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f-DERIVATIONS ON RESIDUATED MULTILATTICES

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ABSTRACT. In this paper, we introduce as a generalization of the concept of derivation, the notion of f-derivation on residuated multilattices and investigate several of its properties. Then, we study good ideal f-derivations and make the connection with the complemented elements. Moreover, special sub-classes like the set of f-fixed points, the Kernel are found to have nice substructures.

Key Words: Multilattice, residuated multilattice, filter, complemented elements, ideal derivation.

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1. INTRODUCTION AND PRELIMINARIES

Residuated multilattices have been introduced by Cabrera et *al* in [4] as a generalization of residuated lattices. It is an algebraic hyperstructure where a residuated operation is combining with a multilattice structure.

The concept of derivation have been introduced on commutative rings [3, 9], lattices [5, 11, 13], hyperlattices [12], BCI-algebras [14] and most recently on residuated lattices [6, 10]. Furthermore, the notion of derivation have been generalized to f-derivation [1, 2, 15].

In this work, we extend the concept of f-derivation to residuated multilattices, formulate the definitions and study its first properties. We show that the set of good ideal f-derivations is a boolean algebra and we characterize every good ideal f-derivation. Moreover, we prove that the set of f-fixed points of a good ideal f-derivation is a full sub-multilattice.

The paper is organizing as follows: In section 2, we define multiplicative ideal f-derivation, study its properties and illustrate each property with examples. In

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section 3, we study good ideal f-derivation and show the connection with the complemented elements of the residuated multilattice. We also prove that the set of f-fixed points of a good ideal f-derivation has the structure of multilattice.

We start by briefly recalling the basic definitions needed in the paper.

Definition 1.1. [4]

A **pocrim** is a quadruple $\mathcal{A} := (\mathbb{A}, \top, \odot, \to)$ consisting of a poset $\mathbb{A} := (A, \leq)$ with a greatest element $\top$ and two binary operations $\odot$ and $\to$ on A such that $(A, \top, \odot)$ is a commutative monoid satisfying $a \odot c \leq b$ if and only if $c \leq a \to b$ for all $a, b, c \in A$.

A pocrim is said to be bounded if it has a least element.

The following properties hold in any pocrim $\mathcal{A}$.

For all $a, b, c \in A$, we have:

P1 $a \odot b \leq a,\ a \odot b \leq b$;

P2 $a \odot (a \to b) \leq a \leq b \to (a \odot b)$ and $b \odot (a \to b) \leq b \leq a \to (a \odot b)$;

P3 If $a \leq b$, then $a \odot c \leq b \odot c$, $c \to a \leq c \to b$, and $b \to c \leq a \to c$;

P4 $a \to (b \to c) = b \to (a \to c)$;

P5 $(a \to b) \odot (b \to c) \leq a \to c$;

P6 $a \to b \leq (a \odot c) \to (b \odot c)$;

P7 $a \to b \leq (c \to a) \to (c \to b)$ and $a \to b \leq (b \to c) \to (a \to c)$.

Let $\mathbb{M} := (M, \leq_{\mathbb{M}})$ be a poset. For $X \subseteq M$, $U_{\mathbb{M}}X$ and $L_{\mathbb{M}}X$ are upper bounds and lower bounds of X in $\mathbb{M}$.

A **multi-supremum** (resp. **multi-infimum**) of X is a minimal (resp. maximal) element of $U_{\mathbb{M}}X$ (resp. $L_{\mathbb{M}}X$). The set of multi-suprema (resp. multi-infima) of X is denoted by Multisup(X) (resp. Multinf(X)).

For simplicity, we write $x \sqcup y$,(resp. $x \sqcap y$) for Multisup($\{x, y\}$), Multinf($\{x, y\}$) in this order for $x, y \in M$.

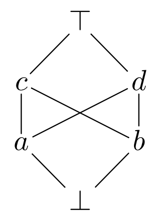
When $x \sqcup y$ (resp. $x \sqcap y$) is singleton $\{a\}$ (resp. $\{b\}$), we write $x \vee y = a$ (resp. $x \wedge y = b$). Note that for every $x, y \in M$, $x \leq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$. We denote the set of natural numbers by $\mathbb{N}$ and we set $x^0 = \top$ and $x^n = x^{n-1} \odot x$, for $n \geq 1$ and $x \in M$.

For a bounded pocrim $\mathcal{M} := (\mathbb{M}, \top, \odot, \to)$ with $\mathbb{M} := (\mathbb{M}, \leq)$ and a least element $\bot$, we set $x^* = x \to \bot$ for every $x \in M$ and let $X^* = \{x^*,\ x \in X\}$ for every $X \subseteq M$. For $a \in M$, $\downarrow_{\mathbb{M}} a = \{x \in M, x \leq a\}$ and $\uparrow_{\mathbb{M}} a = \{x \in M, a \leq x\}$. For $X \subseteq M$, the upper closure of X is $\uparrow_{\mathbb{M}} X = \bigcup\limits_{x \in X} \uparrow_{\mathbb{M}} x$ and the lower closure of X is $\downarrow_{\mathbb{M}} X = \bigcup\limits_{x \in X} \downarrow_{\mathbb{M}} x$.

