b_k|}\left\{(1-t)\,|b_k| - tb_k + b_2 + 2\gamma + |b_2|\right\}\right]$$

$$\ge |b_k|\,|z|^{k-2}\left[|z| - \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}\right]$$

Hence $g(z) > 0$, if $|z| > \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$, this implies that $|g(z)| > 1$ are lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

Since the zeroes of $g(z)$ whose modulus is less than or equal to one lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

It follows that all the zeroes of $g(z)$ lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

Since all the zeroes of $g(z)$ are also the zeroes of $D_\beta{}' f(z)$ lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

Thus all the zeroes of $D_\beta{}' f(z)$ lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

In other words all the zeroes of $D_\beta f(z)$ which does not lie in

$$|z| \le \frac{1}{|b_k|}\left\{|b_k| - t\,(b_k + |b_k|) + b_2 + 2\gamma + |b_2|\right\}$$

are simple, where $b_s = (s-1)\,[s\beta a_s + (k-(s-1))a_{s-1}]$,
for $s = 2, 3, 4, 5, \ldots, k$.

Acknowledgments

The authors express their gratitude to the anonymous referees for their insightful remarks, suggestions, and ideas that aided in the improvement of this paper

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