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FUZZY ESSENTIAL-SMALL SUBMODULES AND FUZZY SMALL-ESSENTIAL SUBMODULES

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Abstract. In this paper, we introduce the concepts of a fuzzy essential-small submodule and a fuzzy small-essential submodule of a module. We investigate various properties of such fuzzy submodules. It is also shown that the Jacobson L-radical is the sum of all essential-small L-submodules of a module. We also prove that the L-socle is the intersection of all small-essential L-submodules of a module.

Key Words: Fuzzy small submodule, fuzzy essential submodule, fuzzy small-essential and fuzzy essential-small submodule.

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1. Introduction

In 1965, Zadeh [7] introduced the concept of a fuzzy subset as a generalization of the characteristic function in classical set theory. Later in 1971, Rosenfled [1] defined the concept of the fuzzy subgroup of a group.

Negoita and Ralescu [2] were the first ones to introduce a fuzzy submodule. Pan [5] studied fuzzy finitely generated modules and fuzzy quotient modules. The study of fuzzy submodules was also carried out by Zahedi [9]. Kalita [8] defined a fuzzy essential submodule and proved some characteristics of such submodules. Rahman and Saikia [10]

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defined a fuzzy small submodule. Basent [3] proved some properties of fuzzy superfluous modules.

In this paper, we introduce the class of all fuzzy essential submodules to generalize fuzzy small submodules and the class of all fuzzy small submodules to generalize fuzzy essential submodules respectively.

2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We use the notations "$\subseteq$" and "$\leq$" to denote inclusion and submodule respectively. We recall some definitions from Moderson and Malik [6].

Definition 2.1. [6] A fuzzy subset of an R-module M, is a mapping $\mu : M \to [0, 1]$.

We denote the set of all fuzzy subsets of an R-module M by $[0, 1]^M$ and μ_* by the set $\mu_* = \{x \in M \mid \mu(x) = 1\}$.
Also, we denote the support of a fuzzy set μ by μ^* and is defined by $\mu^* = \{x \in M \mid \mu(x) > 0\}$.

Definition 2.2. Let L be a complete Heyting algebra. An L-subset of an R-module M is a mapping $\mu : M \to L$.

We denote by L^M, the set of all L-subsets of M.

Definition 2.3. [6] If $N \subseteq M$ and $\alpha \in [0, 1]$, then α_N is defined as,

$$\alpha_N(x) = \begin{cases} \alpha, & if\ x \in N, \\ 0, & otherwise. \end{cases}$$

If $N = \{x\}$, then α_x is often called a fuzzy point and is denoted by χ_α.
If $\alpha = 1$, then 1_N is known as the characteristic function of N and is denoted by χ_N.
We recall the following well known properties.
If $\mu, \sigma \in [0, 1]^M$, then

(i) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$,
(ii) $(\mu \cup \sigma)(x) = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$,
(iii) $(\mu \cap \sigma)(x) = min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$,
(iv) $(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$, for all $x \in M$.

Definition 2.4. [6] Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Let $\mu \in [0,1]^X$ and $\sigma \in [0,1]^Y$. Then the image $f(\mu) \in [0,1]^Y$ and the inverse image $f^{-1}(\sigma) \in [0,1]^X$ are defined as follows: for all $y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x \in X, f(x) = y\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \textit{otherwise}. \end{cases}$$

and $f^{-1}(\sigma)(x) = \sigma(f(x))$ *for all* $x \in X$.

Definition 2.5. [6] Let $\xi \in [0,1]^R$ and $\mu \in [0,1]^M$.
We denote by $\xi \bigodot \mu$, the fuzzy subset of M defined by

$$(\xi \bigodot \mu)(x) = \bigvee \left\{ \bigwedge_{i=1}^{n} (\xi(r_i) \wedge \mu(x_i)) | r_i \in R, x_i \in M, 1 \leq i \leq n, n \in \mathbb{N}, \sum_{i=1}^{n} r_i x_i = x \right\}, \text{ for all } x \in M.$$

Definition 2.6. [6] Let M be an R-module and L be a complete Heyting algebra. An L subset μ of an R-module M is called an L-submodule of M, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu(\theta) = 1$,
(ii) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
(iii) $\mu(rx) \geq \mu(x)$.

We denote the set of all L-submodules of an R-module M by $L(M)$. If $L = [0,1]$, then μ is called a fuzzy submodule of M. The set of all fuzzy submodules of M is denoted by $F(M)$.

Definition 2.7. [6] Let $\mu,\ \nu \in F(M)$ such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$\left(\frac{\nu}{\mu}\right)([x]) = \vee \{\nu(z) \mid z \in [x]\}, \ \ \forall x \in \nu^*,$$

where $[x]$ denotes the coset $x + \mu^*$.

Definition 2.8. [6] Let $\mu \in [0,1]^M$. Then $\cap\{\nu \mid \mu \subseteq \nu, \nu \in F(M)\}$ is a fuzzy submodule of M. It is called the fuzzy submodule generated by the fuzzy subset μ and is denoted by $\langle\mu\rangle$.

