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GENERALIZATIONS OF PRIME FUZZY IDEALS OF A LATTICE

SHRIRAM KHANDERAO NIMBHORKAR AND YOGITA SUBHASH PATIL

Abstract. As a generalization of the concepts of a fuzzy prime ideal and a prime fuzzy ideal, the concepts of a fuzzy 2-absorbing ideal and a 2-absorbing fuzzy ideal of a lattice are introduced. Some results on such fuzzy ideals are proved. It is shown that the radical of a fuzzy ideal of L is a 2-absorbing fuzzy ideal if and only if it is a 2-absorbing primary fuzzy ideal of L. We also introduce and study these concepts in a product of lattices.

Key Words: Lattice, fuzzy lattice, fuzzy ideal, fuzzy prime ideal, fuzzy 2-absorbing ideal, fuzzy primary ideal.

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1. Introduction

Zadeh [14] developed the concept of a fuzzy set. Gugan [13] generalized this concept by taking the evaluation set as a lattice. Ajmal and Thomas [8] defined a fuzzy lattice and a fuzzy sublattice as a fuzzy algebra. Attallah [7], Koguep et.al. [4] and Davvaz and Kazanci [3] have studied fuzzy sublattices, fuzzy ideals, fuzzy prime ideals in lattices.

The notion of a 2-absorbing ideal of a commutative ring was introduced by Badawi [1]. A proper ideal I of a commutative ring R is said to be a 2-absorbing, if whenever $a, b, c \in R, abc \in I$ then either $ab \in I$ or $ac \in I$ or $bc \in I$. This concept was generalized by Anderson and Badawi [6], Payrovi and Babaei [15], Badawi and Darani [2], Chaudhary [12], Yuand and Wu [5] and Wasadikar and Gaikwad [10, 9] in other

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∗Address correspondence to Yogita Subhash Patil; E-mail: saharshyog.143@rediffmail.com

mathematical structures such as semirings, semigroups, submodules and lattices.

In this paper, we introduce the concepts of a fuzzy 2-absorbing ideal and a 2-absorbing fuzzy ideal of a lattice L. This is a generalization of the concepts of a fuzzy prime ideal and a prime fuzzy ideal of L introduced by Koguep et. al. [4]. Also we define a primary fuzzy ideal and the radical of a fuzzy ideal of L. Some properties of these fuzzy ideals are proved. We also introduce and study these concepts in a product of lattices.

2. Preliminaries

Throughout in this paper, $L = (L, \wedge, \vee)$ denotes a lattice with 0. We recall some concepts and results.

Definition 2.1. A fuzzy subset μ of L is a function $\mu : L \to [0, 1]$.

Definition 2.2. [5] A fuzzy subset μ of L is called proper if it is a non-constant function.

Definition 2.3. [4] For any $\alpha \in [0, 1]$ the set $\mu_\alpha = \{x \in L/\mu(x) \geq \alpha\}$ is called the $\alpha-$cut of μ or α-level set and $\mu_{\alpha^+} = \{x \in L/\mu(x) > \alpha\}$ is called the strong $\alpha-$cut of μ.

Definition 2.4. [4] A fuzzy subset μ of L is called a fuzzy sublattice of L if $\mu(x \wedge y) \wedge \mu(x \vee y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in L$.

Definition 2.5. [4] A fuzzy sublattice μ of L is called a fuzzy ideal of L if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

Definition 2.6. [3] For fuzzy subsets μ, η of L, $\mu \subseteq \eta$ means $\mu(x) \leq \eta(x)$ for all $x \in L$.

The following result is from [7].

Lemma 2.7. *Let μ be a fuzzy sublattice of L. Then μ is a fuzzy ideal of L if and only if $\mu(x) \leq \mu(y)$ whenever, $x \geq y$ for all $x, y \in L$.*

3. Fuzzy prime ideals and prime fuzzy ideals of a lattice

The following concept is well-known in lattice theory, see Grätzer [11].

Definition 3.1. A nonempty subset I of a lattice L is called an ideal, if for $a, b \in L$, the following conditions hold.
(i) If $a, b \in I$, then $a \vee b \in I$ and (ii) if $a \leq b$ and $b \in I$, then $a \in I$.
A proper ideal I (i.e. $I \neq L$) is called a prime ideal, if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

Koguep et. al. [4], have defined a fuzzy prime ideal and a prime fuzzy ideal as follows.

Definition 3.2. A proper fuzzy ideal μ of a lattice L is called a fuzzy prime ideal, if for all $a, b \in L$, $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$.

In fact, a proper fuzzy ideal μ of L is fuzzy prime if and only if for all $a, b \in L$, $\mu(a \wedge b) = \mu(a) \vee \mu(b)$.

Definition 3.3. A fuzzy ideal μ of L is called a prime fuzzy ideal of L if for any two fuzzy ideals σ and θ of lattice L if $\sigma \wedge \theta \subseteq \mu$ imply that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

We have the following theorem.

Theorem 3.4. *Let I be an ideal of L and χ_I denote the characteristic function of I.*
(i) I is a prime ideal of L if and only if χ_I is a fuzzy prime ideal of L.
(ii) I is a prime ideal of L if and only if χ_I is a prime fuzzy ideal of L.

Proof. Clearly, χ_I is a fuzzy ideal of L.
(i): Suppose that I is a prime ideal of L.
Let $a, b \in L$. We need to show that

$$\chi_I(a \wedge b) = \chi_I(a) \vee \chi_I(b).$$

If $a, b \in I$, then $a \wedge b \in I$ and we have

$$\chi_I(a \wedge b) = 1 = 1 \vee 1 = \chi_I(a) \vee \chi_I(b).$$

If $a, b \notin I$, then as I is a prime ideal, $a \wedge b \notin I$ and we have

$$\chi_I(a \wedge b) = 0 = 0 \vee 0 = \chi_I(a) \vee \chi_I(b).$$

If only one of a or b is in I, say $a \in I$. Then $a \wedge b \in I$. We have $\chi_I(a) = \chi_I(a \wedge b) = 1$ and $\chi_I(b) = 0$. Thus

$$\chi_I(a \wedge b) = 1 = 1 \vee 0 = \chi_I(a) \vee \chi_I(b).$$

Thus χ_I is a fuzzy prime ideal of L.
Conversely, suppose that χ_I is a fuzzy prime ideal of L.
Let $a \wedge b \in I$. Then

$$\chi_I(a \wedge b) = 1 = \chi_I(a) \vee \chi_I(b). \tag{3.1}$$

If both $a, b \notin I$, then $\chi_I(a) = \chi_I(b) = 0$ implies that $\chi_I(a) \vee \chi_I(b) = 0$, which contradicts (3.1).
Hence I must be a prime ideal of L.

(ii): Suppose that I is a prime ideal of L.
Let σ, θ be fuzzy ideals of L. Suppose that $\sigma \cap \theta \subseteq \chi_I$.
If $\sigma \nsubseteq \chi_I$, $\theta \nsubseteq \chi_I$, then there exist $a, b \in L$ such that
$\chi_I(a) < \sigma(a)$ and $\chi_I(b) < \theta(b)$.
By the definition of χ_I, we conclude that $a, b \notin I$. For, if say $a \in I$, then $\chi_I(a) = 1$ leads to $1 < \sigma(a)$, which is not possible.

Since I is a prime ideal of L, we get $a \wedge b \notin I$. Hence $\chi_I(a \wedge b) = 0$.
Since σ, θ are fuzzy ideals of L, we have
$\sigma(a) \leq \sigma(a \wedge b)$ and $\theta(b) \leq \theta(a \wedge b)$.
As the image of any element under a fuzzy set is a nonnegative number, from the above, we get

$$\begin{aligned}\chi_I(a \wedge b) &= 0 \\ &\leq \chi_I(a) \wedge \chi_I(b) \\ &< \sigma(a) \wedge \theta(b) \\ &\leq \sigma(a \wedge b) \wedge \theta(a \wedge b) \\ &= (\sigma \cap \theta)(a \wedge b) \leq \chi_I(a \wedge b) \\ &= 0.\end{aligned}$$

Thus we get $0 < 0$ which is not possible.
Hence either $\sigma \subseteq \chi_I$ or $\theta \subseteq \chi_I$.

Conversely, suppose that χ_I is a prime fuzzy ideal of L.
Suppose that for some $a, b \in L$, $a \wedge b \in I$ but $a, b \notin I$.
Define fuzzy ideals σ and θ of L as follows.

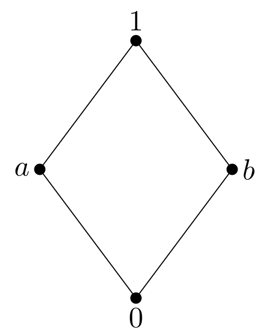
$$\sigma(x) = \begin{cases} 1, & \text{if; } x \in (a]; \\ 0 & \text{otherwise} \end{cases}$$
$$\theta(x) = \begin{cases} 1, & \text{if; } x \in (b]; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma \cap \theta \subseteq \chi_I$ but neither $\sigma \subseteq \chi_I$ nor $\theta \subseteq \chi_I$, a contradiction.
Hence I is a prime ideal of L. □

The following example shows that the condition of "primeness" in Theorem 3.4 is necessary.

Example 3.5. Consider the lattice L shown in Figure 1. We note that the ideal $I = (0]$ is not a prime ideal of L, as $a \wedge b = 0 \in I$ but neither $a \in I$, nor $b \in I$.
(i): We have $\chi_I(a \wedge b) = 1$ and $\chi_I(a) = \chi_I(b) = 0$.

Thus $\chi_I(a \wedge b) \nleq \chi_I(a) \vee \chi_I(b) = 0$.
Hence χ_I is not a fuzzy prime ideal of L.



L
Figure 1

(ii): Define fuzzy ideals σ and θ of L as follows.
$\sigma(0) = 1$, $\sigma(1) = \sigma(b) = 0$, $\sigma(a) = 1/2$.
$\theta(0) = 1$, $\theta(1) = \theta(a) = 0$, $\theta(b) = 1/3$.
Then $\sigma \cap \theta \subseteq \chi_I$ but neither $\sigma \subseteq \chi_I$ nor $\theta \subseteq \chi_I$.
Thus χ_I is not a prime fuzzy ideal of L.

Koguep et. al. [4], have given an example of a fuzzy prime ideal of a lattice, which is not a prime fuzzy ideal. But no example of a prime fuzzy ideal of a lattice is given by them. We pose the following question.

Question: Let L be a lattice with 0 (least element) and 1(greatest element). Whether a prime fuzzy ideal, other than the characteristic function of a prime ideal of L exists?

The following example indicates nonexistence of a prime fuzzy ideal (other than the characteristic function of a prime ideal) of a lattice.

