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ASYMPTOTIC STABILITY OF SOME EQUATIONS

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ABSTRACT. In this paper, we investigate asymptotic stability of several integral and differential equations.

Key Words: Asymptotic stability; Fractional Volterra type integral equation; Fractional differential equation with modification of the argument; Differential equations with fractional integrable impulses.
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1. Introduction

Asymptotic stability is a kind of stability that is studied recently, there are different kind of definition of asymptotic stability [3], [2]. In this paper we have presented and studied one kind of that [1] on three equations below:

Fractional Volterra type integral equation with delay of the form

$$y(x) = \frac{1}{\Gamma(\beta)} \int_c^x (x-s)^{\beta-1} f(x, s, y(s), y(\alpha(s)))ds, \ \beta \in (0, 1).$$

Fractional differential equation with modification of the argument of the form:

$${}^cD_t^\alpha x(t) = f(t, x(t), x(g(t))), \ t \in I \subset \mathbb{R}, \ \alpha \in (0, 1).$$

Differential equations with fractional integrable impulses of the form:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \cdots, m, \\ x(t) = I^\alpha_{t_i,t} g_i(t, x(t)), & t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \ \alpha \in (0, 1). \end{cases}$$

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2. Preparation of manuscript

The concept of the asymptotic stability of a solution of equation is understood in the following sense givin by Banas and Rzepka.

Definition 2.1. Let $B(x, r)$ denotes the closed ball centered at x with radius $r, (r > 0)$ the symbol B_r stands the ball $B(0, r)$. For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $r(\varepsilon) > 0$ such that, if $y_1, y_2 \in B_r$ and $y_1(t)$, $y_2(t)$ are solutions of equation, then $|y_1(t) - y_2(t)| \leq \varepsilon$ for $t \geq T(\varepsilon)$.

Definition 2.2. For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by

$$(D^{\alpha}_{a^+}h)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-s)^{(n-\alpha-1)}h(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, where $\Gamma(.)$ is the Gamma function.

Definition 2.3. For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by

$$^cD^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$.

Definition 2.4. Given an interval $[a, b]$ of $\mathbb{R}$, then the Riemann-Liouville fractional order integral of a function $h \in L^1([a, b], \mathbb{R})$ of order $\gamma \in \mathbb{R}_+$ is defined by

$$I^{\gamma}_{a^+}h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1}h(s)ds.$$

Definition 2.5. Fractional Volterra type integral equation with the delay is defined

$$y(x) = I^{\beta}_{c^+}f(x, x, y(x), y(\alpha(x))) = \frac{1}{\Gamma(\beta)} \int_c^x (x-s)^{\beta-1}f(x, s, y(s), y(\alpha(s)))ds, \tag{2.1}$$

for $\beta \in (0, 1)$, where $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, a, b and c are fixed real numbers such that $-\infty < a \leq x \leq b < +\infty$, and $c \in (a, b)$, and $\alpha : [a, b] \longrightarrow [a, b]$ is a continuous delay function which therefore fulfills $\alpha(x) \leq x$, for all $x \in [a, b]$.

Definition 2.6. Fractional- order delay differential equation with modification of the argument is defined

$$^cD^{\alpha}_t x(t) = f(t, x(t), x(g(t))),\ t \in I \subset \mathbb{R},\ \alpha \in (0, 1). \tag{2.2}$$

where $f \in C(I\times\mathbb{R}^2,\mathbb{R})$, $g \in C(I,[-h,b])$ with $g(t) \le t$, $I=[0,b]$, $b\in\mathbb{R}_+$ and $h>0$.

Definition 2.7. Differential equation with fractional integrable impulses is defined

(2.3)
$$\begin{cases} x'(t) = f(t,x(t)), & t\in(s_i,t_{i+1}],\ i=0,1,2,\cdots,m, \\ x(t) = I^{\alpha}_{t_i,t}g_i(t,x(t)), & t\in(t_i,s_i],\ i=1,2,\cdots,m,\ \alpha\in(0,1). \end{cases}$$

where $0=t_0=s_0<t_1\le s_1\le t_2<\cdots<t_m\le s_m\le t_{m+1}=T$ are prefixed numbers, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and $g_i:[t_i,s_i]\times\mathbb{R}\to\mathbb{R}$ is continuous for all $i=1,2,\cdots,m$ and the symbol $I^{\alpha}_{t_i,t}g_i$ is so-called Riemann- Liouville fractional integrals of the order α and is given by

$$I^{\alpha}_{t_i,t}g_i(t,x(t)) = \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-s)^{\alpha-1}g_i(s,x(s))ds.$$