Definition 1.2. [4] A poset, $(M, \leq)$, is a **multilattice** if and only if it satisfies that, for all $a, b, c \in M$, $a \leq c$ and $b \leq c$ implies that there exists $x \in a \sqcup b$ such that $x \leq c$ and its dual version for $a \sqcap b$.

A multilattice $\mathbb{M}$ is said to be **full** if $a \sqcap b \neq \emptyset$ and $a \sqcup b \neq \emptyset$ for all $a, b \in M$.

An example of a bounded (full) multilattice which is not a lattice is the multilattice with the following Hasse diagram:



ML_6

It is usually denoted by ML_6.

A **residuated multilattice** $\mathcal{M} := (\mathbb{M}, \top, \odot, \to)$ ($\mathcal{RML}$ for short) is a pocrim whose underlying poset is a multilattice.

A $\mathcal{RML}$ is called bounded if it has a lower bound $\bot$.

For convenience and to increase the readability, we summarize the main properties of residuated multilattices needed throughout the paper. These can be either found or derived from some properties in [4].

Proposition 1.3. [4] *The following conditions hold in a $\mathcal{RML}$ $\mathcal{M}$.*

For all $x, y, z \in M$

M1 $x \odot y, x \odot (x \to y) \in \downarrow_{\mathbb{M}} (x \sqcap y)$;

M2 $(x \odot y) \sqcup (x \odot z) \subseteq x \odot (y \sqcup z)$;

M3 $(x \sqcap y) \to z \subseteq \uparrow_{\mathbb{M}} [(x \to z) \sqcup (y \to z)]$;

M4 $(x \sqcup y) \to z \subseteq \downarrow_{\mathbb{M}} [(x \to z) \sqcap (y \to z)]$;

M5 $(x \to z) \sqcap (y \to z) \subseteq (x \sqcup y) \to z$;

M6 $x \to y = \max\{(x \sqcup y) \to y\} = \max\{x \to (x \sqcap y)\}$;

M7 $x \leq x^{**}$, $x^* = x^{***}$, $x^{**} \to y^{**} = y^* \to x^*$, $(x \odot y)^* = x \to y^*$;

M8 $(x \sqcap y)^* \subseteq \uparrow_{\mathbb{M}} (x^* \sqcup y^*)$;

M9 $(x \sqcup y)^* \subseteq \downarrow_{\mathbb{M}} (x^* \sqcap y^*)$;

M10 $(x^* \sqcap y^*) \subseteq (x \sqcup y)^*$.

Definition 1.4. [4] Given a $\mathcal{RML}$ $\mathcal{M}$, a non-empty subset F of M is called:

1) **deductive system** (ds, for short) if (ds-1) $\top \in F$ and (ds-2) for all $x, y \in A$, if $x, x \to y \in F$, then $y \in F$, or equivalently (i) for all $x, y \in F$, $x \odot y \in F$ and (ii) for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$.

3) **m-filter** if
   (i) $x, y \in F$ implies $\emptyset \neq x \sqcap y \subseteq F$;
   (ii) $x \in F$ implies $x \sqcup a \subseteq F$ for all $a \in M$;
   (iii) for all $a, b \in M$, if $(a \sqcup b) \cap F \neq \emptyset$ then $a \sqcup b \subseteq F$.

2) **Filter** if F is a ds satisfying: for all $x, y \in M$, if $x \to y \in F$, then $x \sqcup y \to y \subseteq F$ and $x \to x \sqcap y \subseteq F$.

Definition 1.5. [8] Let $\mathbb{M}$ be a multilattice and X be a non-empty subset of M. (SML-1) X is called a **full sub-multilattice** (f-Sub-multilattice) of M if for all

$x, y \in X$, $x \sqcup y \subseteq X$ and $x \sqcap y \subseteq X$;
(SML-2) X is called a **restricted sub-multilattice** (r-Sub-multilattice) of M if for all $x, y \in X$, $(x \sqcup y) \cap X \neq \emptyset$ and $(x \sqcap y) \cap X \neq \emptyset$.

Definition 1.6. [4] Let $h : M \to M'$ be a map between residuated multilattices, h is said to be a **homomorphism** if h satisfies $h(x \sqcup y) \subseteq h(x) \sqcup h(y)$, $h(x \sqcap y) \subseteq h(x) \sqcap h(y)$, $h(x \odot y) = h(x) \odot h(y)$ and $h(x \to y) = h(x) \to h(y)$ for all $x, y \in M$.

For all homomorphism h between residuated multilattices, one can observe that $h(\top) = \top$ and h is isotone.

Definition 1.7. [7] Let $\mathcal{M}$ be a $\mathcal{RML}$ and X a subset of M. X is a **full sub residuated multilattice** (or f-Sub-$\mathcal{RML}$ for short) if the following conditions hold.

S1. $\top \in X$;
S2. for every $x, y \in X$, $x \odot y \in X$, $x \to y \in X$;
S3. X is an f-Sub-multilattice.

If we replace S3 in the definition above by X is a restricted sub-multilattice, we obtain the definition of restricted sub residuated multilattice (or r-Sub-$\mathcal{RML}$ for short).