Let $\xi \in F(M)$. If $\xi = \langle \mu \rangle$ for some $\mu \in [0,1]^M$, then μ is called a generating fuzzy subset of ξ.

Lemma 2.9. [6] *Let $\mu, \nu \in L(M)$. Then $\mu + \nu \in L(M)$.*

Lemma 2.10. [6] *μ_* is a submodule of an R-module M if and only if μ is a fuzzy submodule of M.*

Proposition 2.11. [3] *Let $\mu, \nu \in F(M)$. Then $(\mu \cap \nu)_* = \mu_* \cap \nu_*$, $(\mu \cup \nu)_* = \mu_* \cup \nu_*$. Further if, μ and σ have finite images, then $(\mu + \sigma)_* = \mu_* + \sigma_*$.*

Proposition 2.12. [3] *If μ and σ are two fuzzy submodules of an R-module M such that $\mu \subseteq \sigma$, then $(\sigma/\mu)_* = \sigma_*/\mu_*$.*

Theorem 2.13. [8] *A submodule A of an R-module M is essential in M if and only if χ_A is an essential fuzzy submodule of M.*

Theorem 2.14. [10] *Let $\mu \in F(M)$. Then $\mu \ll_f M$ if and only if $\mu_* \ll M$.*

Theorem 2.15. [8] *Let μ be a non-zero fuzzy submodule of an R-module M. Then $\mu \trianglelefteq_f M$ if and only if $\mu^* \trianglelefteq M$.*

Theorem 2.16. [8] *Let μ be a fuzzy submodule of M. If σ is a relative complement for μ in M then $\mu \oplus \sigma \trianglelefteq_f M$.*

Theorem 2.17. [8] *For a fuzzy submodule δ, the following are equivalent:*

(i) *δ is semisimple,*
(ii) *δ has no proper essential submodule,*
(iii) *Every submodule of δ is a direct summand of δ.*

Remark 2.18. [10] If $x \in M$, then $\chi_R \bigodot \chi_{\{x\}}$ is a fuzzy submodule of M generated by $\chi_{\{x\}}$ and in this case,

$$\chi_R \bigodot \chi_{\{x\}} = \langle \chi_{\{x\}} \rangle = \chi_{\langle \{x\} \rangle} = \chi_{Rx}.$$

Definition 2.19. [4] Let N be a submodule of an R-module M.

(i) N is said to be e-small in M (*denoted by* $N \ll_e M$), if $N + L = M$ with $L \trianglelefteq M$ implies $L = M$.
(ii) N is said to be s-essential in M (*denoted by* $N \trianglelefteq_S M$), if $N \cap L = 0$ with $L \ll M$ implies $L = 0$.

Definition 2.20. [10] Let M be an R-module and let $\mu \in L(M)$. Then μ is said to be a small L-submodule of M, if for any $\nu \in L(M)$ satisfying

$\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$ and is denoted by $\mu \ll_L M$. If $L = [0, 1]$, then μ is called a fuzzy small submodule of M and is denoted by notation $\mu \ll_f M$.

Definition 2.21. Let M be an R-module and let $\mu \in L(M)$. Then μ is said to be an essential L-submodule of M, if for any $\nu \in L(M)$ satisfying $\mu \cap \nu = \chi_\theta$ implies $\nu = \chi_\theta$ and is denoted by $\mu \trianglelefteq_L M$. If $L = [0, 1]$, then μ is called a fuzzy essential submodule of M and is denoted by $\mu \trianglelefteq_f M$.

3. Fuzzy essential-small submodules

In this section, we introduce the concept of a fuzzy essential-small submodule. We obtain some properties of this concept. Now onwards all the fuzzy sets involved in this paper have finite images.

Definition 3.1. Let M be an R-module and let $\mu \in L(M)$. If for any essential submodule $\sigma \in L(M)$, $\mu + \sigma = \chi_M$ implies that $\sigma = \chi_M$, then μ is said to be a e-small (essential-small) L-submodule of M and is denoted by $\mu \ll_{eL} M$.
If $L = [0, 1]$, then μ is called a fuzzy e-small submodule of M and is denoted by $\mu \ll_{fe} M$.

Theorem 3.2. *Let $\mu \in F(M)$. Then $\mu \ll_{fe} M$ if and only if $\mu_* \ll_e M$.*

Proof. Assume that $\mu \ll_{fe} M$.
Let N be an essential submodule of M such that $\mu_* + N = M$. (I)
We define an fuzzy submodule σ_N as follows:

$$\sigma(N) = \begin{cases} 1, & if\ x \in N, \\ 0, & if\ x \notin N. \end{cases}$$

Then by Theorem 2.13, σ_N is a fuzzy essential submodule of M.
Clearly, $\sigma_* = N$.
Therefore, from (I) we get $\mu_* + \sigma_* = M$.
This implies $(\mu + \sigma)_* = M$ and so $\mu + \sigma = \chi_M$.
But, given that $\mu \ll_{fe} M$.
Hence $\sigma = \chi_M$ and so, $\sigma_* = M$.
Thus, $N = M$ and so $\mu_* \ll_e M$.