Example 3.6. Consider the lattice L, shown in Figure 1. Any fuzzy ideal of L is of the form (or similar form with appropriate changes).
$\mu(0) = 1$, $\mu(1) = 0$, $\mu(a) = 0$, $\mu(b) = \beta$.
Consider the fuzzy ideals σ, θ of L defined by
$\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(a) = 0$, $\sigma(b) = \beta + \gamma$, where $0 < \gamma < 1$.
$\theta(0) = 1$, $\theta(1) = 0$, $\theta(a) = \alpha + \gamma$, $\theta(b) = 0$.
Then $\sigma \cap \theta \subseteq \mu$ but neither $\sigma \subseteq \mu$ nor $\theta \subseteq \mu$.

4. Fuzzy 2-absorbing ideals

The following definition is from Wasadikar and Gaikwad [10].

Definition 4.1. Let L be a lattice with 0. An ideal I of L is called a 2-absorbing ideal, if for $a, b, c \in L$, $a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $b \wedge c \in I$ or $c \wedge a \in I$.

We extend the concept of a 2-absorbing ideal, in the context of a fuzzy ideal of a lattice and prove some properties of fuzzy 2-absorbing ideals of a lattice. We denote the set of all fuzzy ideals of L by $FI(L)$.

Definition 4.2. A proper fuzzy ideal μ of a lattice L is called a fuzzy 2-absorbing ideal of L if for all $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) \leq \max\{\mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)\}.$$

Since $\mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)$ are nonnegative real numbers, the definition of a fuzzy 2-absorbing ideal is equivalent to
μ is a fuzzy 2-absorbing ideal iff for all $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

In fact, μ is a fuzzy 2-absorbing ideal iff for all $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) = \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Lemma 4.3. *Let I be an ideal of L. Then I is a 2-absorbing ideal of L if and only if χ_I is a fuzzy 2-absorbing ideal of L.*

Proof. Suppose that I is a 2-absorbing ideal of L. Let $a, b, c \in L$.
If $a \wedge b \wedge c \in I$, then as I is 2-absorbing, either

$$a \wedge b \in I \text{ or } b \wedge c \in I \text{ or } c \wedge a \in I.$$

Thus in this case,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then clearly, $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$.
Thus in this case also,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a).$$

Hence χ_I is a fuzzy 2-absorbing ideal of L.

Conversely, suppose that χ_I is a fuzzy 2-absorbing ideal of L.
Let $a, b, c \in L$ be such that $a \wedge b \wedge c \in I$, but $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$.
This implies that

$$\chi_I(a \wedge b \wedge c) = 1 \text{ and } \chi_I(a \wedge b) = \chi_I(b \wedge c) = \chi_I(c \wedge a) = 0.$$

Then $\chi_I(a \wedge b \wedge c) \nleq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a)$, a contradiction, as χ_I is fuzzy 2-absorbing. □

The following lemma shows that any level set of a fuzzy 2-absorbing ideal of L is a 2-absorbing ideal of L.

Lemma 4.4. *A fuzzy ideal μ of L is a fuzzy 2-absorbing ideal if and only if for each $t \in Image(\mu)$, the level ideal μ_t is a 2-absorbing ideal of L.*

Proof. (i): Let μ be a fuzzy 2-absorbing ideal of L. Let $t \in Image(\mu)$.
Let $a, b, c \in L$ be such that $a \wedge b \wedge c \in \mu_t$. Then $t \leq \mu(a \wedge b \wedge c)$.
Since μ is a fuzzy 2-absorbing ideal,

$$t \leq \mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a). \tag{4.1}$$

Since $t, \mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)$ are nonnegative real numbers, if

$$\mu(a \wedge b) < t,\ \mu(b \wedge c) < t \text{ and } \mu(c \wedge a) < t,$$

then

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) < t. \tag{4.2}$$

Thus (4.1) and (4.2) lead to $t < t$, which is not possible.
Hence

$$t \leq \mu(a \wedge b) \text{ or } t \leq \mu(b \wedge c) \text{ or } t \leq \mu(c \wedge a).$$

Thus either

$$a \wedge b \text{ or } b \wedge c \text{ or } c \wedge a \in \mu_t;$$

i.e. μ_t is a 2-absorbing ideal of L.
(ii): Let μ_t be a 2-absorbing ideal of L for each $t \in Image(\mu)$.
Let $a, b, c \in L$ and $\mu(a \wedge b \wedge c) = t$.
Then $a \wedge b \wedge c \in \mu_t$. Since μ_t is a 2-absorbing ideal of L, either

$$a \wedge b \text{ or } b \wedge c \text{ or } c \wedge a \in \mu_t.$$

This implies that

$$t \leq \mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Thus μ is a fuzzy 2-absorbing ideal of L. □

Now we show that every fuzzy prime ideal of L is a fuzzy 2-absorbing ideal.

Lemma 4.5. *Let μ be a fuzzy prime ideal of L. Then μ is a fuzzy 2-absorbing ideal of L.*

Proof. Let μ be a fuzzy prime ideal of L. Then for all $a, b \in L$,

$$\mu(a \wedge b) \leq \mu(a) \vee \mu(b).$$

Hence for all $a, b, c \in L$, we have

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(c),$$
$$\mu(a \wedge b \wedge c) \leq \mu(b \wedge c) \vee \mu(a),$$
$$\mu(a \wedge b \wedge c) \leq \mu(c \wedge a) \vee \mu(b).$$

Hence

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(c) \vee \mu(b \wedge c) \vee \mu(a) \vee \mu(c \wedge a) \vee \mu(b). \tag{4.3}$$

By the definition of a fuzzy ideal, it follows that for any $x, y \in L$, $\mu(x) \leq \mu(x \wedge y)$.
Hence (4.3) reduces to

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Thus μ is a fuzzy 2-absorbing ideal of L. □

The following example shows that the converse of Lemma 4.5 does not hold.

Example 4.6. Consider the lattice L shown in Figure 1. Let μ be the fuzzy set defined by $\mu(0) = 1, \mu(a) = 0, \mu(b) = 1/2, \mu(1) = 0$.
Then μ is a fuzzy 2-absorbing ideal of L.
However, μ is not a fuzzy prime ideal as

$$1 = \mu(0) = \mu(a \wedge b) \neq 0 \vee 1/2 = \mu(a) \vee \mu(b).$$

Lemma 4.7. *The intersection of any two distinct fuzzy prime ideals of L is a fuzzy 2-absorbing ideal of L.*

Proof. Let μ, θ be two distinct fuzzy prime ideals of L.
We know that for any $a \in L$, $(\mu \cap \theta)(a) = \mu(a) \wedge \theta(a)$.
Let $a, b, c \in L$. We have

$$(\mu \cap \theta)(a \wedge b \wedge c) = \mu(a \wedge b \wedge c) \wedge \theta(a \wedge b \wedge c) \tag{4.4}$$

Since every fuzzy prime ideal is fuzzy 2-absorbing, from (4.4), we get

$$\begin{aligned} &(\mu \cap \theta)(a \wedge b \wedge c) \\ &\leq [\mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a)] \\ &\quad \wedge [\theta(a \wedge b) \vee \theta(b \wedge c) \vee \theta(c \wedge a)]. \end{aligned} \tag{4.5}$$

Since μ and θ are fuzzy prime ideals, we can write

$$\mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) \leq \mu(a) \vee \mu(b) \vee \mu(c)$$

and

$$\theta(a \wedge b) \vee \theta(b \wedge c) \vee \theta(c \wedge a) \leq \theta(a) \vee \theta(b) \vee \theta(c).$$

We note that all the terms on the right hand side of (4.5) belong to the distributive lattice $[0,1]$. Hence we can write

$$\begin{aligned}(\mu \cap \theta)(a \wedge b \wedge c) &\leq [\mu(a) \vee \mu(b) \vee \mu(c)] \wedge [\theta(a) \vee \theta(b) \vee \theta(c)] \\ &= [\mu(a) \wedge \theta(a)] \vee [\mu(a) \wedge \theta(b)] \vee [\mu(a) \wedge \theta(c)] \\ &\quad \vee [\mu(b) \wedge \theta(a)] \vee [\mu(b) \wedge \theta(b)] \vee [\mu(b) \wedge \theta(c)] \\ &\quad \vee [\mu(c) \wedge \theta(a)] \vee [\mu(c) \wedge \theta(b) \vee [\mu(c) \wedge \theta(c)].\end{aligned} \tag{4.6}$$

For any fuzzy ideal σ, we have $\sigma(x) \leq \sigma(x \wedge y)$, for all $x, y \in L$.
Hence $\mu(x) \leq \mu(x \wedge y)$ and $\theta(y) \leq \theta(x \wedge y)$ for all $x, y \in L$.
This implies

$$\mu(x) \wedge \theta(y) \leq \mu(x \wedge y) \wedge \theta(x \wedge y) = (\mu \cap \theta)(x \wedge y).$$

Applying this to the R. H. S. of (4.6), we get

$$\begin{aligned}(\mu \cap \theta)(a \wedge b \wedge c) \leq (\mu \cap \theta)(a) \vee (\mu \cap \theta)(a \wedge b) \vee (\mu \cap \theta)(b \wedge c) \\ \vee (\mu \cap \theta)(c \wedge a) \vee (\mu \cap \theta)(b) \vee (\mu \cap \theta)(c).\end{aligned} \tag{4.7}$$

Since $\mu \cap \theta$ is a fuzzy ideal, for all $x, y \in L$, we have

$$(\mu \cap \theta)(x) \leq (\mu \cap \theta)(x \wedge y).$$

Applying this to the R. H. S. of (4.7), we get

$$(\mu \cap \theta)(a \wedge b \wedge c) \leq (\mu \cap \theta)(a \wedge b) \vee (\mu \cap \theta)(b \wedge c) \vee (\mu \cap \theta)(c \wedge a).$$

Thus $\mu \cap \theta$ is a fuzzy 2-absorbing ideal of L. □

The following example shows that the condition of "primeness"in Lemma 4.7 is necessary. This example also shows that in general the intersection of two fuzzy 2-absorbing ideals need not be a fuzzy 2-absorbing ideal.

Example 4.8. Consider the lattice shown in Figure 2.

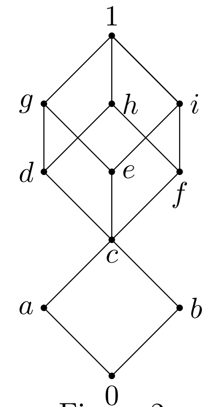


Figure 2

Define $\mu : L \to [0,1]$ and $\theta : L \to [0,1]$ as follows.

