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PROBABILISTIC MODULAR METRIC SPACES

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Abstract. The purpose of this study is to investigate the connection between probabilistic and modular metric spaces. We discuss several important properties such as convergence and completeness, etc, and the relationship among the mentioned properties in the probabilistic metric and modular metric spaces. Also corresponding examples of probabilistic metric space obtained by a metric space is extended to modular metric spaces.

Key Words: Probabilistic metric space, Modular metric space, Luxemburg metric.
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1. Introduction

The problem of devising a suitable theory of probabilistic (or fuzzy) metric space has been investigated by several authors from different points of view (see e. g. [4, 16]). Probabilistic metric spaces were introduced in 1942 by Menger [19]. Then, Kramosil and Michalek [24] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space. The concept of fuzzy metric in its definition includes a parameter, t, that allows to introduce novel (fuzzy metric) concepts with respect to the classical metric concepts. For instance, the concepts of principal fuzzy metric spaces were motivated by the study of the p-convergence [20] and the generalization of non-Archimedean fuzzy metrics [23]. The aim of this study is to present probabilistic modular metric spaces and to

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extend corresponding examples of probabilistic metric space obtained by a metric space (for instance, see [14] and references there in) to modular metric spaces and to introduce the concept of probabilistic (convex) modular metric spaces and construct Luxemburg metric on it. The structure of the paper is as follows. In Section 2 and 3 we explain some well-known and some new definitions and results in the theory of probabilistic metric spaces and modular metric spaces, respectively, which will be used in the following sections. In section 4 we begin by a crucial example of probabilistic metric space obtained from modular metric spaces. This highlights that modular metric spaces are a class of probabilistic metric space. In section 5, we introduce the concept of modular set and the Luxemburg metric on a probabilistic modular space. Finally, in section 6 we construct a new definition for convex probabilistic modular metric spaces.

2. Probabilistic Metric Spaces

A t-norm [24] is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian monoid with unit 1. If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

Let Δ^+ stands for the set of all non-decreasing functions (called distribution function) $F : \mathbb{R} \to \mathbb{R}^+$ satisfying $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$ and $F(0) = 0$. Since any function from Δ^+ is equal zero on $[-\infty, 0]$, we can consider the set Δ^+ consisting of non-decreasing functions F defined on $[0, +\infty]$ that satisfy $F(0) = 0$ and $F(+\infty) = 1$. Denote ϵ_0 the specific distribution function defined by $\epsilon_0(t) = 1$, for all $t > 0$, otherwise 0.

Definition 2.1 (see [16, 19, 25])**.** A Menger (or probabilistic) metric space (briefly in the sequel we use PM-space) is a triple $(X, F, *)$, where X is a non-empty set and F is a mapping from $X \times X$ into Δ^+. We denote the function $F(x, y)(\cdot)$ by $F_{x,y}(\cdot)$, for all $x, y \in X$. The function F is assumed to satisfy the following conditions:

(PM1) $F_{x,y}(t) = 0$ if $t = 0$,
(PM2) $F_{x,y}(t) = \epsilon_0$ if and only if $x = y$,
(PM3) $F_{x,y}(t) = F_{y,x}(t)$,
(PM4) $F_{x,y}(t + s) \geq F_{x,z}(t) * F_{y,z}(s)$,

for all $x, y, z \in X$ and $t, s > 0$. The map F is called PM-metric on X.

Axiom (PM2) is equivalent to the following condition:

(PM2) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

Sometimes (PM2) is replaced by the following condition (for comparison, see [10, 15]):

(PM2) $F_{x,y}(t) = 1$ if and only if $x = y$,

for all $x, y \in X, t > 0$.

In the above definition, if the triangle inequality (PM4) is replaced by the following:

(NA) $F_{x,y}(t) \geq F_{x,z}(t) * F_{y,z}(t)$ for all $x, y, z \in X$ and $t > 0$,

the triple $(X, F, *)$ is then called a non-Archimedean (or strong, see [13] about the terminology) PM-space.

These spaces have been widely studied, for further information and historical background of probabilistic and fuzzy metric spaces see also [4, 8, 10–14]. Also, Kramosil and Michalek [18] additionally is assumed that F satisfies in

(PM5) $F_{x,y}(\cdot) : [0, +\infty) \to [0, 1]$ is left continuous.

Acutely they used the notation $M(x, y, t)$ instead of $F_{x,y}(t)$ and term 'fuzzy metric spaces'.

Remark 2.2 ([11])**.** In a PM-space $(X, F, *)$, $F_{x,y}(\cdot)$ is non-decreasing for all $x, y \in X$.