Conversely, assume that $\mu_* \ll_e M$.
Let $\sigma \trianglelefteq_f M$ be such that $\mu + \sigma = \chi_M$.
Then $(\mu + \sigma)_* = M$ implies that $\mu_* + \sigma_* = M$.
But, given $\mu_* \ll_e M$.

Hence $\sigma_* = M$ and so $\sigma = \chi_M$.
Thus, $\mu \ll_{fe} M$. □

We note that every fuzzy small submodule of an R-module is a fuzzy e-small submodule. However the following example shows that the converse need not be true.

Example 3.3. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$.
Define $\mu : M \to [0,1]$ by,

$$\mu(x) = \begin{cases} 1, & if\ x \in \{0,8,16\}, \\ \alpha, if & x \notin \{0,8,16\}, \text{ where } 0 \leq \alpha < 1. \end{cases}$$

We note that $\mu_* = \{0,8,16\}$ is an e-small submodule of $\mathbb{Z}_{24}$.
Hence, μ is a fuzzy e-small submodule of $\mathbb{Z}_{24}$ by Theorem 3.2.
But, $\mu_* = \{0,8,16\}$ is not a small submodule of $\mathbb{Z}_{24}$ and so by Theorem 2.14, μ is not a fuzzy small submodule of $\mathbb{Z}_{24}$.

Theorem 3.4. *Let $A \leq M$. Then $A \ll_e M$ if and only if $\chi_A \ll_{fe} M$.*

Proof. Assume that $A \ll_e M$.
Let $\mu \in F(M)$ be such that $\mu \trianglelefteq_f M$ and $\mu + \chi_A = \chi_M$.
This implies that $(\mu + \chi_A)_* = M$.
Hence by Proposition 2.11, $\mu_* + A = M$.
But $A \ll_e M$ and so $\mu_* = M$.
Hence, $\mu = \chi_M$.
Thus, $\chi_A \ll_{fe} M$.

Conversely, assume that $\chi_A \ll_{fe} M$.
Suppose that A is not an e-small submodule of M.
Then there exists $B \trianglelefteq M$ such that $A + B = M$ with $B \neq M$.
Clearly, $\chi_A \neq \chi_M$ and $\chi_B \neq \chi_M$.
Any $x \in M$ can be written as $x = a + b$ for some $a \in A$ and $b \in B$.
We have

$$(\chi_A + \chi_B)(x) = \vee\{\chi_A \wedge \chi_B |\, p, q \in M, p + q = x\} \geq \chi_A(a) + \chi_B(b) = 1.$$

This implies $\chi_A + \chi_B = \chi_M$.
Since $\chi_A \ll_{fe} M$, we get $\chi_B = \chi_M$ and so $B = M$, a contradiction.
Hence, $A \ll_e M$. □

Theorem 3.5. *Let $f : M \to N$ be an epimorphism and μ be a fuzzy subset of M. If $\mu \ll_{fe} M$, then $f(\mu) \ll_{fe} N$.*

Proof. Let $\sigma \trianglelefteq_f N$ be such that $f(\mu) + \sigma = \chi_N$.
Then $(f(\mu) + \sigma)_* = N$ and so by Proposition 2.11 , $f(\mu)_* + \sigma_* = N$.

Hence by [[11], Lemma 3.8], $f(\mu_*) + \sigma_* = N$. (I)
As $\mu \ll_{fe} M$ then by Theorem 3.2, $\mu_* \ll_e M$.
Since $f : M \to N$ is an epimorphism, we conclude
from [[4], Proposition 2.5(2)] that $f(\mu_*) \ll_e N$. (II)
Now from (I) and (II), we get $\sigma_* = N$. Thus $\sigma = \chi_N$.
Hence, $f(\mu) \ll_{fe} N$. □

Theorem 3.6. *Let $\mu, \sigma \in F(M)$. Then $\mu \ll_{fe} M$ and $\sigma \ll_{fe} M$ if and only if $\mu + \sigma \ll_{fe} M$.*

Proof. Suppose that $\mu \ll_{fe} M$ and $\sigma \ll_{fe} M$.
Let $\delta \trianglelefteq_f M$ be such that $(\mu + \sigma) + \delta = \chi_M$.
As $\mu \ll_{fe} M$, we get $(\sigma + \delta) = \chi_M$.
Since $\sigma \ll_{fe} M$, we get $\delta = \chi_M$.
Thus, $\mu + \sigma \ll_{fe} M$.