$\mu(0) = 1$	$\theta(0) = 1$
$\mu(a) = 1/2$	$\theta(a) = 1/3$
$\mu(b) = 2/3$	$\theta(b) = 1/3$
$\mu(c) = 1/2$	$\theta(c) = 1/3$
$\mu(d) = 1/2$	$\theta(d) = 0$
$\mu(e) = 0$	$\theta(e) = 1/3$
$\mu(f) = 0$	$\theta(f) = 0$
$\mu(g) = 0$	$\theta(g) = 0$
$\mu(h) = 0$	$\theta(h) = 0$
$\mu(i) = 0$	$\theta(i) = 0$
$\mu(1) = 0$	$\theta(1) = 0$

We note that μ and θ are fuzzy 2-absorbing ideals of L.
For
$\mu(d \wedge e \wedge f) = \mu(c)$ and $\mu(d \wedge e) = \mu(e \wedge f) = \mu(f \wedge d) = \mu(c)$.
$\mu(g \wedge h \wedge i) = \mu(c) = 1/2$ and $\mu(g \wedge h) = \mu(d) = 1/2$, $\mu(h \wedge i) = \mu(f) = 0$, $\mu(i \wedge g) = \mu(e) = 0$.
Similarly for other elements.
$\theta(d \wedge e \wedge f) = \theta(c)$ and $\theta(d \wedge e) = \theta(e \wedge f) = \theta(f \wedge d) = \theta(c)$.
$\theta(g \wedge h \wedge i) = \theta(c) = 1/3$ and $\theta(g \wedge h) = \theta(d) = 0$, $\theta(h \wedge i) = \theta(f) = 0$, $\theta(i \wedge g) = \theta(e) = 1/3$.
Similarly for other elements.
We have

$(\mu \cap \theta)(0) = \min\{\mu(0), \theta(0)\} = \min\{3/4, 3/4\} = 3/4.$
$(\mu \cap \theta)(a) = \min\{\mu(a), \theta(a)\} = \min\{2/3, 1/3\} = 1/3.$
$(\mu \cap \theta)(b) = \min\{\mu(b), \theta(b)\} = \min\{2/3, 1/3\} = 1/3.$
$(\mu \cap \theta)(c) = \min\{\mu(c), \theta(c)\} = \min\{1/2, 1/3\} = 1/3.$
$(\mu \cap \theta)(d) = \min\{\mu(d), \theta(d)\} = \min\{1/2, 0\} = 0.$
$(\mu \cap \theta)(e) = \min\{\mu(e), \theta(e)\} = \min\{0, 1/3\} = 0.$
$(\mu \cap \theta)(f) = \min\{\mu(f), \theta(f)\} = \min\{0, 0\} = 0.$
$(\mu \cap \theta)(g) = \min\{\mu(g), \theta(g)\} = \min\{0, 0\} = 0.$
$(\mu \cap \theta)(h) = \min\{\mu(h), \theta(h)\} = \min\{0, 0\} = 0.$
$(\mu \cap \theta)(i) = \min\{\mu(i), \theta(i)\} = \min\{0, 0\} = 0.$
$(\mu \cap \theta)(1) = \min\{\mu(1), \theta(1)\} = \min\{0, 0\} = 0.$
$(\mu \cap \theta)(g \wedge h \wedge i) = (\mu \cap \theta)(c) = 1/3.$
$(\mu \cap \theta)(g \wedge h) = (\mu \cap \theta)(d) = 0.$
$(\mu \cap \theta)(h \wedge i) = (\mu \cap \theta)(f) = 0.$
$(\mu \cap \theta)(i \wedge g) = (\mu \cap \theta)(e) = 0.$
Thus
$(\mu \cap \theta)(g \wedge h \wedge i) \nleq \max\{(\mu \cap \theta)(g \wedge h), (\mu \cap \theta)(h \wedge i), (\mu \cap \theta)(i \wedge g)\}.$
Hence $\mu \cap \theta$ is not a fuzzy 2-absorbing ideal of L.

5. 2-Absorbing Fuzzy Ideals

Now we introduce the concept of a 2-absorbing fuzzy ideal on the lines of a prime fuzzy ideal.

Definition 5.1. A proper fuzzy ideal μ of L is called a 2-absorbing fuzzy ideal of L if whenever $\theta \cap \eta \cap \nu \subseteq \mu$ for $\theta, \eta, \nu \in FI(L)$, then either $\theta \cap \eta \subseteq \mu$ or $\eta \cap \nu \subseteq \mu$ or $\theta \cap \nu \subseteq \mu$.

The following example shows that the concept of a "fuzzy 2-absorbing ideal"is different from that of a "2-absorbing fuzzy ideal".

Example 5.2. Consider the following fuzzy ideals of the lattice L shown in Figure 1.
$\mu = \{(0, 7/8), (a, 1/3), (b, 3/4), (1, 1/3)\},$
$\eta = \{(0, 1), (a, 1/4), (b, 4/5), (1, 1/4)\},$
$\nu = \{(0, 1), (a, 3/4), (b, 2/3), (1, 2/3)\},$
$\gamma = \{(0, 4/5), (a, 3/4), (b, 4/5), (1, 3/4)\},$
We note that (i) μ is a fuzzy 2-absorbing ideal and (ii) $\eta \cap \nu \cap \gamma \subseteq \mu$.
But $\eta \cap \nu \nsubseteq \mu$, $\eta \cap \gamma \nsubseteq \mu$ and $\gamma \cap \nu \nsubseteq \mu$.
Thus μ is not a 2-absorbing fuzzy ideal.

Lemma 5.3. *Let I be an ideal of L. If χ_I is a 2-absorbing fuzzy ideal of L, then I is a 2-absorbing ideal of L.*

Proof. Suppose that χ_I is a 2-absorbing fuzzy ideal of L.
Let $a \wedge b \wedge c \in I$ for some $a, b, c \in L$. Suppose that $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$.
Then clearly, $a, b, c \notin I$.
Define fuzzy ideals

$$\mu(x) = \begin{cases} 1, & \text{if } x \in (a], \\ 0 & \text{otherwise.} \end{cases}$$
$$\theta(x) = \begin{cases} 1, & \text{if } x \in (b], \\ 0 & \text{otherwise.} \end{cases}$$
$$\eta(x) = \begin{cases} 1, & \text{if } x \in (c], \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$(\mu \cap \theta \cap \eta)(x) = \begin{cases} 1, & \text{if } x \in (a \wedge b \wedge c], \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \cap \theta \cap \eta \subseteq \chi_I$ but $\mu \cap \theta \nsubseteq \chi_I$, $\theta \cap \eta \nsubseteq \chi_I$ and $\mu \cap \eta \nsubseteq \chi_I$.
This contradicts the assumption that χ_I is a 2-absorbing fuzzy ideal. □

Remark 5.4. However, we are unable to prove or disprove that if I is a 2-absorbing ideal of L, then χ_I is a 2-absorbing fuzzy ideal of L.

Lemma 5.5. *Every prime fuzzy ideal of a lattice L is a 2-absorbing fuzzy ideal of L.*

Proof. Let μ be a prime fuzzy ideal of L. Suppose that $\theta, \eta, \nu \in FI(L)$ and $\theta \cap \eta \cap \nu \subseteq \mu$. As μ is a prime fuzzy ideal of L we have either
(1) $\theta \cap \eta \subseteq \mu$ or $\nu \subseteq \mu$, or (2) $\theta \cap \nu \subseteq \mu$ or $\eta \subseteq \mu$, or (3) $\eta \cap \nu \subseteq \mu$ or $\theta \subseteq \mu$.
Without loss of generality, suppose that $\theta \cap \eta \subseteq \mu$ or $\nu \subseteq \mu$.
If $\theta \cap \eta \subseteq \mu$ then the proof is obvious and if $\nu \subseteq \mu$ then $\theta \cap \nu \subseteq \mu$ and $\eta \cap \nu \subseteq \mu$. Thus μ is a 2-absorbing fuzzy ideal of a lattice L. □

We are unable to give an example to show that the converse of Lemma 5.5 does not hold.

Proposition 5.6. *The intersection of two prime fuzzy ideals of L is a 2-absorbing fuzzy ideal of L.*

Proof. Let μ and δ be two distinct prime fuzzy ideals of L. Assume that θ, η, ν are fuzzy ideals of L such that
$\theta \cap \eta \cap \nu \subseteq \mu \cap \delta$ but $\theta \cap \eta \nsubseteq \mu \cap \delta$, $\theta \cap \nu \nsubseteq \mu \cap \delta$ and $\eta \cap \nu \nsubseteq \mu \cap \delta$.
Clearly, $\theta \cap \eta \cap \nu \subseteq \mu$ and $\theta \cap \eta \cap \nu \subseteq \delta$.
Since μ and δ are prime fuzzy ideals, we have
(i) $\theta \cap \eta \subseteq \mu$ or $\nu \subseteq \mu$ and (ii) $\theta \cap \eta \subseteq \delta$ or $\nu \subseteq \delta$.
We have the following cases:
Case(1): If $\theta \cap \eta \subseteq \mu$ and $\theta \cap \eta \subseteq \delta$, then we have $\theta \cap \eta \subseteq \mu \cap \delta$, a contradiction.
Case(2): If $\nu \subseteq \mu$ and $\nu \subseteq \delta$, then we get $\theta \cap \nu \subseteq \mu \cap \delta$, a contradiction.
Case(3): Let $\theta \cap \eta \subseteq \mu$ and $\nu \subseteq \delta$. As μ is a prime fuzzy ideal, we get either $\theta \subseteq \mu$ or $\eta \subseteq \mu$. Hence either $\theta \cap \nu \subseteq \mu \cap \delta$ or $\eta \cap \nu \subseteq \mu \cap \delta$, a contradiction in either case.
Case(4): Let $\nu \subseteq \mu$ and $\theta \cap \eta \subseteq \delta$. As δ is a prime fuzzy ideal, we get either $\theta \subseteq \delta$ or $\eta \subseteq \delta$. Hence either $\theta \cap \nu \subseteq \mu \cap \delta$ or $\eta \cap \nu \subseteq \mu \cap \delta$, a contradiction in either case.
Hence at least one of $\theta \cap \eta$ or $\theta \cap \nu$ or $\eta \cap \nu$ must be a subset of $\mu \cap \delta$.
Therefore $\mu \cap \delta$ is a 2-absorbing fuzzy ideal. □

6. Fuzzy primary ideals

The following definition is from Wasadikar and Gaikwad [10].

Definition 6.1. Let L be a lattice with 0. An ideal I of L is called a primary ideal, if for $a, b \in L$, $a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$, where $\sqrt{I}$ denotes the radical of I (i.e. the intersection of all prime ideals containing I).
If there does not exist a prime ideal containing an ideal I in a lattice L then we define $\sqrt{I} = L$.