It follows that at each point $t > 0$ the right limit $F^+_{x,y} = \lim_{s\to t^+} F_{x,y}(s)$ and the left limit $F^-_{x,y} = \lim_{s\to t^-} F_{x,y}(s)$ exist in $[0, 1]$ and the following two inequalities hold:

$$F^-_{x,y}(t) \leq F_{x,y}(t) \leq F^+_{x,y}(t).$$

Lemma 2.3 (see Lemma 6 of [11])**.** If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

$$F_{x,y}(t - \epsilon) \leq \liminf_{n\to\infty} F_{x_n,y_n}(t) \leq \limsup_{n\to\infty} F_{x_n,y_n}(t) \leq F_{x,y}(t + \epsilon),$$

for all $t > 0$ and $0 < \epsilon < t/2$.

Corollary 2.4. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(i) If $F_{x,y}(\cdot)$ is right continuous, for all $x, y \in X$, then $\limsup_{n\to\infty} F_{x_n,y_n}(t) \leq F_{x,y}(t)$, for all $t > 0$, that is, $F(\cdot,\cdot)(t)$ is an upper semi-continuous function on $X \times X$, for all $t > 0$.

(ii) If $F_{x,y}(\cdot)$ is left continuous, for all $x, y \in X$, then $\liminf_{n\to\infty} F_{x_n,y_n}(t) \geq F_{x,y}(t)$, for all $t > 0$, that is, $F(\cdot,\cdot)(t)$ is a lower semi-continuous function on $X \times X$, for all $t > 0$.

(iii) If $F_{x,y}(\cdot)$ is continuous, for all $x, y \in X$, then $\lim_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$, for all $t > 0$, that is, $F(\cdot,\cdot)(t)$ is a continuous function on $X \times X$, for all $t > 0$ (see also [25]).

The following definitions and results can be seen in [10, 11, 25]. We will make a slight change in the notation to distinguish similar concepts that are already defined.

Definition 2.5. (a) $\{x_n\}$ is called a F-Cauchy sequence in $(X, F, *)$ if for any given $t > 0$ and $\epsilon \in (0, 1]$, there exists $n_0 = n_0(\epsilon, t) \in \mathbb{N}$ such that $F_{x_n,x_m}(t) > 1 - \epsilon$ whenever $n, m \geq n_0$. Equivalently, $\{x_n\}$ is M-Cauchy if $\lim_{n,m} F_{x_n,x_m}(t) = 1$, where $\lim_{n,m}$ denotes the double limit as $n \to \infty$, and $m \to \infty$ (George and Veeramani [10], Schweizer and Sklar [25]).

(b) In [20] the author modified the definition of convergence and obtained a more general concept which is called p-convergence. Let $(X, F, *)$ be a PM-space and $t_0 > 0$ be fixed. A sequence $\{x_n\}$ in X is said to be P-convergent to x_0 in X if $\lim_n F_{x_n,x_0}(t_0) = 1$. A sequence $\{x_n\}$ in X is said to be P-Cauchy if for each $\epsilon \in (0, 1]$ there exists $n_0 \in \mathbb{N}$ such that $F_{x_n,x_m}(t_0) > 1 - \epsilon$ for all $n, m \geq n_0$, i.e. $\lim_{m,n} F_{x_n,x_m}(t_0) = 1$ ([12, 20]).

(c) A PM-space in which every F-Cauchy and P-Cauchy sequence is convergent is called F-complete and P-complete, respectively.

3. Modular Metric Spaces

Chistyakov [6] developed the theory of modular (spaces) and introduced modular metric (spaces). By now this theory has studied including several directions such as fixed point theory (e.g., [5–7, 17]). Our reference for modular metric spaces is [5–7, 17].

Definition 3.1 ([6])**.** A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular (metric) on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_t(x, y) = 0$, for all $t > 0$;
(ii) $\omega_t(x, y) = \omega_t(y, x)$, for all $t > 0$, and $x, y \in X$;
(iii) $\omega_{t+s}(x, y) \leq \omega_t(x, z) + \omega_s(z, y)$, for all $t, s > 0$ and $x, y, z \in X$.

If, instead of (i), the function ω satisfies a weaker condition

(i)' $x = y$ if and only if $\omega_t(x, y) = 0$, for some $t > 0$,

then ω is said to be a strict modular on X.

The modular metric ω_t is said to be convex, if $t\omega_t$ is also a modular metric on X or equivalently it satisfies the inequality

$$\omega_{t+s}(x,y) \leq \frac{t}{t+s}\omega_t(x,z) + \frac{s}{t+s}\omega_s(z,y).$$

for all $t, s > 0$ and $x, y, z \in X$.

Also, like as PM-Spaces, we say ω is strong (or non-Archimedean) if the triangle inequality (iii) is replaced by the following:

(iii)' $\omega_t(x,y) \leq \max\{\omega_t(x,z), \omega_t(z,y)\}$, for all $t > 0$ and $x, y, z \in X$,

and we say that ω is quasi-convex modular metric if the triangle inequality (iii) is replaced by the following:

(iii)" $\omega_{s+t}(x,y) \leq \max\{\omega_t(x,z), \omega_s(z,y)\}$, for all $t, s > 0$ and $x, y, z \in X$.