Conversely, assume that $\mu + \sigma \ll_{fe} M$.
Let $\alpha \trianglelefteq_f M$ be such that $\mu + \alpha = \chi_M$.
Then $\chi_M = \mu + \alpha \subseteq (\mu + \sigma) + \alpha$.
But always $(\mu + \sigma) + \alpha \subseteq \chi_M$.
Therefore, we get $(\mu + \sigma) + \alpha = \chi_M$.
As $\mu + \sigma \ll_{fe} M$, we get $\alpha = \chi_M$.
Hence, $\mu \ll_{fe} M$.
Using similar arguments, we get $\sigma \ll_{fe} M$. □

Theorem 3.7. *Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$.
If $\nu \ll_{fe} M$, then $\mu \ll_{fe} M$ and $\frac{\nu}{\mu} \ll_{fe} \frac{\chi_M}{\mu}$.*

Proof. Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$.
Then μ_* and ν_* are R-submodules of M such that $\mu_* \subseteq \nu_*$.
As $\nu \ll_{fe} M$, then by Theorem 3.2 we conclude that $\nu_* \ll_e M$.
Then by using [[4], Proposition 2.5(1(a))],
we get $\mu_* \ll_e M$ and $\frac{\nu_*}{\mu_*} \ll_e \frac{M}{\mu_*}$.
Thus by Proposition 2.12, we get $\mu_* \ll_e M$ and $(\frac{\nu}{\mu})_* \ll_e (\frac{\chi_M}{\mu})_*$.
Hence by Theorem 3.2, we conclude that $\mu \ll_{fe} M$ and $\frac{\nu}{\mu} \ll_{fe} \frac{\chi_M}{\mu}$. □

We recall some definitions.

Definition 3.8. [10] A fuzzy submodule σ in M is called a fuzzy direct sum of two fuzzy submodules μ and ν if $\sigma = \mu + \nu$ and $\mu \cap \nu = \chi_\theta$.

Definition 3.9. [10] Any $\mu \in F(M)$ is called a fuzzy direct summand of M if there exists a $\gamma \in F(M)$ such that χ_M is a fuzzy direct sum of μ, ν.

Theorem 3.10. *Let $\mu, \delta \in F(M)$. Suppose that μ is a fuzzy submodule of δ. Then the following statements are equivalent:*

(i) $\mu \ll_{fe} \delta$.

(ii) *If $\sigma + \mu = \chi_M$, then σ is a direct summand of δ.*

Proof. $(i) \Rightarrow (ii)$: Let $\beta \in F(M)$ be a relative complement of σ in δ. Then by Theorem 2.16, we get $\beta \oplus \sigma \trianglelefteq_f \delta$.
Since, $\beta + \sigma + \mu = \chi_M$ and $\mu \ll_{fe} \delta$. This implies $\beta + \sigma = \chi_M$.
Thus σ is a direct summand of δ.
$(ii) \Rightarrow (i)$: Let $\xi \trianglelefteq_f \delta$ and $\xi + \mu = \chi_M$. Then ξ is a direct summand of δ, so $\xi = \chi_M$ and thus, $\mu \ll_{fe} \delta$. □

4. Fuzzy small-essential submodules

In this section, we introduce the concept of a fuzzy small-essential submodule and prove some results.

Definition 4.1. Let M be an R-module and $\mu \in L(M)$. If for any small submodule $\sigma \in L(M)$ satisfying $\mu \cap \sigma = \chi_\theta$ implies that $\sigma = \chi_\theta$, then μ is said to be a s-essential (small-essential) L-submodule of M and is denoted by $\mu \trianglelefteq_{sL} M$.
If $L = [0, 1]$, then μ is called a fuzzy s-essential submodule of M and is denoted by $\mu \trianglelefteq_{fs} M$.

Theorem 4.2. *Let $\mu \in F(M)$. Then $\mu \trianglelefteq_{fs} M$ if and only if $\mu^* \trianglelefteq_s M$.*

Proof. Let $\mu \trianglelefteq_{fs} M$ and A be a small submodule of M.
Suppose that $\mu^* \cap A = \{\theta\}$. Then $(\mu \cap \chi_A)^* = \{\theta\}$.
Hence, $\mu \cap \chi_A = \chi_\theta$.
But $\mu \trianglelefteq_{fs} M$ implies that $\chi_A = \chi_\theta$.
Hence $A = \{\theta\}$. Thus $\mu^* \trianglelefteq_s M$.

Conversely, assume that $\mu^* \trianglelefteq_s M$.
Let γ be a fuzzy small submodule of M such that $\mu \cap \gamma = \chi_\theta$.
Then $(\mu \cap \gamma)^* = \{\theta\}$ and so $\mu^* \cap \gamma^* = \{\theta\}$.
Since, $\mu^* \trianglelefteq_s M$ we get $\gamma^* = \{\theta\}$.
Hence $\gamma = \chi_\theta$. Thus $\mu \trianglelefteq_{fs} M$. □

Every fuzzy essential submodule of an R-module M is a fuzzy s-essential submodule of M. However, the following example shows that converse need not be true.