We define the radical of a fuzzy ideal. Since there are two concepts of primeness (namely, a fuzzy prime ideal and a prime fuzzy ideal), we can introduce two concepts of the radical and primariness. For the radical of a fuzzy set, we use the notation $\sqrt{\mu}$. The context will decide the radical (i.e. whether fuzzy prime radical or prime fuzzy radical).

Definition 6.2. Let μ be a fuzzy ideal of a lattice L. We define the fuzzy prime (respectively, prime fuzzy) radical of μ as the intersection of all fuzzy prime (respectively, prime fuzzy) ideals containing μ and we denote it by $\sqrt{\mu}$.

We note that for a fuzzy ideal μ of L always $\mu \subseteq \sqrt{\mu}$.
It can be shown that for an ideal I of L, $\sqrt{\chi_I} = \chi_{\sqrt{I}}$.

Wasadikar and Gaikwad, [10, 9] have introduced and studied the concepts of a primary ideal and a 2-absorbing primary ideal in a lattice. We introduce the concept of a fuzzy primary ideal of a lattice.

Definition 6.3. A proper fuzzy ideal μ of a lattice L is called a fuzzy primary ideal of L, if for $a, b \in L$,

$$\mu(a \wedge b) \leq \mu(a) \vee \sqrt{\mu}(b).$$

Lemma 6.4. *Let I be a proper ideal of L. Then I is a primary ideal of L if and only if χ_I is a fuzzy primary ideal of L.*

Proof. Suppose that I is a primary ideal of L. Let $a, b \in L$.
(i) If $a \wedge b \in I$, then as I is a primary ideal of L, either $a \in I$ or $b \in \sqrt{I}$. Hence

$$\chi_I(a \wedge b) \leq \chi_(a) \vee \sqrt{\chi_I}(b).$$

(ii) If $a \wedge b \notin I$, then clearly $a \notin I$ and $b \notin I$. In this case also

$$\chi_I(a \wedge b) \leq \chi_(a) \vee \sqrt{\chi_I}(b).$$

Thus χ_I is a fuzzy primary ideal of L.

Conversely, suppose that χ_I is a fuzzy primary ideal of L.
Let $a \wedge b \in I$. Then

$$\chi_I(a \wedge b) \leq \chi_(a) \vee \sqrt{\chi_I}(b),$$

implies that either $\chi_I(a) = 1$ or $\sqrt{\chi_I}(b) = 1$.
Thus either $a \in I$ or $b \in \sqrt{I}$. □

Now we give a relationship between a fuzzy prime ideal and a fuzzy primary ideal.

Lemma 6.5. *If μ is a fuzzy prime ideal of L, then μ is a fuzzy primary ideal of L.*

Proof. Let μ be a fuzzy prime ideal of L. For all $a, b \in L$,

$$\mu(a \wedge b) \leq \mu(a) \vee \mu(b).$$

Since $\mu \subseteq \sqrt{\mu}$, we get the result. □

The following example shows that the converse of Lemma 6.5 does not hold.

Example 6.6. Consider the ideal $I = (a]$ of the lattice shown in Figure 3. We note that $J = (d]$ is the only prime ideal of L containing I. Hence $\sqrt{I} = J$. We know that for any ideal A of L, $\sqrt{\chi_A} = \chi_{\sqrt{A}}$.

Hence $\sqrt{\chi_I} = \chi_{\sqrt{I}} = \chi_J$. Since J is a prime ideal, χ_J is a fuzzy prime ideal and so χ_I is a fuzzy primary ideal.
We have $\chi_I(b \wedge c) = 1$ but $\chi_I(b) \vee \chi_I(c) = 0$ as $b, c \notin I$. Thus χ_I is not fuzzy prime.

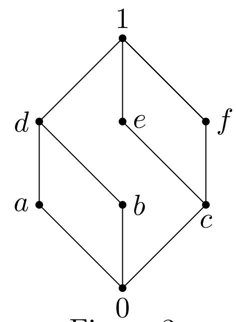


Figure 3

Theorem 6.7. *Let μ be a fuzzy ideal of L. Then μ is fuzzy primary if and only if the level set μ_t, $t \in Image(\mu)$ is a primary ideal of L.*

Proof. Suppose that μ is a fuzzy primary ideal of L.
Let $a, b \in L$ be such that $a \wedge b \in \mu_t$ and $a \notin \mu_t$, $b \notin \sqrt{\mu_t}$.
Then we have

$$t \leq \mu(a \wedge b), t < \mu(a), t < \sqrt{\mu}(b).$$

Since μ is fuzzy primary, we have

$$\mu(a \wedge b) \leq \mu(a) \vee \sqrt{\mu}(b).$$

Thus we get $t < t$, which is not possible.
Hence μ_t is a primary ideal of L.

Conversely, suppose that μ_t is a primary ideal of L.
Let $a, b \in L$ be such that

$$\mu(a \wedge b) \nleq \mu(a) \vee \sqrt{\mu}(b).$$

Let $\mu(a \wedge b) = t$. Then $\mu(a) < t$ and $\sqrt{\mu}(b) < t$.
Since μ_t is a primary ideal, $a \wedge b \in \mu_t$ implies that either $a \in \mu_t$ or $b \in \sqrt{\mu}_t$, i.e. either $\mu(a) \geq t$ or $\sqrt{\mu}(b) \geq t$, a contradiction. □

Definition 6.8. A proper fuzzy ideal μ of a lattice L is called a fuzzy 2-absorbing primary ideal of L, if for $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(c \wedge a).$$

Lemma 6.9. *A proper ideal I of L is a 2-absorbing primary ideal, if and only if χ_I is a fuzzy 2-absorbing primary ideal of L.*

Proof. Suppose that I is a 2-absorbing primary ideal of L. Let $a, b, c \in L$. Consider $\chi_I(a \wedge b \wedge c)$.
If $a \wedge b \wedge c \in I$, then $\chi_(a \wedge b \wedge c) = 1$.
As I is 2-absorbing primary, we have either

$$a \wedge b \in I \text{ or } b \wedge c \in \sqrt{I} \text{ or } c \wedge a \in \sqrt{I}.$$

Hence either

$$\chi_I(a \wedge b) = 1 \text{ or } \chi_{\sqrt{I}}(b \wedge c) = \sqrt{\chi_I}(b \wedge c) = 1 \text{ or } \chi_{\sqrt{I}}(c \wedge a) = \sqrt{\chi_I}(c \wedge a) = 1.$$

Thus

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{I}}(b \wedge c) \vee \chi_{\sqrt{I}}(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then $\chi_(a \wedge b \wedge c) = 0$. Clearly, $a \wedge b \notin I$.
Hence

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{I}}(b \wedge c) \vee \chi_{\sqrt{I}}(c \wedge a).$$

Thus χ_I is a fuzzy 2-absorbing primary ideal.

Conversely, suppose that χ_I is a fuzzy 2-absorbing primary ideal.
Let $a \wedge b \wedge c \in I$. Then $\chi_I(a \wedge b \wedge c) = 1$.
Suppose that $a \wedge b \notin I$, $b \wedge c \notin \sqrt{I}$ and $c \wedge a \notin \sqrt{I}$.
Since χ_I is a fuzzy 2-absorbing primary ideal, we have

$$1 = \chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{I}}(b \wedge c) \vee \chi_{\sqrt{I}}(c \wedge a).$$

Since each of $\chi_I(a \wedge b)$, $\chi_{\sqrt{I}}(b \wedge c)$, $\chi_{\sqrt{I}}(c \wedge a)$ belongs to $[0, 1]$, at least one of these numbers must be 1.
This implies that either

$$a \wedge b \in I \text{ or } b \wedge c \in \sqrt{I} \text{ or } c \wedge a \in \sqrt{I}.$$

Thus I is a 2-absorbing primary ideal. □

Lemma 6.10. *If μ is a fuzzy primary ideal of L, then μ is a fuzzy 2-absorbing primary ideal of L.*

Proof. Let μ be a fuzzy primary fuzzy ideal of L. Let $a, b, c \in L$.
As μ is a fuzzy primary ideal, we have

$$\begin{aligned} \mu(a \wedge b \wedge c) &= \mu(a \wedge b \wedge b \wedge c) \\ &\leq \mu(a \wedge b) \vee \sqrt{\mu}(b \wedge c) \\ &\leq \mu(a \wedge b) \vee \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(c \wedge a). \end{aligned}$$

Thus μ is a fuzzy 2-absorbing primary ideal. □

The following example shows that a fuzzy 2-absorbing primary ideal of L need not be a fuzzy primary ideal.

Example 6.11. Consider the ideal $I = (0]$ of the lattice shown in the Figure 4.
We note that the ideals $(h] = \{0, a, b, c, e, f, g, h\}$ and $(i] = \{0, b, c, d, g, i\}$ are the only prime ideals of L.
Hence $\sqrt{I} = (h] \cap (i] = (g]$.
We note that I is a 2-absorbing primary ideal as for any $x, y, z \in L$, $x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{I}$ or $z \wedge x \in \sqrt{I}$.
Hence by Lemma 6.9, χ_I is a fuzzy 2-absorbing primary ideal of L.
We note that $\chi_I(h \wedge i) = 1$ but $\chi_I(h) = 0$ as well as $\chi_{\sqrt{I}}(i) = 0$.
Thus $\chi_I(h \wedge i) \not\leq \chi_I(h) \vee \chi_{\sqrt{I}}(i)$.
Hence χ_I is not a fuzzy primary ideal of L.

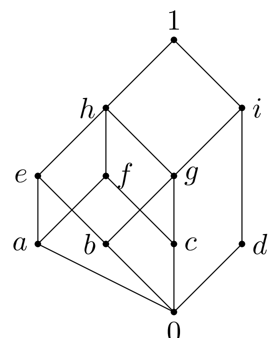


Figure 4

Lemma 6.12. *If μ is a fuzzy 2-absorbing ideal of L, then μ is a fuzzy 2-absorbing primary ideal of L.*

Proof. Let μ be a fuzzy 2-absorbing ideal of L.
Let $a, b, c \in L$. Since μ is a fuzzy 2-absorbing ideal, we get

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Since $\mu \subseteq \sqrt{\mu}$, we get the result. □

The following example shows that a fuzzy 2-absorbing primary ideal of L need not be a fuzzy 2-absorbing ideal.

Example 6.13. Consider the lattice shown in Figure 5. Consider the ideal $I = (0]$. The only prime ideals of L are $(j], (k], (l]$.
We have $\sqrt{I} = (j] \cap (k] \cap (l] = (d]$.
Also $\sqrt{\chi_I} = \chi_{\sqrt{I}} = \chi_J$, where $J = (d]$.
We note that I is a 2-absorbing primary ideal of L. Hence by Lemma 6.9, χ_I is a fuzzy 2-absorbing primary ideal of L.