In [7], the author proved that the set of complemented elements of a bounded residuated multilattice is a Boolean algebra. Here we summarize all the results on the set of complemented elements needed throughout this work. Let $\mathcal{M}$ be a bounded residuated multilattice. An element $c \in M$ is called **complemented** if there is an element c' such that $\top \in c \sqcup c'$ and $\bot \in c \sqcap c'$ (or equivalently $c \vee c' = \top$ and $c \wedge c' = \bot$). We call c' **complement** of c in $\mathcal{M}$. We denote by $C(\mathcal{M})$ the set of all complemented elements of $\mathcal{M}$.

Proposition 1.8. [7] *Let $\mathcal{M}$ be a $\mathcal{RML}$ and $c \in C(\mathcal{M})$ an element which has a complement $c' \in M$.*

(i) *If c'' is another complement of c then $c' = c''$;*
(ii) *$c = c'^*$, $c' = c^*$ and $c = c^{**}$;*
(iii) *$c \odot c = c$;*
(iv) *$e \in C(\mathcal{M})$ and $x \in M$, $e \odot x \in e \sqcap x$, in particular $e \wedge x$ exists in M and $e \wedge x = e \odot x$;*
(v) *for every $e, f \in C(\mathcal{M})$, $e \wedge f, e \vee f$ exist and belong to $C(\mathcal{M})$. Moreover, $e \wedge f = e \odot f \in C(\mathcal{M})$, and $e \vee f = e^* \to f$;*
(vi) *for every $e \in M$, $e \in C(M)$ if and only if $e \vee e^* = \top$.*

2. f-derivations on residuated multilattices

Definition 2.1. Let $\mathcal{M}$ be a $\mathcal{RML}$ and $d : M \to M$ be a map. We call d an **f-multiplicative derivation** (or simply f-derivation) on M, if there exists a homomorphism $f : M \to M$ such that the following condition is satisfied:

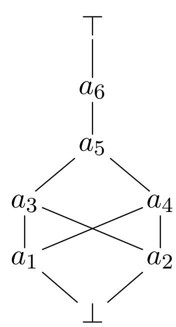
$$d(x \odot y) \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y)) \text{ for all } x, y \in M.$$

It is worth noting that the notion of f-derivation on residuated multilattices generalizes that of derivation on residuated lattices given in [6] and derivation on residuated multilattices given in [7].

Example 2.2. Let $\mathcal{M}$ be a bounded $\mathcal{RML}$. We define $d_\perp, f_\top, d_{id}, d, f$ mappings from M to M by: $d_\perp(x) = \perp$, $f_\top(x) = \top$ and $d_{id}(x) = x$ for all $x \in M$. $d_\perp$ and d_{id} are $f_\top$-derivations on M which shall be called respectively least element $f_\top$-derivation, identity $f_\top$-derivation and d is an f-identity f-derivations on M.

If f is an homomorphism, then f is an f-derivation on M.

Example 2.3. Let $\mathcal{M}$ be the multilattice depicted in the following Hasse Diagram:



We define the operations $\odot$ and $\to$ as follows:

$$x \odot y = \begin{cases} \perp & \text{if } x, y \in M \setminus \{\top\} \\ x & \text{if } y = \top \\ y & \text{if } x = \top \end{cases} \qquad x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x = \top \\ a_6 & \text{otherwise} \end{cases}.$$

Then, $\mathcal{M}$ is a $\mathcal{RML}$. We define functions d and f on M by:

$$d(x) = \begin{cases} \perp & \text{if } x \in M \setminus \{\top\} \\ a_2 & \text{if } x = \top \end{cases} \qquad f(x) = \begin{cases} \perp, & \text{if } x \in M \setminus \{a_6, \top\} \\ \top, & \text{otherwise} \end{cases}$$

Then, it can be easily verified that d is an f-derivation on M.

Example 2.4. Let $f : M \to M$ be a homomorphism of the $\mathcal{RML}$ M and $t \in M$. We define the map $f_t : M \to M$ by: $f_t(x) = f(x) \odot t$ for each $x \in M$. It is easy to prove that f_t is an f-derivation on M which will be called principal f-derivation on M generated by t.

Proposition 2.5. *Let $\mathcal{M}$ be a bounded residuated multilattice and d an f-derivation on M. Then, for all $x, y \in M$,*

(i) *$d(\bot) \leq f(\bot)$, moreover if $f(\bot) = \bot$, then $d(\bot) = \bot$;*
(ii) *$d(x) \geq f(x) \odot d(\top)$;*
(iii) *for $n \geq 2$, $d(x^n) = d(x) \odot f(x^{n-1})$;*
(iv) *If $d(\bot) = \bot$ and $x \leq y^*$, then $d(y) \leq (f(x))^*$ and $d(x) \leq (f(y))^*$;*
(v) *If $d(\bot) = \bot$ then $d(x^*) \leq (f(x))^*$ and $d(x) \leq (f(x^*))^*$.*
(vi) *If $f(\bot) = \bot$ then $d(x) \leq (f(x))^{**}$.*