Remark 3.2 ([6])**.** In a metric pseudo-modular ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is non-increasing on $(0,\infty)$.

Note that every convex modular metric is a quasi-convex modular metric and every quasi-convex modular metric is a modular metric, since the following inequalities hold

$$\begin{aligned}\omega_{t+s}(x,y) &\leq \frac{t\omega_t(x,z) + s\omega_s(z,y)}{t+s} \leq \max\{\omega_t(x,z), \omega_s(z,y)\} \\ &\leq \omega_t(x,z) + \omega_s(z,y),\end{aligned}$$

for all $s, t > 0, x, y \in X$. Also in general a non-Archimedean modular metric need not be a modular metric, since let (X,d) be a non-Archimedean metric space and consider $\omega_\lambda(x,y) = \wedge\{\lambda, d(x,y)\}$ (or $\lambda d(x,y)$), for all $\lambda > 0, x, y \in X$ then the axioms (i), (ii) and (iii)' are satisfied and ω is not a modular metric since $\lambda \to \omega_\lambda$ is not a non-increasing function. Indeed if ω be a non-Archimedean modular metric such that $\lambda \to \omega_\lambda$ be a non-increasing function then ω is quasi-convex modular metric, since without loss of generality assume that $t \geq s > 0$ and for given $x, y \in X$ the following inequalities hold

$$\begin{aligned}\omega_{t+s}(x,y) &\leq \omega_t(x,y) \leq \max\{\omega_t(x,z), \omega_t(z,y)\} \\ &\leq \max\{\omega_t(x,z), \omega_s(z,y)\},\end{aligned}$$

However, there is a non-Archimedean modular metric that is not convex, since let (X,d) be a metric space, setting $\omega_\lambda(x,y) = d(x,y)/\lambda$ for all $\lambda > 0, x, y \in X$ then ω is a modular metric that is not convex [6, Examples 3.4-(b)], but it is easy to see that ω is non-Archimedean modular metric iff (X,d) is non-Archimedean metric.

Also let (X,d) be a non-Archimedean metric space and $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda+d(x,y)}$, for all $\lambda > 0, x,y \in X$, then ω is a quasi-convex modular metric that is not convex modular (see Example 3.10-(a) of [6]). Since ω is non-increasing, based on the above observations, for proving (iii)" it is enough to show (iii)'. It is easy to check that ω_t is a non-Archimedean metric on X, for all $t > 0$ and so for each $x,y \in X$ we have

$$\omega_t(x,y) \leq \vee \left\{ \frac{d(x,z)}{t+d(x,z)}, \frac{d(y,z)}{t+d(y,z)} \right\} = \vee\{\omega_t(x,z), \omega_t(z,y)\}.$$

More examples on modular metric spaces may be found in [7, 17].

Let (X,ω) be a modular metric space. Given $x_0 \in X$, the set $X_\omega = \{x \in X : \lim_{t\to\infty} \omega_t(x,x_0) = 0\}$ is a metric space, called modular space, whose metric is given by $d_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq t\}$ for $x,y \in X_\omega$. Moreover, if ω is convex, the modular set X_ω is equal to $X^*_\omega = \{x \in X : \exists t = t(x) > 0; \omega_t(x,x_0) < 1\}$ and metrizable by $d^*_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq 1\}$ for $x,y \in X^*_\omega$ [6, Theorem 3.6 and Theorem 3.7].

Definition 3.3 ([6])**.** Let X_ω be a modular metric space.

(a) The sequence $\{x_n\}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n,x) \to 0$, as $n \to \infty$, for all $\lambda > 0$. x will be called the ω-limit of $\{x_n\}$.

(b) The sequence $\{x_n\}$ in X_ω is said to be ω-Cauchy if $\omega_\lambda(x_m,x_n) \to 0$, as $m,n \to \infty$, for all $\lambda > 0$.

(c) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belongs to M.

(d) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.

(e) A subset M of X_ω is said to be ω-compact if for any $\{x_n\}$ in M there exists a subset sequence $\{x_{n_k}\}$ and $x \in M$ such that $\omega_\lambda(x_{n_k},x) \to 0$, for all $\lambda > 0$.

(f) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}$ and $\{y_n\}$ in X_ω, ω-convergent to x,y, respectively. We have

$$\omega_\lambda(x,y) \leq \liminf_{n\to\infty} \omega_\lambda(x_n,y_n),$$

for any $x,y \in X_\omega, \lambda > 0$.

In general if $\lim_{n\to\infty} \omega_t(x_n,x) = 0$, for some $t > 0$, then we may not have $\lim_{n\to\infty} \omega_t(x_n,x) = 0$, for all $t > 0$. Therefore, as is done in modular function spaces, ω is said to satisfies the Δ_2-condition if

$\lim_{n\to\infty} \omega_t(x_n, x) = 0$, for some $t > 0$, implies $\lim_{n\to\infty} \omega_t(x_n, x) = 0$, for all $t > 0$ (see [3, 7]).