Example 4.3. Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{24}$.
Define $\mu : M \to [0, 1]$ by,

$$\mu(x) = \begin{cases} 1, & if\ x = 0, \\ 0.9, & if\ x \in \{3, 6, 9, 12, 15, 18, 21\}, \\ 0, & if\ x \notin \{3, 6, 9, 12, 15, 18, 21\}. \end{cases}$$

Then $\mu^* = \{0, 3, 6, 9, 12, 15, 18, 21\}$ is an s-essential submodule of $\mathbb{Z}_{24}$ and so μ is a fuzzy s-essential submodule of M by Theorem 4.2.
Consider, the fuzzy submodule σ of M defined by

$$\sigma(x) = \begin{cases} 1, & if\ x = 0, \\ 0.7, & if\ x \in \{8, 16\}, \\ 0, & if\ x \notin \{8, 16\}. \end{cases}$$

Here, as $\sigma_* = \{0\}$ is small submodule of M, then by Theorem 2.14, σ is a fuzzy small submodule of M. As $\mu \cap \sigma = \chi_\theta$, but $\sigma \neq \chi_\theta$.
Thus μ is not a fuzzy essential submodule of M.

Definition 4.4. A fuzzy submodule μ of an R-module M is said to be a fuzzy simple submodule if for any $\nu \in F(M)$, $\nu \subseteq \mu$ implies that either $\nu = \chi_\theta$ or $\nu = \mu$.

Definition 4.5. [8] Let μ be a fuzzy submodule of an R-module M. The sum of all fuzzy simple submodules of μ is called the socle of μ. It is denoted by $Soc_f(\mu)$.
If μ has no fuzzy simple submodule, then $Soc_f(\mu) = \chi_\theta$.
μ is said to be fuzzy semisimple or fuzzy completely reducible provided $Soc_f(\mu) = \mu$, that is; if μ is the sum of all its simple submodules.

Clearly, χ_θ is a fuzzy simple submodule.

Theorem 4.6. *If δ is a fuzzy semisimple, then χ_θ is the only fuzzy small submodule of δ. Also, χ_M is the only fuzzy essential submodule of δ.*

Proof. As δ is fuzzy semisimple, then by Theorem 2.17 there exists a fuzzy submodule μ of δ such that $\delta \oplus \mu = \chi_M$. (I)
If δ is fuzzy small, then $\mu = \chi_M$ and so $\delta = \chi_\theta$.
If δ is fuzzy essential, then from (I), $\mu = \chi_\theta$ and so $\delta = \chi_M$. □

Theorem 4.7. *Every fuzzy submodule of a semisimple module is an s-essential submodule.*

Proof. Let M be a semisimple module. Let $\mu \in F(M)$ and γ be fuzzy small submodule of M with $\mu \cap \gamma = \chi_\theta$. Since M is semisimple, χ_θ is the only fuzzy small submodule of M. Hence $\gamma = \chi_\theta$.
Thus μ is an s-essential submodule of M. □

Theorem 4.8. *Let M and N are R-modules and $f : M \to N$ be a module homomorphism. Let μ be a fuzzy subset of N. If $\mu \trianglelefteq_{fs} N$, then $f^{-1}(\mu) \trianglelefteq_{fs} M$.*

Proof. First we shall prove: $f^{-1}(\mu^*) = (f^{-1}(\mu))^*$.
We note that if $x \in (f^{-1}(\mu))^*$, then $(f^{-1}(\mu))(x) > 0$.
This implies that $\mu(f(x)) > 0$.
Hence $f(x) \in \mu^*$ and so $x \in f^{-1}(\mu^*)$.
Therefore, $f^{-1}(\mu^*) \subseteq (f^{-1}(\mu))^*$.
By using similar arguments, we get $(f^{-1}(\mu))^* \subseteq f^{-1}(\mu^*)$.
As $\mu \trianglelefteq_{fs} N$, then $\mu^* \trianglelefteq_s N$.
Hence by [[4], Proposition 2.7(2)], we get $f^{-1}(\mu^*) \trianglelefteq_s M$.
This implies that $(f^{-1}(\mu))^* \trianglelefteq_s M$ and so $f^{-1}(\mu) \trianglelefteq_{fs} M$. □

Theorem 4.9. *Let M be an R-module and $A \leq M$. Then $A \trianglelefteq_s M$ if and only if $\chi_A \trianglelefteq_{fs} M$.*

Proof. Suppose that $A \trianglelefteq_s M$.
Let $\mu \ll_f M$ be such that $\mu \cap \chi_A = \chi_\theta$.
This implies that $(\mu \cap \chi_A)^* = \{\theta\}$.
Hence $\mu^* \cap A = \{\theta\}$.
As $A \trianglelefteq_s M$ we get, $\mu^* = \{\theta\}$ and so $\mu = \chi_\theta$.
Hence, $\chi_A \trianglelefteq_{fs} M$.

