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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF EQUATION WITH A NON-SMOOTH POTENTIAL

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Abstract. This paper deals with the existence and multiplicity of solutions for a class of nonlocal $p-$Kirchhoff problem. Using the mountain pass theorem and fountain theorem, we establish the existence of at least one solution and infinitely many solutions for a class of locally Lipschitz functional.

Key Words: p-Kirichhoff problem, Symmetric Mountain Pass theorem, fountain theorem, hemivariational inequality.

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1. Introduction

We are concerned with the study of the nonlocal elliptic problem inclusion

$$\begin{cases} M\Big(\int_\Omega |\nabla u(x)|^p dx\Big)\triangle_p u + \lambda |u|^{p-2}u \in -\partial F(x,u) & in \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$. Assume that $p > 1$ is a real number and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ denote the $p-$Laplace operator. λ is a real parameter and $0 < \lambda < \frac{m_0\lambda_*}{m_1}$. $F : \Omega\times\mathbb{R} \to \mathbb{R}$ is locally Lipschitz function and $\partial F(x,u)$ denote the generalized Clarke gradient of $F(x,u)$.
$u \in X$ is said to be a *weak solution* of problem (1.1), if

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$M\Big(\int_\Omega |\nabla u(x)|^p dx\Big) \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx$
$-\lambda \int_\Omega |u(x)|^{p-2}u(x)v(x)dx \quad - \int_\Omega F^0(x,u(x);v(x))dx \ \geq\ 0, \quad \forall v \in X.$
Problem (1.1) is called *nonlocal* because of the presence of the term M , which implies that the equation in (1.1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particulary interesting. Nonlocal differential equations are also called Kirichhoff-type equations because Kirichhoff (cf. [10]) has investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. The parameters in (1.2), have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.
We point out λ_* is the Rayleigh quotient associated with our problem, that is

$$\lambda_* := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\frac{1}{p}\widehat{M}\Big(\int_\Omega |\nabla u(x)|^p dx\Big)}{\int_\Omega \frac{1}{p}|u|^p dx}.$$

Recently, the investigation of existence and multiplicity of solutions for Kirchhoff-type problems has drawn the attentions of many authors. In (cf. [14]) authors studied the existence of solutions for Kirchhoff type equations with Dirichlet boundary-value condition by using mountain pass theorem in critical point theory, without the (PS) condition; In (cf. [11]) authors studied the existence and multiplicity of solutions to the fractional Kirchhoff-type problem by using critical point theorems and the truncation technique; In (cf. [12]) authors studied the existence and multiplicity of solutions for the problem Kirchhoff-type problem by variational methods.
In the past decades, the existence and uniqueness of solutions to several classes of nonlinear inclusions for variational and hemivariational inequalities has drawn the attentions of many authors (cf. [3], [6], [13], [1]).

These types of inequalities have established a new subject in nonsmooth analysis, due to these fields are based on the subdifferential in the sense of Clarke of locally Lipschitz functionals. The theory of hemivariational inequalities has given significant consequence both in pure and applied mathematics and it is valuable to comprehend many problems of mechanics and engineering for non-convex, non-smooth functionals.

The applications to non-smooth variational problems have been seen in many papers; in (cf. [9]) authors studied the existence of three distinct nontrivial solutions for a nonlocal perturbed Kirchhoff-type variational-hemivariational inequalities based on three critical point theorems for non-smooth functional due to Bonanno and Winkert; in (cf. [8]) authors studied the existence of nontrivial positive solutions for a Kirchhoff type variational inequality based on the non-smooth critical point theory for Szulkin-type functionals; in (cf. [16]) authors studied a class of p(x)-Kirchhoff type problem with a subdifferential term by the method of lower-upper solutions, penalization techniques, truncations; in (cf. [15]) author studied the existence of solutions to a variational inequality involving nonlocal elliptic operators based on Schauders fixed point theorem. In the present paper, we consider the existence and multiplicity of solutions for Kirchhoff-type problem to a class of locally Lipschitz functionals. The main tool used in our paper is applying the non-smooth version of the symmetric mountain pass theorem is the principle of criticality and the fountain theorem for a locally Lipschitz functional.

We start the paper by giving in Section 2 the basic notions and facts of the Sobolev space, definitions and properties for the generalized gradient of the locally Lipschitz functionals. The main part is concerned with the existence at least one solution and infinitely many solutions for a class of nonlinear hemivariational inequalities on bounded domain by using non-smooth symmetric Mountain pass theorem and fountain theorem.