4. Probabilistic Metrics Induced by Modular Metrics

The following extend many examples in the related articles for probabilistic metric spaces (see e.g. [9, 10, 14]).

Example 4.1 (Schweizer and Sklar [24])**.** Let (X, ω) be a (strong), modular metric space. Supose $G : [0, \infty] \to [0, 1]$ be a non-decreasing, function with $G(0) = 0$ and $G(x) = 1$ iff $x = \infty$. Define fuzzy map $F^{(G,\omega)} : X \times X \times [0, \infty) \to [0, 1]$ as

$$F^{(G,\omega)}_{x,y}(t) = \begin{cases} G\left(\dfrac{t}{\omega_t(x, y)}\right), & t > 0; \\ 0, & t = 0, \end{cases} \tag{4.1}$$

for all $x, y \in X, t > 0$, then for any choice of t-norm, $(X, F^{(G,\omega)}, *)$ is a (strong), PM-space.

Proof. Let (X, ω) be a modular metric space. Given $x \in X$, (i)' implies $\omega_\lambda(x, x) = 0$, for all $\lambda > 0$, and so, $F^{(G,\omega)}_{x,x}(t) = G(\infty) = 1$. Conversely let $x, y \in X, t > 0$ and $F^{(G,\omega)}_{x,y}(t) = 1$, then from (4.1) we have $\omega_t(x, y) = 0$, and so, $x = y$. The proof of (PM2) is complete.

Due to axiom (ii), the equality (PM3) is clear.

Finally, it is enough to prove (PM4) for $* = \wedge$ since it is the strongest possible choice of $*$. For distinct points $x, y, z \in X$ and $s, t > 0$, from the triangle inequality of the modular metric ω it follows that

$$\frac{t + s}{\omega_{t+s}(x, y)} \geq \frac{t + s}{\omega_t(x, z) + \omega_s(z, y)} \geq \min\left\{\frac{t}{\omega_t(x, z)}, \frac{s}{\omega_s(z, y)}\right\},$$

which, since G is non-decreasing, implies (PM4). So $(X, F^{(G,\omega)}, *)$ is a PM-space. □

The following obtained as spacial case of Example 4.1 and extend many examples in [14].

Example 4.2. Let (X, ω) be a modular metric space and $x, y \in X, s, t > 0$.

(i) Define the map $G : [0, \infty] \to [0, 1]$ as $G(r) = r/(r+1)$ or $e^{-1/r}$ for any $r \in [0, \infty]$. Eq. (4.1) becomes

$$F^\omega_{x,y}(t) = \frac{t}{t + \omega_t(x,y)} \qquad \text{or} \qquad e^{-\dfrac{\omega_t(x,y)}{t}}, \tag{4.2}$$

and $(X, F^\omega, *)$ is a PM-space for any choice of t-norm $*$ (see also Example 1.8 of [1]).

(ii) Let $F^\omega_{x,y}(t) = \frac{1}{1+\omega_t(x,y)}$, then $(X, F^\omega, \cdot)$ is a PM-space.

(iii) Let $F^\omega_{x,y}(t) = e^{-\omega_t(x,y)}$, then $(X, F^\omega, \cdot)$ is a PM-space.

(iv) If $0 \le \omega \le 1$, setting $F^\omega_{x,y}(t) = 1 - \omega_t(x,y)$, then $(X, F^\omega, \mathfrak{L})$ is a PM-space.

(v) Define the map $F : X \times X \to D^+$ by $F_{x,y}(t) = \epsilon_0(t - \omega_t(x,y))$, then $(X, F, \wedge)$ is a PM-space.

(vi) Let ω be a non-Archimedean modular metric on X and F^ω is defined by equation (4.1) in Example 4.1 or by (i), then $(X, F^\omega, \wedge)$ is a non-Archimedean PM-space.

(vii) Let ω be a convex modular metric or quasi-convex modular metric on X and F^ω is defined by equation (4.1) in Example 4.1 or by (i), then $(X, F^\omega, \wedge)$ is a PM-space.

Proof. Part (i)-(iv) are straight forward. (v) is similar to [26]. To show (vi) we only prove the inequality (NA). Suppose that $\lambda > 0, x, y, z \in X$, then we have

$$\begin{aligned} F^\omega_{x,y}(\lambda) &= G\left(\frac{\lambda}{\omega_\lambda(x,y)}\right) \\ &\ge G\left(\wedge\left\{\frac{\lambda}{\omega_\lambda(x,z)}, \frac{\lambda}{\omega_\lambda(y,z)}\right\}\right) \\ &= \wedge\left\{G\left(\frac{\lambda}{\omega_\lambda(x,z)}\right), G\left(\frac{\lambda}{\omega_\lambda(y,z)}\right)\right\} \\ &= \wedge\{F^\omega_{x,z}(\lambda), F^\omega_{y,z}(\lambda)\}. \end{aligned}$$