Conversely, suppose that $\chi_A \trianglelefteq_{fs} M$.
If possible, assume that A is not an s-essential submodule of M.
Then there exists a small submodule B of M with $B \neq \{\theta\}$ such that $A \cap B = \{\theta\}$. This implies that $\chi_A \cap \chi_B = \chi_\theta$.
Since $\chi_A \trianglelefteq_{fs} M$, we conclude that $\chi_B = \chi_\theta$.
Hence $B = \{\theta\}$, a contradiction.
Thus $A \trianglelefteq_s M$. □

Theorem 4.10. *Let μ, σ be fuzzy submodules of an R-module M. Suppose that $\mu \subseteq \sigma$. If $\mu \trianglelefteq_{fs} M$, then $\mu \trianglelefteq_{fs} \sigma$ and $\sigma \trianglelefteq_{fs} M$.*

Proof. Suppose that β is a fuzzy small submodule of M such that $\beta \subseteq \sigma$ and $\mu \cap \beta = \chi_\theta$.
Since $\mu \trianglelefteq_{fs} M$, we get $\beta = \chi_\theta$.
Hence, $\mu \trianglelefteq_{fs} \sigma$.
Let η be a fuzzy submodule of M such that $\sigma \cap \eta = \chi_\theta$.
Then $\mu \cap \eta = \chi_\theta$. Since $\mu \trianglelefteq_{fs} M$, we get $\eta = \chi_\theta$.
This implies that $\sigma \trianglelefteq_{fs} M$. □

The following example shows that the converse of Theorem 4.10 does not hold.

Example 4.11. Consider the ring $R = \mathbb{Z}$ and its module $M = \mathbb{Z}_{12}$. Define μ, σ and δ from M to $[0,1]$ as follows:

$$\mu(x) = \begin{cases} 1, \text{ if } x = 0, \\ 0.5, \text{ if } x \in \{4,8\}, \\ 0, \text{ if } x \notin \{4,8\}. \end{cases}$$

$$\sigma(x) = \begin{cases} 1, \text{ if } x = 0, \\ 0.7, \text{ if } x \in \{2,4,6,8\}, \\ 0, \text{ if } x \notin \{2,4,6,8\}. \end{cases}$$

$$\delta(x) = \begin{cases} 1, \text{ if } x = 0, \\ 0.4, \text{ if } x = 6, \\ 0, \text{ if } x \neq 6. \end{cases}$$

We note that $\mu \subseteq \sigma$, $\mu \trianglelefteq_{fs} \sigma$ and $\sigma \trianglelefteq_{fs} M$, μ is not an s-essential submodule of M and δ is a fuzzy small submodule of M.
We have $\delta \neq \chi_\theta$ but $\mu \cap \delta = \chi_\theta$, which implies that $\mu \ntrianglelefteq_{fs} M$.

Theorem 4.12. *Let μ, $\sigma \in F(M)$. Then $(\mu \cap \sigma) \trianglelefteq_{fs} M$ if and only if $\mu \trianglelefteq_{fs} M$ and $\sigma \trianglelefteq_{fs} M$.*

Proof. Assume that $(\mu \cap \sigma) \trianglelefteq_{fs} M$.
Let δ be a fuzzy small submodule of M such that $\mu \cap \delta = \chi_\theta$.
We have, $(\mu \cap \sigma) \cap \delta \subseteq \mu \cap \sigma = \chi_\theta$ which implies that $(\mu \cap \sigma) \cap \delta = \chi_\theta$.
Since $(\mu \cap \sigma) \trianglelefteq_{fs} M$, we get $\delta = \chi_\theta$.
Hence, $\mu \trianglelefteq_{fs} M$.
Suppose that α is a fuzzy small submodule of M such that $\alpha \cap \sigma = \chi_\theta$.
We have $(\mu \cap \sigma) \cap \alpha = \mu \cap (\alpha \cap \sigma) \subseteq \alpha \cap \sigma = \chi_\theta$ which implies that $(\mu \cap \sigma) \cap \alpha = \chi_\theta$.
Hence, $\alpha = \chi_\theta$ and so, $\sigma \trianglelefteq_{fs} M$.

Conversely, assume that $\mu \trianglelefteq_{fs} M$ and $\sigma \trianglelefteq_{fs} M$.
Let γ be a fuzzy small submodule of M such $(\mu \cap \sigma) \cap \gamma = \chi_\theta$.
We can write $\mu \cap (\sigma \cap \gamma) = \chi_\theta$.
This implies that $\sigma \cap \gamma = \chi_\theta$ and so $\gamma = \chi_\theta$.
Hence, $(\mu \cap \sigma) \trianglelefteq_{fs} M$. □

5. **Jacobson L-radical and L-socle L-submodules**

In this section, we characterize the generalized L-radical and L-socle as the sum of all essential-small L-submodules of an R-module M and as the intersection of all small-essential L-submodules of an R-module M respectively.

Definition 5.1. [6] Let L be a complete Heyting algebra. An element $a \in L-\{1\}$ is called a maximal element, if there does not exist $c \in L-\{1\}$ such that $a < c < 1$.

Theorem 5.2. [10] *Let $\mu \in L^M$. Then μ is a maximal L-submodule of M if and only if μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M and α is a maximal element of $L - \{1\}$.*

The proof of the following lemma is similar to the proof of [[10], Lemma 5.3] and so it is omitted.