Theorem 2.8. *Assume that $\alpha:[a,b]\longrightarrow[a,b]$ is a continuous function such that $\alpha(x)\le x$, for all $x\in[a,b]$ and $f:[a,b]\times[a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function which additionally satisfies the Lipschitz condition*

$$|f(x,s,y_1(s),y_1(\alpha(s))) - f(x,s,y_2(s),y_2(\alpha(s)))| \le L_f|y_1(s)-y_2(s)|,$$

for any $x,s\in[a,b]$ and $y_1,y_2\in\mathbb{R}$, then equatin (2.1) is asymptotic atable.

Proof. Soppose y_1 and $y_2\in B_r$ be two solutions of equation (2.1), i.e. $|y_1|<r$ and $|y_2|<r$, hence $|y_1-y_2|<2r$ which $r=\dfrac{\beta\Gamma(\beta)}{2L_f(x-c)^{\beta}}\varepsilon$, then

$$|y_1(s)-y_2(s)| = |\frac{1}{\Gamma(\beta)}\int_c^x(x-s)^{\beta-1}f(x,s,y_1(s),y_1(\alpha(s)))ds$$

$$-\frac{1}{\Gamma(\beta)}\int_c^x(x-s)^{\beta-1}f(x,s,y_2(s),y_2(\alpha(s)))ds|$$

$$\le \frac{1}{\Gamma(\beta)}\int_c^x(x-s)^{\beta-1}|f(x,s,y_1(s),y_1(\alpha(s))) - f(x,s,y_2(s),y_2(\alpha(s)))|ds$$

$$\le \frac{L_f}{\Gamma(\beta)}\int_c^x(x-s)^{\beta-1}|y_1(s)-y_2(s)|ds$$

$$\le \frac{L_f2r}{\Gamma(\beta)}\int_c^x(x-s)^{\beta-1}ds = \frac{L_f2r}{\Gamma(\beta)}.\frac{(x-c)^{\beta}}{\beta} = \varepsilon$$

so that $|y_1(s) - y_2(s)| < \varepsilon$, which implies that the the solution of (2.1) are asymptotically stable. $\square$

Example 2.9. Let

$$y(x) = \frac{1}{\Gamma(\frac{1}{3})} \int_0^t \frac{|x(s)|e^{-3t-s} + \frac{1}{1+5t^{\frac{7}{3}}}}{(t-s)^{\frac{2}{3}}} ds$$

that

$$f(x, s, y(s)) = |y(s)|e^{-3x-s} + \frac{1}{1+5x^{\frac{7}{3}}},\ x \neq -1$$

is continuous function that satisfies Lipschitz condition, because

$$|f(x, s, y_1(s)) - f(x, s, y_2(s))|$$
$$= \left| |y_1(s)|e^{-3x-s} + \frac{1}{1+5x^{\frac{7}{3}}} - |y_2(s)|e^{-3x-s} - \frac{1}{1+5x^{\frac{7}{3}}} \right|$$
$$= e^{-3x-s}\Big| |y_1(s)| - |y_2(s)| \Big| \leq e^{-3x-s}|y_1(s) - y_2(s)|,$$

so f satisfies Lipschitz condition with $L = e^{-3x-s}$, according to the previous theorem we put

$$r = \frac{\frac{1}{3}\Gamma(\frac{1}{3})}{2e^{-3x-s}(x-c)^{\frac{1}{3}}}\varepsilon$$

then this equation is asymptotic stable.