To show (vii) we only prove inequality (PM4) for a quasi-convex modular metric. For all $x, y, z \in X$ and $s, t > 0$ we have

$$\begin{aligned} F^\omega_{x,y}(t+s) &= G\left(\frac{1}{\omega_{t+s}(x,y)}\right) \\ &\ge G\left(\wedge\left\{\frac{1}{\omega_t(x,z)}, \frac{1}{\omega_s(y,z)}\right\}\right) \end{aligned}$$

$$\geq \wedge \left\{ G\left(\frac{1}{\omega_t(x,z)}\right), G\left(\frac{1}{\omega_s(y,z)}\right) \right\}$$
$$= \wedge\{F^{\omega}_{x,z}(t), F^{\omega}_{y,z}(s)\}.$$

□

Conversely, we can easily construct a modular metric space (X, ω) from a given PM-space $(X, F, *)$, for a t-norm satisfies $* \geq \mathfrak{L}$. To see this it suffices to consider modular metric

$$\omega_t(x, y) = 1 - F_{xy}(t), \qquad \text{for all} \qquad x, y \in X, t > 0.$$

Using Lemma [2, lemma 3.2], it is easy to see that axioms of Definition 3.1 is satisfied. So, it is natural to change the topological notions, convergence sequences, completeness and etc. between the sets of PM-spaces and modular metric spaces.

In the sequel, we explain some definitions and theorems in set of the modular metric spaces and showed that they can be adapted into the realm probabilistic metric space and vice versa.

At first, consider Example 4.2-(b), it is easy to check that Remark 3.2 is a conclusion of Remark 2.2 and (X, ω) has a Δ_2-condition iff $(X, F^{\omega}, \cdot)$ is a principle PM-space and taking a look at Corollary 2.4 we concluded that if ω has a Fatou property then $F_{x,y}(\cdot)$ is a right continuous function on $(0, \infty)$, for all $x, y \in X$.

Theorem 4.3. Let (X, ω) be a (complete) modular metric space.

- (i) Let ω be a non-Archimedean modular metric on X and F^{ω} is defined by equation (4.1) in Example 4.1 or by (a)-(b) in Example 4.2, then $(X, F^{\omega}, \wedge)$ is a (complete) non-Archimedean PM-space.
- (ii) Let ω be a convex modular metric or quasi-convex modular metric and G and F be the same in Example 4.1. $(X, F^{\omega}, \wedge)$ is a PM-space and so when F^{ω} is given by equations in Example 4.2-(b) and (c) then $(X, F^{\omega}, \wedge)$ is a (complete) PM-space.

Proof. To show (i) we only prove (2.1) inequality. Suppose that $\lambda > 0, x, y, z \in X$, then we have

$$F^{\omega}_{x,y}(\lambda) = G\left(\frac{\lambda}{\omega_\lambda(x,y)}\right)$$
$$\geq G\left(\wedge\left\{\frac{\lambda}{\omega_\lambda(x,z)}, \frac{\lambda}{\omega_\lambda(y,z)}\right\}\right)$$
$$= \wedge\left\{ G\left(\frac{\lambda}{\omega_\lambda(x,z)}\right), G\left(\frac{\lambda}{\omega_\lambda(y,z)}\right) \right\}$$

$$= \wedge\{F^{\omega}_{x,z}(\lambda), F^{\omega}_{y,z}(\lambda)\}.$$

Also Example 4.2-(a) and (b) are special cases of (4.1) when G is defined by $G(r) = r/(r+1)$ and $G(r) = e^{-1/r}$, for all $r \in [0,\infty]$, respectively. The proof of (ii) is complete. To show (ii) it is enough to prove (PM4) inequality for quasi-convex modular metric (see Section 1). For all $x, y, z \in X$ and $s, t > 0$ we have

$$\begin{aligned} F^{\omega}_{x,y}(t+s) &= G\left(\frac{1}{\omega_{t+s}(x,y)}\right) \\ &\geq G\left(\wedge\left\{\frac{1}{\omega_t(x,z)}, \frac{1}{\omega_s(y,z)}\right\}\right) \\ &\geq \wedge\left\{G\left(\frac{1}{\omega_t(x,z)}\right), G\left(\frac{1}{\omega_s(y,z)}\right)\right\} \\ &= \wedge\{F^{\omega}_{x,z}(t), F^{\omega}_{y,z}(s)\}. \end{aligned}$$

Also Example 4.2-(b) and (c) are special cases of (4.1) when G is defined by $G(r) = r/(r+1)$ and $G(r) = e^{-1/r}$, for all $r \in [0,\infty]$, respectively. The proof of (ii) is complete. □

5. Probabilistic Modular Metric Spaces

In this section we introduce the modular set and the Luxemburg metric on a probabilistic modular space.

Definition 5.1. Let $(X, F, *)$ be a PM-space on X. Fix $x_0 \in X$. The two sets

$$X_F = X_F(x_0) = \{x \in X; F_{x,x_0}(t) \to 1 \quad \text{as} \quad t \to \infty\}$$

and

$$X^*_F = X^*_F(x_0) = \{x \in X; \exists t = t(x), F_{x,x_0}(t) > 0\},$$

are said to be probabilistic modular metric spaces (around x_0).