Lemma 5.3. *Let M be an R-module and $x \in M$.*
Then $\chi_{\{x\}}$ is in the sum of all essential-small L-submodules of M if and only if $\chi_R \bigodot \chi_{\{x\}} \ll_{eL} M$.

Definition 5.4. Let M be an R-module. We define
(i) $Rad_{eL}(M)$ as the intersection of all essential L-submodules of M which are maximal in M, and
(ii) $Soc_{sL}(M)$ as the sum of all small L-submodules of M which are minimal in M.

Theorem 5.5. *Let M be an R-module. Then $Rad_{eL}(M)$ is the sum of all essential-small L-submodules of M.*

Proof. Let ξ be the sum of all essential-small L-submodules of M.
Let $\delta \in L(M)$ be an essential-small submodule of M.
Suppose that $\sigma \trianglelefteq_L M$ is maximal in M.
Then $\delta \leq \sigma$, for otherwise, $\sigma + \delta = \chi_M$.
Since $\delta \ll_{eL} M$, we have $\sigma = \chi_M$, a contradiction.
Thus, δ is contained in every essential L-submodule of M which is also a maximal L-submodule of M.
Hence, $\xi \subseteq Rad_{eL}(M)$. (I)
On the other hand, it follows from Lemma 5.3, that $\chi_{\{x\}} \subseteq Rad_{eL}(M)$.
Suppose that $\chi_R \bigodot \chi_{\{x\}}$ is not an essential-small L-submodule of M.
Let,

$$\tau = \{\nu \in L(M) |\ \nu \neq \chi_M,\ \nu \trianglelefteq_L M, \chi_R \bigodot \chi_{\{x\}} + \nu = \chi_M\}.$$

Then $\tau \neq \phi$, as $\nu \in \tau$.
Now, for each $\sigma \in \tau$, $\sigma \neq \chi_M$, $\sigma \trianglelefteq_L M$ and $\chi_{\{x\}} \nsubseteq \sigma$.
We note that any proper L-submodule containing ν is also in τ and $(\tau, \subseteq)$ forms a poset. Clearly, the union of members of a chain in τ is again a member of τ.
Hence by Zorn's lemma, τ has a maximal element, say μ.
Claim: μ is a maximal L-submodule in M.
If not, there is an L-submodule δ in M such that $\mu \subseteq \delta \subseteq M$.
Thus, $\chi_M \subseteq \delta + \chi_R \bigodot \chi_{\{x\}}$ and $\delta \trianglelefteq_L M$.
Hence, $\delta \in \tau$, a contradiction to the maximality of μ.
So μ is maximal in M and $\mu \trianglelefteq_L M$.
Thus, $\chi_{\{x\}} \subseteq Rad_{eL}(M) \subseteq \mu$.
Now $\chi_R \bigodot \chi_{\{x\}} + \mu = \chi_M$ implies that $\mu = \chi_M$, a contradiction.
So, $\chi_R \bigodot \chi_{\{x\}} \ll_{eL} M$.
Hence, $Rad_{eL}(M) \subseteq \xi$. (II)
Therefore from (I) and (II), $Rad_{eL}(M)$ is the sum of all essential-small L-submodules of M. □

Corollary 5.6. *If any proper essential L-submodule of M is contained in a maximal L-submodule of M, then $Rad_{eL}(M)$ is the unique largest essential-small L-submodule of M.*

Proof. Let μ be a proper essential L-submodule of M.
Suppose that $\mu \neq \chi_M$.
Let σ be a maximal L-submodule of M such that $\mu \subseteq \sigma$, then $\sigma \trianglelefteq_L M$.
By the definition of $Rad_{eL}(M)$, $Rad_{eL}(M) \subseteq \sigma$.
This implies $Rad_{eL}(M) + \mu \subseteq \sigma + \mu$. (I)
Since $\mu \subseteq \sigma$ from (I) we get $Rad_{eL}(M) + \mu \subseteq \sigma \subsetneq \chi_M$.
Thus, $Rad_{eL}(M) \ll_{eL} M$. □

Remark 5.7. The existence of the $Rad_{eL}(M)$ depends on the existence of an essential L-submodule which is maximal.
If we take $L = [0, 1]$, then $L - \{1\}$ does not have a maximal element. Hence by Theorem 5.2, a maximal L-submodule of M does not exist. Therefore, the assumption of the existence of a maximal element in $L - \{1\}$, in Theorem 5.5 is necessary.

Example 5.8. Let $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Then L is a complete Heyting algebra and 0.8 is a maximal element of $L - \{1\}$.
Consider the module $M = R \oplus R$ of the ring $R = \mathbb{Z}_9$ of integers modulo 9.