We note that I is not a 2-absorbing ideal of L, as $d \wedge e \wedge f = 0 \in I$, but $d \wedge e \notin I$, $e \wedge f \notin I$ and $d \wedge f \notin I$.
We have

$$\chi_I(d \wedge e \wedge f) = 1 \nleq \chi_I(d \wedge e) \vee \chi_I(e \wedge f) \vee \chi_I(d \wedge f) = 0.$$

Thus χ_I is not a fuzzy 2-absorbing ideal of L.

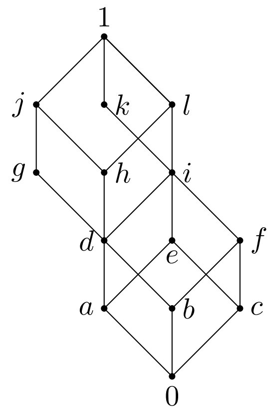


Figure 5

Lemma 6.14. *Let μ be a fuzzy ideal of L. If $\sqrt{\mu}$ is a fuzzy prime ideal, then μ is a fuzzy 2-absorbing primary ideal.*

Proof. Let μ be a fuzzy ideal of L. Suppose that $\sqrt{\mu}$ is a fuzzy prime ideal. If μ is not a fuzzy 2-absorbing primary ideal, then there exist $a, b, c \in L$ such that

$$\mu(a \wedge b \wedge c) \nleq \mu(a \wedge b) \vee \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(a \wedge c). \tag{6.1}$$

This implies that

$$\mu(a \wedge b) \vee \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(a \wedge c) < \mu(a \wedge b \wedge c).$$

Since $\sqrt{\mu}$ is fuzzy prime, we have

$$\sqrt{\mu}(a \wedge b \wedge c) = \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(a) = \sqrt{\mu}(a \wedge c) \vee \sqrt{\mu}(b).$$

Hence

$$\sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(a \wedge c) = \sqrt{\mu}(b \wedge c) \vee \sqrt{\mu}(a) \vee \sqrt{\mu}(c) = \sqrt{\mu}(a \wedge b \wedge c) \vee \sqrt{\mu}(c).$$

Thus from (6.1),

$$\mu(a \wedge b) \vee \sqrt{\mu}(a \wedge b \wedge c) \vee \sqrt{\mu}(c) < \mu(a \wedge b \wedge c).$$

This implies that

$$\sqrt{\mu}(a \wedge b \wedge c) < \mu(a \wedge b \wedge c),$$

which is not possible. Hence μ is fuzzy 2-absorbing primary. □

The following example shows that the converse of Lemma 6.14 does not hold.

Example 6.15. Consider the lattice shown in Figure 6.

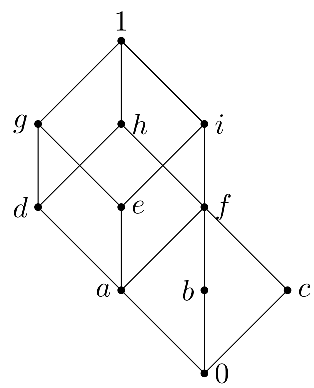


Figure 6

The only prime ideals of L containing the ideal $I = (c]$ are $(h]$ and $(i]$. Hence $\sqrt{I} = (h] \cap (i] = (f]$.
For any $x, y, z \in I$, $x \wedge y \wedge z \in I$ implies that either

$$x \wedge y \in I \text{ or } y \wedge z \in \sqrt{I} \text{ or } x \wedge z \in \sqrt{I}.$$

Hence I is a 2-absorbing primary ideal and so by Lemma 6.9, χ_I is a fuzzy 2-absorbing primary ideal
We note that $d \wedge e = a \in \sqrt{I}$ but $d \notin \sqrt{I}$ and $e \notin \sqrt{I}$. Thus $\sqrt{I}$ is not a prime ideal of L. Hence by Theorem 3.4, $\sqrt{\chi_I} = \chi_{\sqrt{I}}$ is not a fuzzy prime ideal of L.

We omit the easy proof of the following lemma.

Lemma 6.16. *Let μ be a fuzzy ideal of L. Then $\sqrt{\mu} = \sqrt{\sqrt{\mu}}$.*

Theorem 6.17. *Let μ be a fuzzy ideal of L. Then $\sqrt{\mu}$ is fuzzy prime if and only if $\sqrt{\mu}$ is fuzzy primary.*

Proof. It follows from Lemma 6.5, that if $\sqrt{\mu}$ is fuzzy prime, then $\sqrt{\mu}$ is fuzzy primary.

The converse follows form the definition of a fuzzy primary ideal and by Lemma 6.16. □

The proof of the following theorem follows from the definition of a fuzzy 2-absorbing ideal, a fuzzy 2-absorbing primary ideal and Lemma 6.16.

Theorem 6.18. *Let μ be a fuzzy ideal of L. Then $\sqrt{\mu}$ is fuzzy 2-absorbing if and only if $\sqrt{\mu}$ is fuzzy 2-absorbing primary.*

7. Primary Fuzzy Ideals

In the previous section we have defined the prime fuzzy radical of a fuzzy ideal (Definition 6.2). Using this, we define, a primary fuzzy ideal and prove some results.

We note that for a fuzzy ideal μ of L always $\mu \subseteq \sqrt{\mu}$.

Definition 7.1. A proper fuzzy ideal μ of a lattice L is called a primary fuzzy ideal of L if for $\sigma, \theta \in FI(L)$, $\sigma \cap \theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \sqrt{\mu}$.

Now we give a relationship between a prime fuzzy ideal and a primary fuzzy ideal.

Lemma 7.2. *If μ is a prime fuzzy ideal of L, then μ is a primary fuzzy ideal of L.*

Proof. Let μ be a prime fuzzy ideal of L. Let $\theta \cap \eta \subseteq \mu$ for some $\theta, \eta \in FI(L)$. Since μ is a prime fuzzy ideal, either $\theta \subseteq \mu$ or $\eta \subseteq \mu$. Since $\mu \subseteq \sqrt{\mu}$, we get the result. □

The following result gives the existence of primary fuzzy ideals which are not prime fuzzy.

Theorem 7.3. *Let I be a primary ideal of L, $I \neq L$. The fuzzy subset μ of L defined by*

$$\mu(x) = \begin{cases} 1, & \text{if } x \in I, \\ \alpha & \text{if } x \in L - I. \end{cases}$$

is a fuzzy primary ideal of L.

Proof. Clearly, μ is a fuzzy ideal of L.
Since $\mu \subseteq \sqrt{\mu}$, we have $\mu(x) \leq \sqrt{\mu}(x)$ for all $x \in L$.
Hence if $x \in I$, then $\sqrt{\mu}(x) = 1$ and if $x \notin I$, then $\sqrt{\mu}(x) = t \geq \alpha$.

Let σ, θ be fuzzy ideals of L such that $\sigma \cap \theta \subseteq \mu$.
Suppose that $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \sqrt{\mu}$.
Let $x \in L$ be such that $\sigma(x) > \mu(x)$. This implies that $x \notin I$, for otherwise, $\sigma(x) > 1$, which is not possible.
Let $\sigma(x) = k_1 > \alpha = \mu(x)$.
Let $y \in L$ be such that $\theta(y) > \sqrt{\mu}(y)$. Clearly, $y \notin \sqrt{I}$, otherwise, $\theta(y) > \sqrt{\mu}(y) \geq \mu(y) = 1$, which is not possible.
Let $\theta(y) = k_2$. Then $k_2 > \alpha$.
Since I is primary, $x \wedge y \notin I$. Hence $\mu(x \wedge y) = \alpha$.
We have

$$(\sigma \cap \theta)(x \wedge y) \geq \min\{\sigma(x), \theta(y)\} = \min\{k_1, k_2\} > \alpha = \mu(x \wedge y),$$

which is not possible. Thus μ is a primary fuzzy ideal of L. □

Theorem 7.4. *If μ is a primary fuzzy ideal of L, then the level set μ_t, $t \in Image(\mu)$ is a primary ideal of L.*

Proof. Let $a, b \in L$ be such that $a \wedge b \in \mu_t$ and $a \notin \mu_t$.
Define fuzzy ideals σ and θ of L as follows.

$$\sigma(x) = \begin{cases} t, & \text{if } x \leq a, \\ 0 & \text{if } x \nleq a \end{cases}$$

and

$$\theta(x) = \begin{cases} t, & \text{if } x \leq b, \\ 0 & \text{if } x \nleq b. \end{cases}$$

Then $\sigma \cap \theta \subseteq \mu$.
Also $\sigma \nsubseteq \mu$ as $a \notin \mu_t$ implies $\mu(a) < t = \sigma(a)$.
Since μ is a primary fuzzy ideal, we have $\theta \subseteq \sqrt{\mu}$.
Hence $t = \theta(b) \leq \sqrt{\mu}(b)$ and so $b \in \sqrt{\mu_t}$.
Thus μ_t is a primary ideal of L. □

The following example shows that the converse of Theorem 7.4 does not hold.

Example 7.5. We note that set $\mathbb{N}$ of natural numbers with divisibility as the partial order is a lattice. Let p be any prime number. Let $t_i \in (0, 1)$, $0 \leq i \leq m$ be such that $t_0 > t_1 > \ldots > t_m$.
Consider the fuzzy ideal μ of $\mathbb{N}$ defined by

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in (p^m], \\ t_i & \text{if } x \in (p^{m-i}] - (p^{m-i+1}], i = 1, 2, \ldots, m. \end{cases}$$

We have $\sqrt{\mu}(x) = \begin{cases} t_0, & \text{if } x \in (p], \\ t_m & \text{if } x \in \mathbb{N} - (p]. \end{cases}$

Define fuzzy ideals σ and θ of $\mathbb{N}$ by

$$\sigma(x) = \begin{cases} \alpha, & \text{if } x \in (p^m], \text{ where } t_0 < \alpha \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $\theta(x) = t_0$, for all $x \in \mathbb{N}$.
Then

$$(\sigma \cap \theta)(x) = \begin{cases} t_0, & \text{if } x \in (p^m], \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma \cap \theta \subseteq \mu \subseteq \sqrt{\mu}$ and $\sigma \nsubseteq \mu$.
We note that if $x \in \mathbb{N} - (p]$, then $\sqrt{\mu}(x) = t_m < t_0 = \theta(x)$.
Hence $\theta \nsubseteq \sqrt{\mu}$.
Thus μ is not primary fuzzy. However, each level ideal μ_i of μ is primary, $i = 0, \ldots, m$.