The following is immediate.

Theorem 5.2. (i) If F^{ω} be a standard PM-space or considering the Examples 4.2 (i)-(iv) then $X_{F^{\omega}} = X_{\omega}$ and $X^*_{F^{\omega}} = X^*_{\omega}$.

(ii) If $(X, F, *)$ be a PM-space such that condition (KM6) holds then $X = X_F = X^*_F$.

(iii) If sequence $\{x_n\}$ in X_{ω} be an (ω_{t_0}-convergent) ω-convergent to $x \in X_{\omega}$ then $\{x_n\}$ is (P-convergence at t_0) convergence in the standard PM-space X_F.

(iv) If sequence $\{x_n\}$ in X_ω be an (ω_{t_0}-Cauchy) ω-Cauchy then $\{x_n\}$ is a (P-Cauchy at t_0) Cauchy in the standard PM-space X_F.

(v) If X_ω be (ω_{t_0}-compact) compact, (ω_{t_0}-closed) closed and (ω_{t_0}-complete) complete then the standard PM-space X_F is (P- compact) compact, (P-closed) closed and (P-complete at t_0) complete.

In the rest of this section we study the connection between F- convergence and the Luxemburg metric (see Theorem 5.6 below). The following is found in [16, 21, 22], and we give another proof by using [2, Lemma 3.2].

Theorem 5.3 (see [21])**.** If $(X, F, *)$ be a PM-space such that $* \geq \mathfrak{L}$, then the fuzzy modular set X_F is a metric space with metric given by

$$d_F^\circ(x, y) = \inf\{0 < t; F_{x,y}(t) \geq 1 - t\}, \qquad x, y \in X_F.$$

Proof. Given $x \in X_F$. (PM2) implies $F_{x,x}(t) = 1$ for all $t > 0$, and so, $d_F(x, x) = 0$. Let $x, y \in X_F$ and $d_F(x, y) = 0$. Then $F_{x,y}(s)$ does not less than $1 - s$ for all $1 > s > 0$. Hence for any $0 < s < t$, in view of Remark 2.2, we have $F_{x,y}(t) \geq F_{x,y}(s) \geq 1 - s \to 1$ as $s \to 0^+$. It follows that $F_{x,y}(t) = 1$ for all $t > 0$, and so, axiom (PM2) implies $x = y$.

Due to axiom (ii), the equality $d_F^\circ(x, y) = d_F^\circ(y, x)$, $x, y \in X_F$ is clear.

Let us show the triangle equality for d_F°. By the definition of d_F° , for any $t > d_F(x, z)$ and $s > d_F^\circ(y, z)$ and axiom (PM4) we find

$$\begin{aligned} F_{x,z}(t + s) &\geq F_{x,y}(t)\mathfrak{L}F_{y,z}(s) \\ &\geq (1 - t)\mathfrak{L}(1 - s) \\ &\geq 1 - (t + s). \end{aligned}$$

It follows from the definition of d_F° that $d_F^\circ \leq t + s$, and it remains to pass to the limits as $t \to d_F^\circ(x, z)$ and $s \to d_F^\circ(y, z)$. □

Theorem 5.4. If $(X, F, *)$ be a PM-space such that $* \geq \mathfrak{L}$, then the fuzzy modular set X_F is a metric space with metric given by

$$d_F^1(x, y) = \inf_{0<t}(1 + t - F_{x,y}(t)).$$

such that $d_F^\circ \leq d_F^1 \leq 2d_F^\circ$ on $X_F \times X_F$.

Proof. If $x \in X_F$, then, by (PM2), $F_{x,x}(t) = 1$ for all $t > 0$, and so, $d_F^1(x, x) = 0$. Conversely let $d_F^1(x, y) = 0$. The equality $x = y$ will follow from (PM2) if we show that $F_{x,y}(t) = 1$ for all $t > 0$. On the contrary, suppose that $F_{x,y}(t_0) < 1$ for some $t_0 > 0$. Then for all $t \geq t_0$ we find $1 - F_{x,y}(t) + t \geq t_0$, and by Remark 2.2 for all $t < t_0$ we have

$$0 < 1 - F_{x,y}(t_0) \leq 1 - F_{x,y}(t) \leq 1 - F_{x,y}(t) + t.$$

Thus $1 - F_{x,y}(t) + t \geq t_1 = \wedge\{t_0, 1 - F_{x,y}(t_0)\}$, for all $t > 0$, and so by the definition of d_F^1, $d_F^1 \geq t_1 > 0$, which contradicts the assumption.

Axiom (PM3) implies the symmetry property of d_F^1.