Define $\sigma : M \longrightarrow [0,1]$ as follows:

$$\sigma(x) = \begin{cases} 1, & if\ x \in \{(0,0),(0,3),(0,6),(3,0),(6,0),(3,6),(6,3),(3,3),\\ & \quad (6,6)\}, \\ 0.8, & if\ x \in \{M-((0,0),(0,3),(0,6),(3,0),(6,0),(3,6),(6,3),\\ & \quad (3,3),(6,6))\}. \end{cases}$$

Here, $\sigma^* = M$ is an essential L-submodule of M.
Hence by Theorem 2.15, $\sigma \trianglelefteq_L M$. (I)
Also, $\sigma_* = \{(0,0),(0,3),(0,6),(3,0),(6,0),(3,6),(6,3),(3,3),(6,6)\}$ is a maximal submodule of M and $\sigma = \chi_{\sigma_*} \cup 0.8_M$.
By Theorem 5.2, σ is a maximal L-submodule of M. (II)
Thus, from (I) and (II), σ is an essential L-submodule which is maximal and in fact, it is the only essential L-submodule which is maximal and so $Rad_{eL}(M) = \sigma$.
As, $\sigma_* \ll_e M$, it follows from Theorem 3.2 that $\sigma \ll_{eL} M$.
Thus, $Rad_{eL}(M) \ll_{eL} M$.

Theorem 5.9. *Let M be an R-module. $Soc_{sL}(M)$ is the intersection of all small-essential L-submodules of an R-module M.*

Proof. Let ζ be the intersection of all small-essential L-submodules of M.
Let $\mu, \sigma \in L(M)$ be such that $\mu \trianglelefteq_{sL} M$ and $\sigma \ll_L M$ is minimal in M.
Then $\sigma \leq \mu$, otherwise, $\sigma \cap \mu = \chi_\theta$.
Hence, $\sigma = \chi_\theta$, a contradiction.
Thus $Soc_{sL}(M) \subseteq \zeta$.
Also, $\zeta \subseteq Soc_{sL}(M)$, thus $Soc_{sL}(M)$ and ζ are semisimple L-modules.
If $\zeta \nsubseteq Soc_{sL}(M)$, there exists a simple L-submodule δ such that $\delta \leq \zeta$ and δ is not small in M.
Let $\nu \neq \chi_M$ be a L-submodule such that $\delta + \nu = \chi_M$.
(i): If $\delta \cap \nu \neq \chi_\theta$, then $\delta \cap \nu = \delta$ (since, δ is a simple L-submodule).
Hence $\delta \subseteq \nu$. Thus, $\nu = \chi_M$, a contradiction.
(ii): If $\delta \cap \nu = \chi_\theta$, then $\chi_M = \delta \oplus \nu$.
To show that $\nu \trianglelefteq_{sL} M$.
Let $\sigma \ll_L M$ and $\nu \cap \sigma = \chi_\theta$.
Then $\mu + \nu$ is a proper L-submodule of M.
As $\sigma \ll_L M$ and $\nu \cap \sigma = \chi_\theta$, we have $\sigma^* \ll M$ and $\nu^* \cap \sigma^* = \{\theta\}$.
Hence $\sigma^* + \nu^*$ is a proper submodule of M.
Hence $\sigma^* \cong (\sigma + \nu)^*/\nu^*$ is a submodule of M/ν^*,
where $M/\nu^* \cong \delta^*$ is a simple submodule.

Thus, $\sigma^* = \{\theta\}$, which implies that $\sigma = \chi_\theta$.
Therefore, $\nu \trianglelefteq_{sL} M$. Hence, $\delta \subseteq \zeta \subseteq \nu$.
Since $\delta + \nu = \chi_M$ we get $\nu = \chi_M$, a contradiction.
From (i) and (ii), we have, $\delta \cap \nu = \chi_\theta$ and $\delta \cap \nu \neq \chi_\theta$ which is a contradiction.
Thus, $\zeta = Soc_{sL}(M)$. □

Corollary 5.10. *If any non-zero small L-submodule of an R-module M contains a minimal L-submodule of M, then $Soc_{sL}(M)$ is the unique least small-essential L-submodule of M.*

Proof. Let ν be a non-zero small L-submodule of M.
If $\nu \neq \chi_\theta$ and if δ is a minimal L-submodule of M such that $\delta \subseteq \nu$, then $\delta \ll_L M$.
By the definition of $Soc_{sL}(M)$, $\delta \subseteq Soc_{sL}(M)$.
Hence $\delta \cap \nu \subseteq Soc_{sL}(M) \cap \nu$. (I)
But, $\delta \subseteq \nu$ gives $\delta \subseteq \nu \cap \delta$.
Thus, (I) becomes, $\chi_\theta \neq \delta \subseteq Soc_{sL}(M) \cap \nu$.
Hence $Soc_{sL}(M) \trianglelefteq_{sL} M$. □

6. Conclusion

In this paper, we have studied various properties of fuzzy essential-small and fuzzy small-essential submodules of an R-module M and also characterize the generalize L-radical and L-socle of an R-module M. Further, these concepts can be used to study fuzzy essential-small submodules relative to an arbitrary submodule and fuzzy small-essential submodules relative to an arbitrary submodule.

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