Definition 7.6. A proper fuzzy ideal μ of a lattice L is called a 2-absorbing primary fuzzy ideal of L, if whenever, $\theta \cap \eta \cap \nu \subseteq \mu$ for $\theta, \eta, \nu \in FI(L)$, then either

$$\theta \cap \eta \subseteq \mu \text{ or } \eta \cap \nu \subseteq \sqrt{\mu} \text{ or } \theta \cap \nu \subseteq \sqrt{\mu}.$$

It known that $\sqrt{\chi_I} = \chi_{\sqrt{I}}$.

Lemma 7.7. *Let I be an ideal of L. If χ_I is a 2-absorbing primary fuzzy ideal of L, then I is a 2-absorbing ideal of L.*

Proof. Suppose that χ_I is a 2-absorbing primary fuzzy ideal of L.
Let $a \wedge b \wedge c \in I$ for some $a, b, c \in L$. Suppose that

$$a \wedge b \notin I, \ b \wedge c \notin \sqrt{I} \text{ and } c \wedge a \notin \sqrt{I}.$$

Then clearly, $a \notin I$ and $b, c \notin \sqrt{I}$.
Define fuzzy ideals

$$\mu(x) = \begin{cases} 1, & \text{if } x \in (a], \\ 0 & \text{otherwise.} \end{cases}$$

$$\theta(x) = \begin{cases} 1, & \text{if } x \in (b], \\ 0 & \text{otherwise.} \end{cases}$$
$$\eta(x) = \begin{cases} 1, & \text{if } x \in (c], \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$(\mu \cap \theta \cap \eta)(x) = \begin{cases} 1, & \text{if } x \in (a \wedge b \wedge c], \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \cap \theta \cap \eta \subseteq \chi_I$ but $\mu \cap \theta \nsubseteq \chi_I$, $\theta \cap \eta \nsubseteq \chi_{\sqrt{I}}$ and $\mu \cap \eta \nsubseteq \chi_{\sqrt{I}}$. This contradicts the assumption that χ_I is a 2-absorbing primary fuzzy ideal. □

Remark 7.8. However, we are unable to prove or disprove that if I is a 2-absorbing ideal of L, then χ_I a is a 2-absorbing fuzzy ideal of L.

Lemma 7.9. *If μ is a primary fuzzy ideal of L, then μ is a 2-absorbing primary fuzzy ideal of L.*

Proof. Let μ be a primary fuzzy ideal of L. Let $\theta \cap \eta \cap \nu \subseteq \mu$ for some $\theta, \eta, \nu \in FI(L)$. Since μ is a primary fuzzy ideal of L, either

(1) $\theta \cap \eta \subseteq \mu$ or $\nu \subseteq \sqrt{\mu}$ or (2) $\theta \subseteq \mu$ or $\eta \cap \nu \subseteq \sqrt{\mu}$ or
(3) $\theta \subseteq \sqrt{\mu}$ or $\eta \cap \nu \subseteq \mu$ or (4) $\eta \subseteq \mu$ or $\theta \cap \nu \subseteq \sqrt{\mu}$.

These possibilities imply that either

(i) $\theta \cap \eta \subseteq \mu$ or (ii) $\eta \cap \nu \subseteq \sqrt{\mu}$ or (iii) $\theta \cap \nu \subseteq \sqrt{\mu}$.

Hence μ is a 2-absorbing primary fuzzy ideal of L. □

Lemma 7.10. *If μ is a 2-absorbing fuzzy ideal of L, then μ is a 2-absorbing primary fuzzy ideal of L.*

Proof. Let μ be a 2-absorbing fuzzy ideal of L. Let $\theta, \eta, \nu \in FI(L)$ be such that $\theta \cap \eta \cap \nu \subseteq \mu$.
Since μ is a 2-absorbing fuzzy ideal of L, either

$$\theta \cap \eta \subseteq \mu \text{ or } \theta \cap \nu \subseteq \mu \text{ or } \eta \cap \nu \subseteq \mu.$$

Since $\mu \subseteq \sqrt{\mu}$, we get the result. □

Definition 7.11. Let μ be a fuzzy ideal of L. If δ is the only prime fuzzy ideal containing μ, then we say that μ is a δ- primary fuzzy ideal of L.

Theorem 7.12. *Let μ_1, μ_2 be fuzzy ideals and δ_1, δ_2 be prime fuzzy ideals of L. Suppose that μ_1 is a δ_1- primary fuzzy ideal and μ_2 is a δ_2- primary fuzzy ideal. Then $\mu_1 \cap \mu_2$ is a 2-absorbing primary fuzzy ideal of L.*

Proof. Since μ_1 is a δ_1-primary fuzzy ideal, we get $\sqrt{\mu_1} = \delta_1$.
As μ_2 is a δ_2-primary fuzzy ideal, we get $\sqrt{\mu_2} = \delta_2$.
Let $\mu = \mu_1 \cap \mu_2$. Then $\sqrt{\mu} = \delta_1 \cap \delta_2$.
Now suppose that $\theta \cap \eta \cap \nu \subseteq \mu$ for some $\theta, \eta, \nu \in FI(L)$.
Assume that $\theta \cap \eta \nsubseteq \sqrt{\mu}$ and $\eta \cap \nu \nsubseteq \sqrt{\mu}$.
Then $\theta, \eta, \nu \nsubseteq \sqrt{\mu} = \delta_1 \cap \delta_2$.
By Proposition 3.1, $\sqrt{\mu} = \delta_1 \cap \delta_2$ is a 2-absorbing fuzzy ideal of L.
Since $\theta \cap \eta \nsubseteq \sqrt{\mu}, \eta \cap \nu \nsubseteq \sqrt{\mu}$ we have $\theta \cap \nu \subseteq \sqrt{\mu}$.
We show that $\theta \cap \nu \subseteq \mu$.
Since $\theta \cap \nu \subseteq \sqrt{\mu} \subseteq \delta_1$, we assume that $\theta \subseteq \delta_1$.
As $\theta \nsubseteq \sqrt{\mu}$ and $\theta \cap \nu \subseteq \sqrt{\mu} \subseteq \delta_2$, we conclude that $\theta \nsubseteq \delta_2$ and $\nu \subseteq \delta_2$.
Since $\nu \subseteq \delta_2$ and $\nu \nsubseteq \sqrt{\mu}$ we have $\nu \nsubseteq \delta_1$.
If $\theta \subseteq \mu_1$ and $\nu \subseteq \mu_2$, then $\theta \cap \nu \subseteq \mu$ and we are done.
We may assume that $\theta \nsubseteq \mu_1$.
Since μ_1 is a δ_1-primary fuzzy ideal and $\theta \nsubseteq \mu_1$, we have $\eta \cap \nu \subseteq \delta_1$.
Since $\nu \subseteq \delta_2$ and $\eta \cap \nu \subseteq \sqrt{\mu}$ which is a contradiction.
Thus, $\theta \subseteq \mu_1$.
Since μ_2 is a δ_2-primary fuzzy ideal of L and $\nu \nsubseteq \mu_2$, we get $\theta \cap \eta \subseteq \delta_2$.
Since $\theta \subseteq \delta_1$ and $\theta \cap \eta \subseteq \delta_2$, we have $\theta \cap \eta \subseteq \sqrt{\mu}$ which is a contradiction.
Thus, $\nu \subseteq \mu_2$.
Hence $\theta \cap \nu \subseteq \mu$. □

Theorem 7.13. *Suppose that μ is a non-constant fuzzy ideal of L such that $\sqrt{\mu}$ is a prime fuzzy ideal. Then μ is a 2-absorbing primary fuzzy ideal.*

Proof. Suppose that for some $\theta, \eta, \nu \in FI(L), \theta \cap \eta \cap \nu \subseteq \mu$ and $\theta \cap \eta \nsubseteq \mu$.
(i): We note that $\theta \cap \eta \cap \nu \subseteq \mu \subseteq \sqrt{\mu}$. Hence, if $\theta \cap \eta \nsubseteq \sqrt{\mu}$, then as $\sqrt{\mu}$ is prime fuzzy we get $\nu \subseteq \sqrt{\mu}$ and so $\eta \cap \nu \subseteq \sqrt{\mu}$.
(ii): If $\theta \cap \eta \subseteq \sqrt{\mu}$, then as $\sqrt{\mu}$ is prime fuzzy, either $\theta \subseteq \sqrt{\mu}$ or $\eta \subseteq \sqrt{\mu}$.
Hence either $\theta \cap \nu \subseteq \sqrt{\mu}$ or $\nu \cap \eta \subseteq \sqrt{\mu}$.
Thus, μ is a 2-absorbing primary fuzzy ideal of L. □

Now we give a characterization for $\sqrt{\mu}$ to be a prime fuzzy ideal.

Theorem 7.14. *Let μ be a non-constant fuzzy ideal of a lattice L. Then $\sqrt{\mu}$ is a prime fuzzy ideal of L if and only if $\sqrt{\mu}$ is a primary fuzzy ideal of L.*

Proof. Let $\sqrt{\mu}$ be a prime fuzzy ideal of L. Let $\theta, \eta \in FI(L)$ be such that $\theta \cap \eta \subseteq \sqrt{\mu}$. As $\sqrt{\mu}$ is a prime fuzzy ideal of L, either $\theta \subseteq \sqrt{\mu}$ or $\eta \subseteq \sqrt{\mu}$. Since $\sqrt{\mu} = \sqrt{\sqrt{\mu}}$ we conclude that $\sqrt{\mu}$ is a primary fuzzy ideal of L.
Conversely, suppose that $\sqrt{\mu}$ is a primary fuzzy ideal of L.
Let $\theta, \eta \in FI(L)$ be such that $\theta \cap \eta \subseteq \sqrt{\mu}$. As $\sqrt{\mu}$ is primary fuzzy ideal, either $\theta \subseteq \sqrt{\mu}$ or $\eta \subseteq \sqrt{\sqrt{\mu}} = \sqrt{\mu}$. Hence $\sqrt{\mu}$ is a prime fuzzy ideal of L. □

Now we prove the following characterization.

Theorem 7.15. *Let μ be a non-constant fuzzy ideal of a lattice L. Then $\sqrt{\mu}$ is a 2-absorbing fuzzy ideal of L if and only if $\sqrt{\mu}$ is a 2-absorbing primary fuzzy ideal of L.*

Proof. Let $\sqrt{\mu}$ be a 2-absorbing fuzzy ideal of L. Let $\theta, \eta, \nu \in FI(L)$ be such that $\theta \cap \eta \cap \nu \subseteq \sqrt{\mu}$. Since $\sqrt{\mu}$ is a 2-absorbing fuzzy ideal of L, either

$$\theta \cap \eta \subseteq \sqrt{\mu} \text{ or } \eta \cap \nu \subseteq \sqrt{\mu} \text{ or } \theta \cap \nu \subseteq \sqrt{\mu}.$$

Using $\sqrt{\mu} = \sqrt{\sqrt{\mu}}$, we conclude that $\sqrt{\mu}$ is a 2-absorbing primary fuzzy ideal of L.