Let us establish the triangle inequality. By the definition of d_F^1, for any $\epsilon > 0$ we find $t = t(\epsilon) > 0$ and $s = s(\epsilon) > 0$ such that

$$1 - F_{x,z}(t) + t \leq d_F^1(x, z) + \epsilon, \qquad 1 - F_{y,z}(s) + s \leq d_F^1(y, z) + \epsilon$$

applying [2, Lemma 3.2], we have

$$\begin{aligned} d_F^1(x, y) &\leq 1 - F_{x,y}(t + s) + t + s \\ &\leq 1 - F_{x,z}(t) + t + 1 - F_{y,z}(s) + s \\ &\leq d_F^1(x, z) + d_F^1(y, z) + 2\epsilon, \end{aligned}$$

and it remains to take into account the arbitrariness of $\epsilon > 0$. Let us prove that metrics d_F° and d_F^1 are equivalent on X_F. Suppose that $0 < t$, if $F_{x,y}(t) \geq 1 - t$ then we have $1 + t - F_{x,y}(t) \leq 2t$, so we get $d_F^1(x, y) \leq 2d_F^\circ(x, y)$, conversly, for any $0 < t$, we have $t \leq 1+t-F_{x,y}(t)$, so we get $d_F^\circ(x, y) \leq d_F^1(x, y)$, for all $x, y \in X$. □

Theorem 5.5. (a) Let $(X, F^\omega, \cdot)$ be a PM-space in Example 4.2-(e), then $d_F^\circ = d_\omega^\circ$ and $d_F^1 = d_\omega^1$.
(b) Let $(X, F^\omega, \cdot)$ be a PM-space in Example 4.2-(b) and (c), then $d_F^\circ \leq d_\omega^\circ$ and $d_F^1 \leq d_\omega^1$.

Proof. (a) is obvious and (b) is immediately concluded from the following statements

$$\omega_t(x, y) \leq t \qquad \Longrightarrow \qquad e^{-\omega_t(x,y)} \quad \text{and} \quad \frac{1}{1 + \omega_t(x, y)} \geq 1 - t,$$

and both

$$1 + t - e^{-\omega_t(x,y)} \quad \text{and} \quad 1 + t - \frac{1}{1 + \omega_t(x, y)} \leq t + \omega_t(x, y),$$

for all $x, y \in X_F = X_\omega, 0 < t \leq 1$. □

Similar to [6, Theorem 2.13] next theorem shows that the convergence in metric d_F° and F-convergence for sequences of X_F are equivalent.

Theorem 5.6. Given a sequence $\{x_n\}$ in X_F then $x_n \to_F x$ iff $x_n \to x$ in (X_F, d_F°). A similar assertion holds for Cauchy sequences.

Proof. Suppose that $x_n \to_F x$. Thus for all $1 \geq t > 0$, $F_{x_n,x}(t) \to 1$, and so, there is a number $n_0(t)$ such that $F_{x_n,x}(t) \geq 1 - t$, for all $n \geq n_0(t)$, hence $d_F^\circ(x_n, x) \leq t$, thus $x_n \to x$ in (X, d_F°). Conversely, fix $t > 0$. For each $1 \geq \epsilon > 0$, setting $\epsilon' = \wedge\{\epsilon, t\}$, by the assumption, there is a number $n_0(\epsilon')$ such that $d_F^\circ(x_n, x) < \epsilon'$, for all $n \geq n_0(\epsilon')$. Thus $F_{x_n,x}(\epsilon') \geq 1 - \epsilon'$, for all $n \geq n_0(\epsilon')$. So by Remark 2.2 we have $F_{x_n,x}(t) \geq F_{x_n,x}(\epsilon') \geq 1 - \epsilon' \geq 1 - \epsilon$, for all $n \geq n_0(\epsilon')$. This means $F_{x_n,x}(t) \to 1$, i.e., $x_n \to_F x$. □

6. Convex Modular Metric Spaces

In this section we investigate the analogy convex formulation of probabilistic modular metric spaces. To be precise, what condition on F is given such that $X_F = X_F^*$?

Definition 6.1. A PM-space $(X, F, *)$ is called convex if $(X, \hat{F}, *)$ is also a PM-space, where $\hat{F}$ is defined by $\hat{F}_{x,y}(t) = F_{x,y}(t)^t, t > 0$, $\hat{F}(0) = 0$.

From this definition it is easy to concluded that if (X, ω) be a convex modular metric space then the PM-space $(X, F^\omega, *)$ in Example 4.2-(b) become a convex PM-space. Also if $(X, F, *)$ is convex, then by Remark 2.2, the function $t \to F_{x,y}(t)^t$ is nondecreasing on $(0, +\infty)$, so

$$\text{if} \qquad 0 < t \leq s, \qquad \text{then} \qquad F_{x,y}(t)^{t/s} \leq F_{x,y}(s),$$

if $x \in X_F^*$, then $F_{x,x_0}(t) > 0$ for some number $t > 0$, by virtue of previous inequality $F_{x,x_0}(s) \geq F_{x,x_0}(t)^{t/s} \to 1$ as $s \to +\infty$, implying $X_F = X_F^*$.