Proof. (i) $d(\bot) = d(\bot \odot \bot) \in (d(\bot) \odot f(\bot)) \sqcup (f(\bot) \odot d(\bot)) = d(\bot) \odot f(\bot)$. Then, $d(\bot \odot \bot) = d(\bot) \odot f(\bot)$ and $d(\bot) \leq f(\bot)$.
(ii) Let $x \in M$, since $\top$ is the neutral element, we have $d(x) = d(x \odot \top) \in (d(x) \odot f(\top)) \sqcup (f(x) \odot d(\top))$. Hence, $d(x) \geq f(x) \odot d(\top)$.
(iii) Straightforward by induction
(iv) Given $x, y \in M$, it follows from the inequality $x \leq y^*$ that $x \odot y = \bot$. Applying the definition of the derivation, $\bot = d(\bot) = d(x \odot y) \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y))$. We obtain $d(x) \odot f(y) = f(x) \odot d(y) = \bot$ and $d(y) \leq (f(x))^*$, $d(x) \leq (f(y))^*$.
(v) and (vi) Follow from (iv).

□

Definition 2.6. Let $\mathcal{M}$ be a $\mathcal{RML}$ and d an f-derivation on M. We say that d is:
(i) **isotone** if $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in M$;
(ii) **f-contractive** if $d(x) \leq f(x)$ for all $x \in M$;
(iii) an **ideal f-derivation** if d is both isotone and f-contractive.

Example 2.7. Let $\mathcal{M}$ be the residuated multilattice depicted in Example 2.3. Then, it is easy to verify that d is an ideal f-derivation on M. We can also observe that every homomorphism f of $\mathcal{M}$ $f : M \to M$ is an ideal f-identity ($f(x) = x$ for all $x \in M$) f-derivation on M.

Proposition 2.8. *Let $\mathcal{M}$ be a $\mathcal{RML}$ and d an isotone derivation on M, we have:*
(i) *if $z \leq x \to y$, then $f(z) \leq d(x) \to d(y)$ and $f(x) \leq d(z) \to d(y)$ for all $x, y, z \in M$;*
(ii) *$f(x \to y) \leq d(x) \to d(y)$, $d(x \to y) \leq f(x) \to d(y)$ for all $x, y, z \in M$.*

Proof. (i) Let $x, y, z \in M$, $z \leq x \to y$ implies $x \odot z \leq y$ by the isotonicity of d, we have $d(x \odot z) \leq d(y)$ since $d(x \odot z) \in (d(x) \odot f(z)) \sqcup (f(x) \odot d(z))$, we obtain that $d(x) \odot f(z), f(x) \odot d(z) \leq d(x \odot z) \leq d(y)$. Hence, by the adjointness conditions, $f(z) \leq d(x) \to d(y)$ and $f(x) \leq d(z) \to d(y)$;
(ii) It is similar to the proof of (i) using the inequality $x \odot (x \to y) \leq y$.

□

Proposition 2.9. *Let $\mathcal{M}$ be a $\mathcal{RML}$ and d an f-contractive derivation on M. We have the following properties:*
(i) *for all $x, y \in M$, $d(x) \odot d(y) \leq d(x \odot y)$;*

(ii) *let* $x, y \in M$*, for all* $b \in d(x) \sqcup d(y)$*, there exists* $a \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y))$ *such that* $a \leq b$*;*

(iii) *if* d *is isotone, then* $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to f(y)$ *for all* $x, y \in M$*;*

(iv) *if* $d(\top) = \top$*, then* d *is an* f*-identity* f*-derivation on* M*;*

(v) *if for every* $x \in M$*,* $f(x) \odot d(\top) = d(\top)$*, then* $d(x) \geq d(\top)$*.*

Proof. (i) Since d is an f-contractive derivation and $\odot$ is monotone in both arguments, we obtain $d(x) \odot d(y) \leq d(x) \odot f(y)$ and $d(x) \odot d(y) \leq f(x) \odot d(y)$ for all $x, y \in M$. Therefore, $d(x) \odot d(y) \leq a$, for any $a \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y))$. Hence, $d(x) \odot d(y) \leq d(x \odot y)$.

(ii) Let $b \in d(x) \sqcup d(y)$. Since $d(x) \odot f(y) \leq d(x)$ and $f(x) \odot d(y) \leq d(y)$, there exists $a \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y))$ such that $a \leq b$.

(iii) Let $x, y \in M$. Then, by **P2** and the fact that d is isotone, we obtain $d(x \odot (x \to y)) \leq d(y)$. It follows from (i) that $d(x) \odot d(x \to y) \leq d(x \odot (x \to y))$. Therefore, $d(x \to y) \leq d(x) \to d(y)$. Moreover, combining the fact that d is contractive and **P3**, we have $d(x) \to d(y) \leq d(x) \to f(y)$ and $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to f(y)$.

(iv) It follows from Proposition 2.5 (ii).

(v) Since $d(x) = d(x \odot \top) \in (d(x) \odot f(\top)) \sqcup d(\top) = d(x) \sqcup d(\top)$, we obtain $d(\top) \leq d(x)$.