Conversely, suppose that $\sqrt{\mu}$ is a 2-absorbing primary fuzzy ideal of L.
Let $\theta, \eta, \nu \in FI(L)$ be such that $\theta \cap \eta \cap \nu \subseteq \sqrt{\mu}$.
As $\sqrt{\mu}$ is a 2-absorbing primary fuzzy ideal of L, either

$$\theta \cap \eta \subseteq \sqrt{\mu} \text{ or } \eta \cap \nu \subseteq \sqrt{\sqrt{\mu}} = \sqrt{\mu} \text{ or } \theta \cap \nu \subseteq \sqrt{\sqrt{\mu}} = \sqrt{\mu}.$$

Hence $\sqrt{\mu}$ is a 2-absorbing fuzzy ideal of L. □

8. Fuzzy ideals in a direct product of lattices

In this section, we consider fuzzy ideals in a direct product of lattices. It is known that if $L_1, \ldots, L_k$ are lattices, then their Cartesian product $L = L_1 \times L_2 \times \ldots \times L_k$ is a lattice under componentwise operations of meet and join and if $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$ then $a \leq b$ iff $a_i \leq b_i$ for $i = 1, \ldots, k$.

Definition 8.1. Let $L = L_1 \times L_2 \times \ldots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. A mapping $\mu : L \to [0, 1]$ is called a fuzzy set of L.

We note the following.

Theorem 8.2. *Let $L = L_1 \times L_2 \times \ldots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. If $\mu_i, 1 \leq i \leq k$ are fuzzy ideals of L_i respectively, then $\mu : L \to [0,1]$ defined by*
$\mu(a_1, \ldots, a_k) = \mu_1(a_1) \wedge \ldots \wedge \mu_k(a_k)$ is a fuzzy ideal of L.

Proof. The proof follows from the definition of the lattice operations in a direct product of lattices and that of μ. □

Notation: We call the fuzzy set μ in Theorem 8.2 as a product of the fuzzy sets μ_i, $1 \leq i \leq k$ and write $\mu = \mu_1 \times \ldots \times \mu_k$.

Theorem 8.3. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If $\mu : L \to [0,1]$ is a fuzzy ideal of L, then there exist fuzzy ideals μ_1, μ_2 of L_1 and L_2 respectively, such that $\mu = \mu_1 \times \mu_2$. Moreover, if μ is fuzzy prime, then so are μ_1 and μ_2.*

Proof. Define $\mu_i : L_i \to [0,1]$ by $\mu_1(x) = \mu(x,0)$ and $\mu_2(y) = \mu(0,y)$.
Let $x, y \in L_1$. We have

$$\mu[(x,0) \wedge (y,0)] = \mu(x \wedge y, 0) = \mu_1(x \wedge y)$$

and

$$\mu[(x,0) \vee (y,0)] = \mu(x \vee y, 0) = \mu_1(x \vee y).$$

Hence

$$\mu_1(x \wedge y) \wedge \mu_1(x \vee y) = \mu[(x,0) \wedge (y,0)] \wedge \mu[(x,0) \vee (y,0)].$$

As μ is a fuzzy ideal, we get

$$\begin{aligned} \mu_1(x \wedge y) \wedge \mu_1(x \vee y) &= \mu[(x,0) \wedge (y,0)] \wedge \mu[(x,0) \vee (y,0)] \\ &\geq \mu(x,0) \wedge \mu(y,0) \\ &= \mu_1(x) \wedge \mu_1(y). \end{aligned}$$

Also

$$\mu_1(x \vee y) = \mu[(x,0) \vee (y,0)] = \mu(x,0) \wedge \mu(y,0) = \mu_1(x) \wedge \mu_1(y).$$

Thus μ_1 is a fuzzy ideal of L_1.
Similarly, we can show that μ_2 is a fuzzy ideal of L_2.
The second part follows from the definition of a fuzzy prime ideal.
We have

$$\mu(x,y) = \mu(x,0) \vee \mu(0,y) = \mu(x,0) \wedge \mu(0,y) = \mu_1(x) \wedge \mu_2(y).$$

□

Example 8.4. Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy prime ideals of L_1 and L_2 respectively. Then $\mu = \mu_1 \times \mu_2$ need not be a fuzzy prime ideal of L.

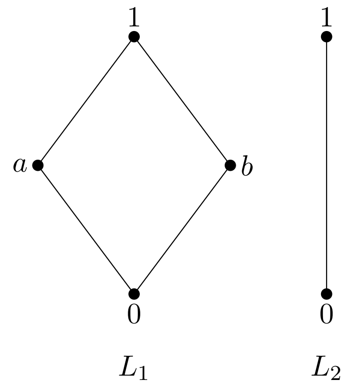


Figure 6

Consider the lattices L_1 and L_2 as shown in Figure 6.

Define $\mu : L_1 \to [0,1]$ and $\theta : L_2 \to [0,1]$ as follows
$\mu(0) = 1$, $\mu(a) = 1/2$, $\mu(b) = 1$, $\mu(1) = 0$ and $\theta(0) = 1$, $\theta(1) = 0$.
We note that μ is a fuzzy prime ideal of L_1 and θ that of L_2.
We consider $\eta : L_1 \times L_2 \to [0,1]$ defined by $\eta(x, y) = \mu(x) \wedge \theta(y)$, i.e. $\eta = \mu \times \theta$.
We have

$\eta(0,0) = \mu(0) \wedge \theta(0) = 1$
$\eta(a,0) = \mu(a) \wedge \theta(0) = 1/2$
$\eta(b,0) = \mu(b) \wedge \theta(0) = 1$
$\eta(1,0) = \mu(1) \wedge \theta(0) = 0$
$\eta(0,1) = \mu(0) \wedge \theta(1) = 0$
$\eta(a,1) = \mu(a) \wedge \theta(1) = 0$
$\eta(b,1) = \mu(b) \wedge \theta(1) = 0$
$\eta(1,1) = \mu(1) \wedge \theta(1) = 0$

We have $\eta[(0,1) \wedge (1,0)] = \eta(0,0) = 1$, $\eta(0,1) = 0$, $\eta[(1,0) = 0$.
Thus $\eta[(0,1) \wedge (1,0)] \nleq \eta(0,1) \vee \eta(1,0)$.
Hence η is not a fuzzy prime ideal of L.

Remark 8.5. From Example 8.4, we conclude that in general,

$$\sqrt{\mu \times \theta} \neq \sqrt{\mu} \times \sqrt{\theta}$$

.

In Example 8.4, we have shown that a product of two fuzzy prime ideals need not be a fuzzy prime ideal. However we have the following theorem.

Theorem 8.6. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ be a fuzzy ideal of L_1. Then $\mu \times \chi_{L_2}$ is a fuzzy prime ideal of L, iff μ is a fuzzy prime ideal of L_1.*

Proof. Suppose that μ is a fuzzy prime ideal of L_1.
We have

$$\begin{aligned}[\mu \times \chi_{L_2}][(x_1, y_1) \wedge (x_2, y_2)] &= [\mu \times \chi_{L_2}](x_1 \wedge x_2, y_1 \wedge y_2)\\ &= \mu(x_1 \wedge x_2) \wedge \chi_{L_2}(y_1 \wedge y_2)\\ &= \mu(x_1 \wedge x_2), \text{ as } \chi_{L_2}(y_1 \wedge y_2) = 1.\end{aligned}$$

Since μ is fuzzy prime,

$$\mu(x_1 \wedge x_2) = \mu(x_1) \vee \mu(x_2).$$

Thus

$$\begin{aligned}[\mu \times \chi_{L_2}](x_1 \wedge x_2, y_1 \wedge y_2) &= [\mu(x_1) \wedge \chi_{L_2}](y_1)] \vee [\mu(x_2) \wedge \chi_{L_2}](y_2)]\\ &= [\mu \times \chi_{L_2}](x_1, y_1) \vee [\mu \times \chi_{L_2}](x_1, y_2).\end{aligned}$$

Hence $\mu \times \chi_{L_2}$ is a fuzzy prime ideal of L.
The converse can be similarly proved. □

Theorem 8.7. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy ideals of L_1 and L_2 respectively. Suppose that $\mu_1(0_1) = \mu_2(0_2) = 1$, where 0_1 is the least element of L_1 and 0_2 that of L_2. If $\mu = \mu_1 \times \mu_2$ is a fuzzy 2-absorbing ideal of L, then μ_1 is a fuzzy 2-absorbing ideal of L_1 and μ_2 that of L_2.*

Proof. Let $a, b, c \in L_1$. Since μ is a fuzzy 2-absorbing ideal of L, we have

$$\mu(a \wedge b \wedge c, 0_2) \leq \mu(a \wedge b, 0_2) \vee \mu(b \wedge c, 0_2) \vee \mu(a \wedge c, 0_2). \tag{8.1}$$

By the definition of μ, we can write (8.1) as

$$\begin{aligned}&\mu_1(a \wedge b \wedge c) \wedge \mu_2(0_2)\\ &\leq [\mu_1(a \wedge b) \wedge \mu_2(0_2)] \vee [\mu_1(b \wedge c) \wedge \mu_2(0_2)] \vee [\mu_1(a \wedge c) \wedge \mu_2(0_2).\end{aligned}$$

By using $\mu_2(0_2) = 1$, we get

$$\mu_1(a \wedge b \wedge c) \leq \mu_1(a \wedge b) \vee \mu_1(b \wedge c) \vee \mu_1(a \wedge c).$$

Thus μ_1 is a fuzzy 2-absorbing ideal of L_1.
Similarly, we can prove that μ_2 is a fuzzy 2-absorbing ideal of L_2. □

By using similar steps, we can prove the following theorem.

Theorem 8.8. *Let $L = L_1 \times L_2 \times \ldots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. Let $\mu_i, 1 \leq i \leq k$ be fuzzy ideals of L_i respectively. Suppose that for each $i = 1, \ldots, k$, $\mu_i(0_i) = 1$, where 0_i is the least element of L_i. If $\mu = \mu_1 \times \ldots \times \mu_k$ is a fuzzy 2-absorbing ideal of L, then μ_i, is a fuzzy 2-absorbing ideal of L_i, $i = 1, \ldots, k$.*

The following example shows that the converse of Theorem 8.7 need not hold.

Example 8.9. Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Figure 4.