Remark 6.2. Let $\epsilon > 0$ and $(X, M, *)$ be a PM-space such that the t-norm $*$ satisfies the following inequality,

$$(a * b)^\epsilon \geq a^\epsilon * b^\epsilon \qquad \text{for all} \qquad a, b \in [0, 1]. \tag{6.1}$$

Then $(X, M^\epsilon, *)$ is a PM-space where the fuzzy set M^ϵ has been defined by $M^\epsilon(x, y, t) = M(x, y, t)^\epsilon$, for all $x, y \in X$ and $t > 0$ (see [2, Proposition 3.10]). Also inequality (6.1) holds for minimum and product t-norms and does not hold for Lukasiewicz t-norm (see [2, Remark 3.11]). This means that the convexity of $(X, F, *)$ depend on the t-norm $*$ and so it is differ from modular metric spaces.

Like the same argument in [6] the Luxemburg distances can be construct in X_F^* in the following way.

Theorem 6.3. If $(X, F, *)$ be a convex PM-space, then the fuzzy modular set X_F^* is a metric space with metric given by

$$d_F^*(x, y) = \inf\{0 < t; F_{x,y}(t) \geq c\}, \qquad x, y \in X_F^*,$$

where $0 < c < 1$ is a constant such that

$$c^t * c^s \geq c^{t+s}, \qquad \forall s, t > 0. \tag{6.2}$$

Proof. Given $x \in X_F$. (PM2) implies $F_{x,x}(t) = 1$ for all $t > 0$, and so, $d_F^*(x, x) = 0$. Let $x, y \in X_F$ and $d_F^*(x, y) = 0$. Then $F_{x,y}(s)$ does not less than c for all $s > 0$. Hence for any $0 < t < s$, in view of Remark 2.2, we have $F_{x,y}(s) \geq F_{x,y}(t)^{t/s} \geq c^{t/s} \to 1$ as $t \to 0^+$. It follows that $F_{x,y}(s) = 1$ for all $s > 0$, and so, axiom (PM2) implies $x = y$. We now establish the triangle inequality. Suppose that $d_F^*(x, y) < t$ and $d_F^*(y, z) < s$, $x, y, z \in X_F^*, t, s > 0$. By definition of d_F^* we obtain $F_{x,y}(t) \geq c$ and $F_{y,z}(s) \geq c$ and by convexity of F and (6.2) we have

$$F_{x,z}(t + s)^{t+s} \geq F_{x,y}(t)^t * F_{y,z}(s)^s \geq c^t * c^s \geq c^{t+s}.$$

Thus, $d_F^*(x, z) \leq t + s$. The arbitrariness of t and s as above implies the triangle inequality. □

Note that if $* \geq \cdot$ then inequality (6.2) holds, but it does not hold for Lukasiewicz t-norm.

Theorem 6.4. If $(X, F, *)$ be a convex PM-space, then the fuzzy modular set X_F^* is a metric space with metric given by

$$d_F^{**}(x, y) = \inf_{0<t}(1 + t - F_{x,y}(t)^t),$$

and $d_F^*(x, y) \leq d_F^{**}(x, y) \leq 2d_F^*(x, y)$, for all $x, y \in X_F^*$.

Proof. Let $x, y \in X_F^*$. Since $t \leq 1 + t - F_{x,y}(t)^t$, for all $t > 0$ and $1 + t - F_{x,y}(t)^t \leq 1 + t - c^t \leq 2t$, for all $t > 0$ and $F_{x,y}(t) \geq c$, the equivalence between metrics and also the first axiom of the metricness are obtained. The symmetry property is obvious. We now establish the triangle inequality. By the definition of d_F^{**}, for any $\epsilon > 0$ we find $t = t(\epsilon) > 0$ and $s = s(\epsilon) > 0$ such that

$$1 - F_{x,z}(t)^t + t \leq d_F^{**}(x, z) + \epsilon, \qquad 1 - F_{y,z}(s)^s + s \leq d_F^{**}(y, z) + \epsilon$$

using the convexity of F and applying [2, Lemma 3.2], we have

$$\begin{aligned} d_F^{**}(x, y) &\leq 1 - F_{x,y}(t + s)^{t+s} + t + s \\ &\leq 1 - F_{x,z}(t)^t + t + 1 - F_{y,z}(s)^s + s \\ &\leq d_F^{**}(x, z) + d_F^{**}(y, z) + 2\epsilon, \end{aligned}$$

and it remains to take into account the arbitrariness of $\epsilon > 0$. □

7. Conclusion

In this paper based on the concept of modular metric we have extended examples of probabilistic metric and introduced probabilistic (convex) modular metric spaces. The above results show that the notions of PM-spaces and modular metric spaces are closely related. Hence, some of the basic definitions and theorems in the both theories (such as their topologies and P-convergences, fixed point theory, e.t.c.) can be reformulate to each other.

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