□

Proposition 2.10. *Let* $\mathcal{M}$ *be a* $\mathcal{RML}$ *and* d *an* f*-contractive* f*-derivation on* M*. We have the following properties:*

(i) *For* $x, y \in M$*. If* $y \leq x$*,* $d(x) = f(x)$ *and there exists* $u \in M$ *such that* $x \odot u = y$ *then* $d(y) = f(y)$*;*

(ii) $\mathrm{Fix}_d(M) = \{x \in M, d(x) = f(x)\}$ *is closed under* $\odot$*;*

(iii) *if* $d(\top) = \top$ *and* $f(\bot) = \bot$*, then* $\mathrm{Fix}_d(C(\mathcal{M})) = \{x \in C(\mathcal{M}), d(x) = f(x)\}$ *is a full sub residuated multilattice of* M*;*

(iv) $d(\top) = \top$ *if and only if* $\mathrm{Fix}_d(M) = M$*.*

Proof. (i) Let $x, y \in M$ and $y \leq x$ with $u \in M$ such that $x \odot u = y$. Assume that $d(x) = f(x)$. By definition $d(y) = d(x \odot u) \in (d(x) \odot f(u)) \sqcup (f(x) \odot d(u)) = (f(x) \odot f(u)) \sqcup (d(x) \odot d(u)) = f(x) \odot f(u)$ since $d(x) = f(x)$ and d is f-contractive. We obtain $d(y) = f(x \odot u) = f(y)$.

(ii) Let $x, y \in \mathrm{Fix}_d(M)$, $d(x \odot y) \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y)) = f(x) \odot f(y) = f(x \odot y)$.

(iii) It is obvious that $\top \in \mathrm{Fix}_d(C(\mathcal{M}))$. Proposition 1.8 assure that $\odot$ and $\wedge$ coincide and by (ii), $\mathrm{Fix}_d(C(\mathcal{M}))$ is closed under $\wedge$. Let $x, y \in C(\mathcal{M})$, we have $x \sqcup y = x \vee y = x^{**} \vee y^{**} = (x^* \wedge y^*)^* \in \mathrm{Fix}_d(C(\mathcal{M}))$ by (iii). Moreover, by Proposition 1.8 (iv) and (v) $x \to y = x^* \vee y = (x \wedge y^*)^* = (x \odot y^*)^*$ which implies that $(x \odot y^*)^* \in \mathrm{Fix}_d(C(\mathcal{M}))$. Therefore, $d((x \odot y^*)^*) = f((x \odot y^*)^*) = f(x^* \vee y) = f(x^*) \vee f(y) = f(x)^* \vee f(y) = f(x) \to f(y) = d(x) \to d(y)$. Hence, $\mathrm{Fix}_d(C(\mathcal{M}))$ is a full sub residuated multilattice of M.

(v) Straightforward.

□

Proposition 2.11. *Let $\mathcal{M}$ be a bounded $\mathcal{RML}$ and d an f-derivation on $C(\mathcal{M})$. We have the following properties:*

(i) *$d(x) = d(x) \odot f(x)$, for all $x \in C(\mathcal{M})$;*
(ii) *if $f(\bot) = \bot$ and $d(\top) = \top$, then the f-derivation in the set of all complemented elements of a residuated multilattice $\mathcal{M}$ coincide with the f-derivation in lattice.*

Proof. (i) It follows from Proposition 2.5 (iii).
(ii) Firstly, we will prove that $f(C(\mathcal{M})) \subseteq C(\mathcal{M})$. From Proposition 1.8 (vi), we need only to prove that $f(x) \sqcup f(x)^* = \top$ for all $x \in C(\mathcal{M})$ and it is obvious since $f(x) \sqcup f(x)^* \supseteq f(x \sqcup x^*) = f(\top) = \top$. Secondly by Definition 2.1 $d(x) \in d(x) \sqcup f(x)$ and $f(x) \leq d(x)$. Moreover, because $d(x) = d(x) \odot f(x)$, we have $d(x) \leq f(x)$. Hence, $d(x) = f(x) \in C(\mathcal{M})$. $d(x \odot y) \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y)) = (d(x) \wedge f(y)) \sqcup (f(x) \wedge d(y)) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) = d(x \wedge y)$.

□

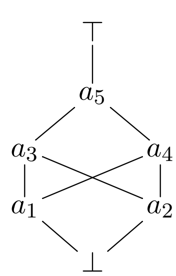
3. Principal, ideal and good ideal f-derivations

In this section, we study the importance of complemented elements in the study of principal, ideal and good ideal f-derivations of a residuated multilattice.

We denote by M^M the set of all maps from M to M. We define a binary relation $\leq$ by $f \leq g \Leftrightarrow f(x) \leq g(x)$ for every $x \in M$. It is easy to see that $(M^M, \leq)$ is a poset. We define the hyperoperations $\sqcup$ and $\sqcap$ by $f \sqcup g, f \sqcup g : M \to 2^M$ by $(f \sqcup g)(x) = f(x) \sqcup g(x)$ and $(f \sqcap g)(x) = f(x) \sqcap g(x)$, for every $f, g \in M^M$. It is clear that $(M^M, \sqcup, \sqcap)$ is a multilattice.

Definition 3.1. An ideal f-derivation d is said to be good if $d(\top) \in C(\mathcal{M})$.