Define $\mu : L_1 \to [0, 1]$ and $\theta : L_2 \to [0, 1]$ as follows

$\mu(0) = 1$	$\theta(0) = 1$
$\mu(a) = 1/6$	$\theta(1) = 0$
$\mu(b) = 1/4$	
$\mu(1) = 1/4$	

We note that μ is a fuzzy 2-absorbing ideal of L_1 and θ that of L_2.
We consider $\eta : L_1 \times L_2 \to [0, 1]$ defined by $\eta(x, y) = \mu(x) \wedge \theta(y)$.
We have

$\eta(0, 0) = \mu(0) \wedge \theta(0) = 1$
$\eta(a, 0) = \mu(a) \wedge \theta(0) = 1/6$
$\eta(b, 0) = \mu(b) \wedge \theta(0) = 1/4$
$\eta(1, 0) = \mu(1) \wedge \theta(0) = 1/4$
$\eta(0, 1) = \mu(0) \wedge \theta(1) = 0$
$\eta(a, 1) = \mu(a) \wedge \theta(1) = 0$
$\eta(b, 1) = \mu(b) \wedge \theta(1) = 0$
$\eta(1, 1) = \mu(1) \wedge \theta(1) = 0$

We have

$$\eta[(a, 1) \wedge (1, 0) \wedge (b, 1)] = \eta(0, 0) = 1.$$
$$\eta[(a, 1) \wedge (1, 0)] = \eta(a, 0) = 1/6.$$
$$\eta[(1, 0) \wedge (b, 1)] = \eta(b, 0) = 1/4.$$
$$\eta[(a, 1) \wedge (b, 1)] = \eta(a \wedge b, 1) = \eta(0, 1) = 0.$$

Thus

$\eta[(a,1)\wedge(1,0)\wedge(b,1)] \nleq \eta[(a,1)\wedge(1,0)]\vee\eta[(1,0)\wedge(b,1)]\vee\eta[(a,1)\wedge(b,1)]$.

Hence η is not a fuzzy 2-absorbing ideal of L.

Theorem 8.10. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy ideals of L_1 and L_2 respectively. Suppose that (i) $\mu_1(0_1) = \mu_2(0_2) = 1$, where 0_1 is the least element of L_1 and 0_2 that of L_2 and (ii) $\mu_1(1_1) = \mu_2(1_2) = 0$, where 1_1 is the greatest element of L_1 and 1_2 that of L_2. If $\mu = \mu_1 \times \mu_2$ is a fuzzy 2-absorbing ideal of L, then μ_1 is a fuzzy prime ideal of L_1 and μ_2 that of L_2.*

Proof. Suppose that μ_1 is not a fuzzy prime ideal of L_1.
Then there exist $a, b \in L_1$ such that

$$\mu(a \wedge b) \nleq \mu(a) \vee \mu(b).$$

Consider the elements $x = (a, 1)$, $y = (1, 0)$, $z = (b, 1)$ from L.
We note the following.

$$\begin{aligned}
\mu(x \wedge y \wedge z) &= \mu(a \wedge b, 0) = \mu_1(a \wedge b) \wedge \mu_2(0) = \mu_1(a \wedge b).\\
\mu(x \wedge y) &= \mu(a, 0) = \mu_1(a) \wedge \mu_2(0) = \mu_1(a).\\
\mu(y \wedge z) &= \mu(b, 0) = \mu_1(b) \wedge \mu_2(0) = \mu_1(b).\\
\mu(z \wedge x) &= \mu(a \wedge b, 1) = \mu_1(a \wedge b) \wedge \mu_2(1) = 0.
\end{aligned}$$

Since μ is a fuzzy 2-absorbing ideal, we have

$$\mu(x \wedge y \wedge z) \leq \mu(x \wedge y) \vee \mu(y \wedge z) \vee \mu(z \wedge x).$$

i.e.

$$\mu_1(a \wedge b) \leq \mu_1(a) \vee \mu_1(b) \vee 0 = \mu_1(a) \vee \mu_1(b),$$

a contradiction.
Hence μ_1 is a fuzzy prime ideal.
Similarly, we can show that μ_2 is a fuzzy prime ideal. □

Theorem 8.11. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy prime ideals of L_1 and L_2 respectively. If $\mu = \mu_1 \times \mu_2$, then μ is a fuzzy 2-absorbing ideal of L.*

Proof. Let $(a, x), (b, y), (c, z) \in L$. To show that μ is fuzzy 2-absorbing, we need to show that

$\mu[(a,x)\wedge(b,y)\wedge(c,z)] \leq \mu[(a,x)\wedge(b,y)]\vee\mu[(b,y)\wedge(c,z)]\vee\mu[(a,x)\wedge(c,z)]$.

i.e. to show that

(8.2) $\mu(a\wedge b\wedge c, x\wedge y\wedge z) \leq \mu(a\wedge b, x\wedge y)\vee\mu(b\wedge c, y\wedge z)\vee\mu(a\wedge c, x\wedge z)$.

We have

$$\mu(a \wedge b \wedge c, x \wedge y \wedge z) = \mu_1(a \wedge b \wedge c) \wedge \mu_2(x \wedge y \wedge z).$$

As μ_1, μ_2 are fuzzy prime ideals, we can write

$$\mu_1(a \wedge b \wedge c) = \mu_1(a) \vee \mu_1(b) \vee \mu_1(c)$$

and

$$\mu_2(x \wedge y \wedge z) = \mu_2(x) \vee \mu_2(y) \vee \mu_2(z).$$

Also we have

$$\begin{aligned} &\mu(a \wedge b, x \wedge y) \vee \mu(b \wedge c, y \wedge z) \vee \mu(a \wedge c, x \wedge z) \\ &= [\mu_1(a \wedge b) \wedge \mu_2(x \wedge y)] \vee [\mu_1(b \wedge c) \wedge \mu_2(y \wedge z)] \\ &\quad \vee [\mu_1(a \wedge c) \wedge \mu_2(x \wedge z). \end{aligned} \tag{8.3}$$

Since μ_1, μ_2 are fuzzy prime ideals, we can write the R. H. S. of (8.2) as

$$\begin{aligned} &\{[\mu_1(a) \vee \mu_1(b)] \wedge [\mu_2(x) \vee \mu_2(y)]\} \\ &\vee \{[\mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(y) \vee \mu_2(z)]\} \\ &\vee \{[\mu_1(a) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(z)]\}. \end{aligned} \tag{8.4}$$

By applying distributivity, (8.4) can be written as

$$[\mu_1(a) \vee \mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(y) \vee \mu_2(z)]. \tag{8.5}$$

Thus (8.2) holds and μ is fuzzy 2-absorbing. □

Theorem 8.12. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_i, θ_j be fuzzy ideals of L_1 and L_2 respectively. Let $\sigma_{i,j} = \mu_i \times \theta_j$. Then $\cap\sigma_{i,j} = \cap\mu_i \times \cap\theta_j$.*

Proof. Let $(x, y) \in L$. We have

$$\begin{aligned} \cap\sigma_{i,j}(x, y) &= \wedge_{i,j}(\mu_i \times \theta_j)(x, y) \\ &= \wedge_{i,j}(\mu_i(x) \wedge \theta_j(y)) \\ &= \wedge_i\mu_i(x) \wedge \wedge_j\theta_j(y) \\ &= (\wedge_i\mu_i \times \wedge_j\theta_j)(x, y). \end{aligned}$$

Thus $\cap\sigma_{i,j} = \cap\mu_i \times \cap\theta_j$. □

Theorem 8.13. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2.*
(i) Let μ be a fuzzy ideal of L_1. Then $\sqrt{\mu \times \chi_{L_2}} = \sqrt{\mu} \times \chi_{L_2}$.
(ii) Let θ be a fuzzy ideal of L_2. Then $\sqrt{\chi_{L_1} \times \theta} = \chi_{L_1} \times \sqrt{\theta}$.

Proof. (i): Let η be a fuzzy prime ideal of L such that $\mu \times \chi_{L_2} \subseteq P$.
By Theorem 8.3, $\eta = \theta \times \sigma$ for some fuzzy prime ideal θ of L_1 and σ of L_2.
Then $\mu \subseteq \theta$ and $\chi_{L_2} \subseteq \sigma$. It follows that $\sigma = \chi_{L_2}$. Thus $\eta \subseteq \theta \times \chi_{L_2}$.
This shows that $\sqrt{\mu \times \chi_{L_2}} = \sqrt{\mu} \times \chi_{L_2}$.
(ii) Can be similarly proved. □

Theorem 8.14. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ be a fuzzy ideal of L_1. Then $\mu \times \chi_{L_2}$ is a 2-absorbing fuzzy primary ideal of L, if and only if μ is a 2-absorbing fuzzy primary ideal of L_1.*

Proof. Suppose that $\mu \times \chi_{L_2}$ is a 2-absorbing fuzzy primary ideal of L. Let $\theta_1, \theta_2, \theta_3 \in FI(L_1)$ be such that

$$\theta_1 \cap \theta_2 \cap \theta_3 \subseteq \mu.$$

Consider $\theta_i \times \chi_{L_2}$. Then

$$(\theta_1 \cap \theta_2 \cap \theta_3) \times \chi_{L_2} \subseteq \mu \times \chi_{L_2}.$$

This implies that

$$(\theta_1 \times \chi_{L_2}) \cap (\theta_2 \times \chi_{L_2}) \cap (\theta_3 \times \chi_{L_2}) \subseteq \mu \times \chi_{L_2}.$$

Since $\mu \times \chi_{L_2}$ is a 2-absorbing fuzzy primary ideal of L, we get either

$$(\theta_1 \times \chi_{L_2}) \cap (\theta_2 \times \chi_{L_2}) \subseteq \mu \times \chi_{L_2}$$

or

$$(\theta_2 \times \chi_{L_2}) \cap (\theta_3 \times \chi_{L_2}) \subseteq \sqrt{\mu \times \chi_{L_2}}$$

or

$$(\theta_1 \times \chi_{L_2}) \cap (\theta_3 \times \chi_{L_2}) \subseteq \sqrt{\mu \times \chi_{L_2}}.$$

Thus either

$$\theta_1 \cap \theta_2 \subseteq \mu$$

or

$$\theta_2 \cap \theta_3 \subseteq \sqrt{\mu}$$

or

$$\theta_1 \cap \theta_3 \subseteq \sqrt{\mu}.$$

Hence μ is a 2-absorbing fuzzy primary ideal of L_1. The converse follows by retracing similar steps. □

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Shriram Khanderao Nimbhorkar,
Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India
Email: sknimbhorkar@gmail.com

Yogita Subhash Patil,
Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India
Email: saharshyog.143@rediffmail.com