Lemma 2.10. *From theorem 3.4 [6] for $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, the following equations:*

$$\begin{cases} {}^cD_t^\alpha x(t) = f(t, x(t), x(g(t))), & t \in I, \\ x(t) = \psi(t), & t \in [-h, 0]. \end{cases}$$

are equivalent to the singular integral system:

$$\begin{cases} x(t) = \psi(t), & t \in [-h, 0], \\ \psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(g(s)))ds, & t \in I. \end{cases}$$

Theorem 2.11. *Assume that $f \in C(I \times \mathbb{R}^2, \mathbb{R})$ is a continuous function which additionally satisfies the Lipschitz condition*

$$|f(t, y_1(t), y_1(g(t))) - f(t, y_2(t), y_2(g(t)))| \leq L_f|y_1(t) - y_2(t)|,$$

and $g \in C(I, [-h, b])$, $g(t) \leq t$ and $h > 0$, then (2.2) is asymptotic stable.

Proof. Soppose y_1 and $y_2 \in B_r$ be two solutions of equation (2.2), i.e. $|y_1| < r$ and $|y_2| < r$, hence $|y_1 - y_2| < 2r$ which $r = \dfrac{\varepsilon\alpha\Gamma(\alpha)}{2L_f t^\alpha}$, then

$$
\begin{aligned}
&|y_1(s) - y_2(s)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (x-s)^{\alpha-1} |f(s, y_1(s), y_1(g(s))) - f(s, y_2(s), y_2(g(s)))| ds \\
&\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (x-s)^{\alpha-1} |y_1(s) - y_2(s)| ds \\
&\leq \frac{L_f 2r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{L_f 2r}{\Gamma(\alpha)} . \frac{t^\alpha}{\alpha} = \varepsilon
\end{aligned}
$$

so that $|y_1(s) - y_2(s)| < \varepsilon$, which implies that the solution of (2.2) are asymptotically stable. □

Lemma 2.12. [5] *A function x is called a classical solution of the problem*

$$
\begin{cases}
x'(t) = f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, m, \\
x(t) = I^\alpha_{t_i, t} g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, 2, \cdots, m,\ \alpha \in (0, 1), \\
x(0) = x_0 \in \mathbb{R},
\end{cases}
$$

if x satisfies

$$
\begin{cases}
x(0) = x_0; \\
x(t) = I^\alpha_{t_i, t} g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, 2, \cdots, m; \\
x(t) = x_0 + \int_0^t f(s, x(s)) ds, & t \in (0, t_1); \\
x(t) = I^\alpha_{t_i, s_i} g_i(s_i, x(s_i)) + \int_{s_i}^t f(s, x(s)) ds, & t \in (s_i, t_{i+1}],\ i = 1, 2, \cdots, m.
\end{cases}
$$

Theorem 2.13. *Assume that $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $g_i : [t_i, s_i] \times \mathbb{R} \to \mathbb{R}$ are continuous for all $i = 1, 2, \cdots, m$ where $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = T$ are pre-fixed numbers, which additionally satisfies the Lipschitz condition*

$$
\begin{aligned}
|f(t, y_1(t)) - f(t, y_2(t))| &\leq L_f |y_1(t) - y_2(t)|, \\
|g_i(t, y_1(t)) - g_i(t, y_2(t))| &\leq L_{g_i} |y_1(t) - y_2(t)|,
\end{aligned}
$$

and let

$$
M := Lg_i \frac{(s_i - t_i)^\alpha}{\alpha} + L_f (t - s_i)
$$

then (2.3) is asymptotic stable.

Proof. Soppose y_1 and $y_2 \in B_r$ be two solutions of equation (2.3), i.e. $|y_1| < r$ and $|y_2| < r$, hence $|y_1 - y_2| < 2r$ which $r := \dfrac{\varepsilon \Gamma(\alpha)}{2M}$, then

$$
|y_1(s) - y_2(s)|
$$

$$\leq \frac{1}{\Gamma(\alpha)}\Big(Lg_i\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}|y_1(s)-y_2(s)|ds + L_f\int_{s_i}^{t}|y_1(s)-y_2(s)|ds\Big)$$

$$\leq \frac{1}{\Gamma(\alpha)}\Big(2rLg_i\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}ds + 2rL_f\int_{s_i}^{t}ds\Big)$$

$$= \frac{2r}{\Gamma(\alpha)}\Big(Lg_i\frac{(s_i-t_i)^{\alpha}}{\alpha} + L_f(t-s_i)\Big) = \frac{2r}{\Gamma(\alpha)}M = \varepsilon$$

so that $|y_1(s)-y_2(s)| < \varepsilon$, which implies that the the solution of (2.3) are asymptotically stable. □

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