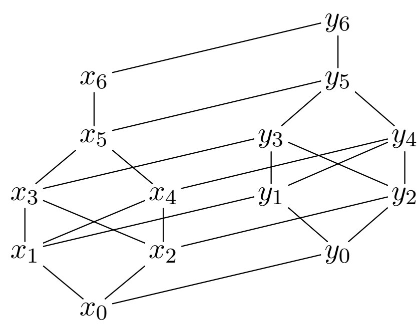
Example 3.2. Let **2** be the two-element Boolean algebra and M be the residuated multilattice depicted in the following figure:



Consider the subsets: $C = \{a_2, a_3, a_4, a_5\}$. The operations $\odot$ and $\to$ are defined as follows:

$$x \odot y = \begin{cases} a_2 & \text{if } x, y \in C \\ x & \text{if } y = \top \\ y & \text{if } x = \top \\ \bot & \text{otherwise} \end{cases} \qquad x \to y = \begin{cases} \top & \text{if } x \leq y \\ a_1 & \text{if } x \in C \text{ and } y \in \{\bot, a_1\} \\ y & \text{if } x = \top \\ a_5 & \text{otherwise} \end{cases}$$

The direct product $M \times \mathbf{2}$ of M and $\mathbf{2}$ is a $\mathcal{RML}$. Note that if one sets $x_0 = (\bot, 0), x_i = (a_i, 0)(1 \leq i \leq 5), x_6 = (\top, 0), y_0 = (\bot, 1), y_i = (a_i, 1)(1 \leq i \leq 5), y_6 = (\top, 1)$, then the multilattice structure of $M \times \mathbf{2}$ is described in the following Hasse diagram.



Multiplication by x_6, i.e., $d_{x_6}(x) = x_6 \odot f(x)$ yields a principal f-derivation on $M \times \mathbf{2}$.

We denote by $PD_f(M^M)$ and $GID_f(M^M)$ respectively the set of all principal f-derivations from M to M and the set of all good ideal f-derivations from M to M.

Remark 3.3. Let $\mathcal{M}$ be a bounded $\mathcal{RML}$ and $f : M \to M$ a homomorphism. We can notice that every principal f-derivation is isotone and the homomorphism f is the greatest element in $PD_f(M^M)$.

Proposition 3.4. *Let $f : M \to M$ be a map. Then:*

(i) *$GID_f(M^M)$ is an f-Sub-multilattice of M^M;*

(ii) *$(GID_f(C(\mathcal{M})^{C(\mathcal{M})}), \odot, \mapsto, \sqcup, \sqcap, d_\bot, f)$ is a bounded residuated multilattice where, $d_\bot(x) = \bot$, $(d_1 \mapsto d_2)(x) = d_1(x) \to d_2(x)$ and $(d_1 \odot d_2)(x) = d_1(x) \odot d_2(x)$ for all $x \in C(\mathcal{M})$.*

Proof. (i) Let d_1, d_2 be two good ideal f-derivations on M and $x \in M$. By the definition of f-derivation and Theorem 1.8 (iv) we have,

$$\begin{aligned}(d_1 \sqcap d_2)(x) &= d_1(x) \sqcap d_2(x) = (f(x) \odot d_1(\top)) \sqcap (f(x) \odot d_2(\top)) \\ &= (f(x) \wedge d_1(\top)) \sqcap (f(x) \wedge d_2(\top)).\end{aligned}$$

Using Theorem 1.8 (iv) $f(x) \odot (d_1 \sqcap d_2)(\top) = f(x) \odot ((d_1(\top) \sqcap d_2(\top)) = f(x) \odot ((d_1(\top) \wedge d_2(\top)) = f(x) \wedge ((d_1(\top) \wedge d_2(\top)) = f(x) \wedge f(x) \wedge ((d_1(\top) \wedge d_2(\top)) = (f(x) \wedge d_2(\top)) \wedge (f(x) \wedge d_1(\top)) = (f(x) \wedge d_2(\top)) \sqcap (f(x) \wedge d_1(\top)) = (d_1 \sqcap d_2)(x)$.

Hence, $d_1 \sqcap d_2$ is a good ideal f-derivation on M.

In addition,

$$\begin{aligned}(d_1 \sqcup d_2)(x) &= d_1(x) \sqcup d_2(x) = (f(x) \odot d_1(\top)) \sqcup (f(x) \odot d_2(\top)) \\ &\subseteq f(x) \odot (d_1(\top) \sqcup d_2(\top)) = f(x) \odot (d_1(\top) \vee d_2(\top)) \\ &= f(x) \odot (d_1 \sqcup d_2)(\top) \text{ By } \mathbf{M2} \text{ Proposition 1.3 .}\end{aligned}$$

We have $d_1 \sqcup d_2$ is a good ideal f-derivation on M and we conclude that $GID_f(M^M)$ is an f-Sub-multilattice of M^M.

(ii) From (i) $(GID_f(C(\mathcal{M})^{C(\mathcal{M})}), \sqcup, \sqcap)$ is a multilattice. It is easy to prove that $(GID_f(C(\mathcal{M})^{C(\mathcal{M})}, \odot, f)$ is a commutative monoid with the neutral element f and $(\odot, \rightarrow)$ is an adjoint pair.

□

Theorem 3.5. *Let $\mathcal{M}$ be a $\mathcal{RML}$ and d an f-contractive derivation on M with $d(x) \in C(\mathcal{M})$ for all $x \in M$. The following propositions are equivalent:*

(i) *d is an isotone f-derivation;*
(ii) *for all $x, y \in M$, $d(x \odot y) = d(x) \odot d(y) = d(x) \odot f(y)$.*

Proof. (i)⇒(ii) Let $x, y \in M$; by the hypothesis $d(x \odot y) \leq d(x), d(y)$. Then, there exists $a \in d(x) \sqcap d(y)$ such that $d(x \odot y) \leq a$. As $d(x), d(y) \in C(\mathcal{M})$, we obtain by Proposition 1.8 (iv) that $d(x) \sqcap d(y) = d(x) \wedge d(y) = d(x) \odot d(y)$ and $d(x \odot y) \leq d(x) \odot d(y)$. Furthermore, by Proposition 2.9 we have $d(x \odot y) = d(x) \odot d(y)$. Hence, $d(x) \odot d(y) = d(x \odot y) \in (d(x) \odot f(y)) \sqcup (f(x) \odot d(y))$. Therefore $d(x) \odot f(y), f(x) \odot d(y) \leq d(x) \odot d(y)$. By the fact that d is f-contractive and $\odot$ monotone, we obtain, $d(x) \odot f(y), f(x) \odot d(y) \leq d(x) \odot d(y) \leq d(x) \odot f(y), f(x) \odot d(y)$. Hence, $d(x \odot y) = d(x) \odot f(y)$ for all $x, y \in M$.

(ii)⇒(i) Let $x, y \in M$ and $a \in x \sqcap y$, by the hypothesis $d(a) = d(\top) \odot f(a) \in d(\top) \odot (f(x) \sqcap f(y))$. Furthermore, $d(\top) \odot (f(x) \sqcap f(y)) \subseteq\downarrow_{\mathbb{M}} [(d(\top) \odot f(x)) \sqcap (d(\top) \odot f(y))] =\downarrow_{\mathbb{M}} [d(x) \sqcap d(y)]$. So, there exists $z \in d(x) \sqcap d(y)$ such that $d(a) = d(\top) \odot f(a) \leq z$. In particular, for $x \leq y$ we have $d(x) = d(\top) \odot f(x) \leq z$ where $z \in d(x) \sqcap d(y)$. Hence, $d(x) \leq d(y)$.

□

Theorem 3.6. *Let $\mathcal{M}$ be a $\mathcal{RML}$ and d an f-contractive derivation on M which satisfies $d(\top) \in C(\mathcal{M})$ for all $x \in M$. Then, the following are equivalent:*

(i) *d is an ideal f-derivation;*
(ii) *$d(x) \leq d(\top)$ for all $x \in M$;*
(iii) *for all $x \in M$, $d(x) = f(x) \odot d(\top)$;*

(iv) *for all* $x, y \in M$, *if* $a \in x \sqcap y$ *then there exists* $z \in d(x) \sqcap d(y)$ *such that* $d(a) \leq z$;

(v) *for all* $x, y \in M$, $d(x \sqcup y) \subseteq d(\top) \odot (f(x) \sqcup f(y))$;

(vi) $d(x \odot y) = d(x) \odot d(y)$ *for all* $x, y \in M$.

Proof. (i)⇒(ii) straightforward.

(ii)⇒(iii) Let $x \in M$, $d(x) \leq d(\top)$ implies $d(x) = d(x) \wedge d(\top) = d(x) \odot d(\top) \leq d(\top) \odot f(x)$. Proposition 2.5 (ii) yields, $d(\top) \odot f(x) \leq d(x)$; so $d(x) = d(\top) \odot f(x)$.

(iii)⇒(i) Follow from the fact that $\odot$ is increasing in both arguments and f is isotone.

(iii)⇒(iv) Let $x, y \in M$ and $a \in x \sqcap y$, from (ii) we have $d(a) = d(\top) \odot f(a)$. Furthermore, $d(\top) \odot f(a) \in d(\top) \odot (f(x) \sqcap f(y)) \subseteq_{\downarrow \mathbb{M}} [(d(\top) \odot f(x)) \sqcap (d(\top) \odot f(y))] =_{\downarrow \mathbb{M}} [d(x) \sqcap d(y)]$. So, there exists $z \in d(x) \sqcap d(y)$ such that $d(a) = d(\top) \odot a \leq z$.

(iv)⇒(i) Let $x, y \in M$ such that $x \leq y$, thus $x \in x \sqcap y$. Using hypothesis, there exists $z \in d(x) \sqcap d(y)$ such that $d(x) \leq z$. Hence, $d(x) \leq d(y)$.

(iii)⇒(v) By the definition of d and the hypothesis, $d(x \sqcup y) = d(\top) \odot f(x \sqcup y) \subseteq d(\top) \odot (f(x) \sqcup f(y))$

(v)⇒(i) Let $x \leq y$, then $x \sqcup y = y$. By hypothesis $d(y) = d(x \sqcup y) \subseteq d(\top) \odot (f(x) \sqcup f(y))$ which implies that there exists $a \in (f(x) \sqcup f(y))$ such that $d(y) = d(\top) \odot a \geq d(\top) \odot f(x) = d(x)$. Hence, $d(x) \leq d(y)$.

(iii)⇒(vi) Let $x, y \in M$ $d(x \odot y) = d(\top) \odot f(x \odot y) = d(\top) \odot d(\top) \odot f(x) \odot f(y) = (d(\top) \odot f(x)) \odot (d(\top) \odot f(y)) = d(x) \odot d(y)$.

(vi)⇒(ii) Let $x \in M$, we have $d(x) = d(x \odot \top) = d(x) \odot d(\top)$, hence $d(x) \leq d(\top)$.

□

The next result shows that good ideal f-derivations on a $\mathcal{RML}$ are in one-to-one correspondence with the complemented elements of the $\mathcal{RML}$.

Proposition 3.7. *A f-derivation d on M is a good ideal f-derivation if and only if there exists a unique $a \in C(M)$ such that $d = d_a$*

Proof. Let d be a good ideal f-derivation on M. By Theorem 3.6 (iii) $d(x) = f(x) \odot d(\top)$. It is obvious that $a = d(\top)$ is unique. Conversely, assume that there exists a unique $a \in C(M)$ such that $d = d_a$. By definition we have $d_a(x) = a \odot f(x)$ which is an f-contractive and an isotone f-derivation. We can conclude that $d = d_a$ is a good ideal f-derivation on M due to $a \in C(M)$.

□

Let $GID_f(M^M)$ denotes the set of all good ideal f-derivations on M. Then by the preceding Proposition, $GID_f(M^M) = \{d_x : x \in C(M)\}$. In addition, define $\preceq, \otimes, \twoheadrightarrow$ on $GID_f(M^M)$ by $d_x \preceq d_{x'}$ if $x \leq x'$, $d_x \otimes d_{x'} = d_{x \odot x'}$ and $d_x \twoheadrightarrow d_{x'} = d_{x \rightarrow x'}$. Then it is straightforward to see that $(GID_f(M^M), \preceq, \otimes, \twoheadrightarrow, d_\perp, d_\top)$ is a Boolean algebra that is naturally isomorphic to the Boolean algebra $C(M)$.

In the remaining section, we show that the set of f-fixed points of an ideal f-derivation is a multilattice.

Proposition 3.8. *Let $\mathcal{M}$ be a $\mathcal{RML}$ and d a good ideal f-derivation on M. Then,* $\mathrm{Fix}_d(M)$ *is closed under the product. Moreover,* $\mathrm{Fix}_d(M)$ *is a full sub-multilattice.*

Proof. Let $x, y \in \mathrm{Fix}_d(M)$. Since d is f-contractive, $d(x \odot y) \leq x \odot y$. It follows from Proposition 2.9 that $f(x \odot y) = f(x) \odot f(y) = d(x) \odot d(y) \leq d(x \odot y)$. Hence $x \odot y \in \mathrm{Fix}_d(M)$.

Let $x, y \in \mathrm{Fix}_d(M)$, $d(x \sqcup y) = d(\top) \odot f(x \sqcup y) \subseteq d(\top) \odot (f(x) \sqcup f(y)) = d(\top) \odot (d(x) \sqcup d(y)) = d(\top) \wedge (d(x) \sqcup d(y)) = d(x) \sqcup d(y)$. Furthermore, $f(x \sqcup y) \subseteq f(x) \sqcup f(y) = d(x) \sqcup d(y)$. Using the fact that d is a contractive f-derivation, we have $d(a) \leq f(a)$ with $d(a), f(a) \in d(x) \sqcup d(y)$ and $d(a) = f(a)$ for all $a \in x \sqcup y$. Therefore, $x \sqcup y \subseteq \mathrm{Fix}_d(M)$.

For all $x, y \in M$, $d(x \sqcap y) = d(\top) \odot f(x \sqcap y) \subseteq d(\top) \odot (f(x) \sqcap f(y)) = d(\top) \odot (d(x) \sqcap d(y))$. For every $a \in d(x) \sqcap d(y)$, $a \leq d(x), d(y) \leq d(\top)$ and $d(\top) \odot a = d(\top) \wedge a = a$. This implies that $d(\top) \odot (d(x) \sqcap d(y)) = d(x) \sqcap d(y)$ and $d(x \sqcap y) \subseteq d(x) \sqcap d(y)$. But, $f(x \sqcap y) \subseteq f(x) \sqcap f(y) = d(x) \sqcap d(y)$. Using the fact that d is f-contractive, we obtain $d(x \sqcap y) = f(x \sqcap y)$. □

4. Conclusion and Final Remarks

The notion of derivation is a powerful tool for studying structural properties of different algebras. We initiated the study of f-derivations on residuated multilattices as a natural generalization of derivations on residuated multilattices and residuated lattices by introducing definitions with clear examples. The set of all complemented elements of a residuated multilattices enabled us to characterize good ideal f-derivations by using the image of the top element as a generator, which offers us a complete description of good ideal f-derivations. Various sets related to derivations were investigated and found to carry nice substructures.

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