2. Preliminaries

Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space, equipped with the norm

$$\|u\| := \Big(\int_\Omega (|\nabla u(x)|^p - |u(x)|^p) dx \Big)^{\frac{1}{p}}.$$

Let p^* denote the critical exponent related to p, on $\bar{\Omega}$

$$p^* = \frac{Np}{N-p}.$$

For p, q in which $q \in [1, p^*]$ for each $x \in \overline{\Omega}$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \qquad \forall u \in W_0^{1,p}(\Omega).$$

For every $u \in W_0^{1,p}(\Omega)$ the Poincaré inequality holds, i.e., there exists a positive constant C_p in which

$$\|u\|_{L^{p}((\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}.$$

In this part we give a brief overview on some prerequisites on non-smooth analysis which are needed in the sequel. Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $< \cdot, \cdot >$ the duality brackets for the pair $(X, X^\star)$. A function $f : X \to \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$ there exists a neighbourhood U of x and a constant $K > 0$ depending on U such that $|f(y) - f(z)| \leq K\|y - z\|$ for all $y, z \in U$.
Let $f : X \to \mathbb{R}$ be a *locally Lipschitz* functional and $u, v \in X, v \neq 0$. The generalized directional derivative of f in u with respect to the direction v is

$$f^0(u; v) = \limsup_{w \to u, \lambda \to 0^+} \frac{f(w + \lambda v) - f(w)}{\lambda}.$$

The generalized gradient of f at $u \in X$ is defined by

$$\partial f(u) = \{x^\star \in X^\star \ : < x^\star, v >_X \leq f^0(u; v), \ \forall v \in X\},$$

which is a nonempty, convex and $w^\star-$compact subset of $X^\star$, where $< \cdot, \cdot >_X$ is the duality pairing between $X^\star$ and X.

Definition 2.1. A point $x \in X$ is said to be a critical point of a locally Lipschitz functional $f : X \to \mathbb{R}$ if $0 \in \partial f(x)$, that is, $f^0(x, v) \geq 0$, for every $v \in X \backslash \{0\}$.

Proposition 2.2. *(cf. [4]) Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for every $u, v \in X$ the following conditions hold:*
(1) *$h^0(u; \cdot)$ is subadditive, positively homogeneous;*
(2) *∂h is convex and weak* compact;*
(3) $(-h)^0(u; v) = h^0(u; -v)$;
(4) *the set-valued mapping $h : X \to 2^{X^*}$ is weak* u.s.c.;*
(5) $h^0(u; v) = \max\{< \xi, v > : \ \xi \in \partial h(u)\}$;
(6) *$\partial(\lambda h)(u) = \lambda \partial h(u)$ for every $\lambda \in \mathbb{R}$;*
(7) $(h + g)^0(u; v) \leq h^0(u; v) + g^0(u; v)$;
(8) *the function $m(u) = \min_{\nu \in \partial h(u)} \nu_{X^*}$ exists and is lower semi continuous; i.e.,*
$\liminf_{u \to u_0} m(u) \geq m(u_0)$;
(9) $h^0(u; v) = \max_{u^* \in \partial h(u)} \langle u^*, v \rangle \leq L\|v\|$.

Definition 2.3. The functional $I : X \to X^\star$ verifies the property $(\mathcal{S}_+)$ if for any weakly convergence sequence $\{u_n\}_n \subset X$ to u in X and

$$\limsup_{n\to\infty} < I(u_n), u_n - u > \leq 0,$$

implies the strong convergence of $\{u_n\}_n$ to u in X.

Proposition 2.4. *(cf. [5]) For*

$$A(u) = \int_\Omega \frac{1}{p} |\nabla u(x)|^p dx, \quad \forall u \in X,$$

then $A(u) \in C^1(X, \mathbb{R})$ and the derivative operator A' of A by

$$< A'(u), v >= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx,$$

is of type (S_+).

To indicate the existence for solution of (1.1), we consider a functional $\mathcal{I}(u) = \Phi(u) - \lambda\varphi(u) - \mathcal{F}(u)$ associated to (1.1), where

$$\Phi(u) := \frac{1}{p} \widehat{M}\Big(\int_\Omega |\nabla u(x)|^p dx \Big), \tag{2.1}$$

and $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. The Fréchet derivative of Φ is $\Phi' : X \to X^*$, where

$$\langle \Phi'(u), v \rangle = M\Big(\int_\Omega |\nabla u(x)|^p dx \Big) \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx,$$

$$\varphi(u) = \int_\Omega \frac{1}{p} |u|^p dx, \qquad \mathcal{F}(u) = \int_\Omega F(x, u(x)) dx.$$

Proposition 2.5. *(cf. [4]) Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz functional. Then $\mathcal{F}$ is a well-defined locally Lipschitz map. Moreover,*

$$\mathcal{F}^0(u; v) \leq \int_\Omega F^0(x, u(x); v(x)) dx, \qquad \forall u, v \in W_0^{1,p}(\Omega).$$

3. Main Result

Here, $F(x, s)$ and $M(t)$ are always supposed to verify the following assumption:
We assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, it is locally Lipschitz in the second variable and satisfying the following properties:

(F_1) $|F(x,s)| \leq a|s|^\theta + b$ for all $(x,s) \in \Omega \times \mathbb{R}$ with $a, b \geq 0$ and $1 < p < \theta \leq p^*$;

(F_2) there exists $\gamma > -\lambda$ such that

$$\limsup_{|s|\to 0} F(x,s) \leq \frac{-\gamma|s|^p}{p}, \quad \text{uniformly for almost all } x \in \Omega;$$

(F_3) There exists $u \in X\backslash\{0\}$ such that

$$\upsilon\|u\|^p \leq \int_\Omega F(x,u(x))dx;$$

where $\upsilon = \max\{\frac{m_1^{\beta-1}}{p}, \frac{\lambda}{p}\}$.

(F_4) $F(x,s) \geq d_1|s|^\mu - d_2$, $\forall (x,s) \in \Omega \times \mathbb{R}$, where d_1, d_2 are positive constants and $\mu > p$;

(M_1) There exist $m_1 \geq m_0 > 0$ such that for all $t \in \mathbb{R}^+$, $m_0 \leq M(t) \leq m_1$;

(M_2) For all $t \in \mathbb{R}^+$, $\widehat{M}(t) \geq M(t)t$.

For a given locally Lipschitz function $f : X \to \mathbb{R}$, we set

$$m(x_n) = \min\{\|x^*\|_* : x^* \in \partial f(x_n)\}.$$

We say that f satisfies the non-smooth Cerami condition at the level $c \in \mathbb{R}$, if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{f(x_n)\}_{n\geq 1} \to c$ and $(1 + \|x_n\|_X)m_f(x_n) \to 0$, has a strongly convergent subsequence. If this property holds at every level $c \in \mathbb{R}$, then we say that f satisfies the *Cerami condition*.

Next theorem is due to Ambrosetti-Rabinowitz and (cf. [2]) and extends to a non-smooth setting the well known mountain pass theorem.

Theorem 3.1. *If X is a reflexive Banach space, $\mathcal{I} : X \to \mathbb{R}$ a locally Lipschitz functional satisfying the Cerami condition and for some $r > 0$, $w \in X$ in which $\|w\| > r$,*

$$\max\{\mathcal{I}(0), \mathcal{I}(w)\} < \inf_{\|x\|=r}\{\mathcal{I}(x)\} =: \gamma$$

then $\mathcal{I}$ has a nontrivial critical point $x \in X$ such that the critical value $c = \mathcal{I}(x) \geq \gamma$ is characterized by the following minimax expression

$$c = \inf_{f\in\Gamma}\sup_{t\in[0,1]} \mathcal{I}(f(t)),$$

where

$$\Gamma = \{f \in C([0,1], X) : f(0) = 0, f(1) = e\}.$$

For proving our main result, we need some lemmas and a proposition.

Lemma 3.2. *Suppose that conditions M_1, M_2 satisfy and $\lambda \in (0, \frac{m_0\lambda_*}{m_1})$. Then every Cerami sequence for the functional $\mathcal{I}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $\{\mathcal{I}(u_n)\}_{n\geq 1}$ is bounded and $(1+\|u_n\|_X)m(u_n) \to 0$ as $n \to \infty$. We will show that the sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.
Assume by contradiction, there is a subsequence, we can suppose that $\|u_n\| \to \infty$ as $n \to \infty$.
Let $z_n = \frac{u_n}{\|u_n\|}$ for all $n \geq 1$. Up to a subsequence, we assume that

$$\begin{aligned} z_n \to z \quad &\text{in} \quad L^p, \\ z_n(x) \to z(x) \quad &\text{a.e. in} \quad \Omega, \\ z_n \rightharpoonup z \quad &\text{in} \quad X, \end{aligned}$$

as $n \to \infty$.
Due to $(1+\|u_n\|_X)m(u_n) \to 0$,

$$\begin{aligned} |\mathcal{I}^\circ(u_n; v)| &= |\langle \Phi'(u_n), v\rangle - \lambda\langle \varphi'(u_n), v\rangle - \int_\Omega F^\circ(x, u_n; v)dx| \\ &\leq \frac{\epsilon_n\|v\|}{1+\|u_n\|}, \end{aligned} \tag{3.1}$$

for every $u \in W_0^{1,p}(\Omega)$, with $\epsilon_n \searrow 0$.
Choosing $v := u_n$ in (3.1), we obtain

$$m_0 \int_\Omega |\nabla u_n|^p dx - \int_\Omega \lambda |u_n|^p dx - \int_\Omega F^\circ(x, u_n; u_n)dx| \leq \epsilon_n. \tag{3.2}$$

Boundedness of $\{\mathcal{I}(u_n)\}_{n\geq 1}$, implies that

$$\widehat{M}(\int_\Omega |\nabla u_n|^p dx) - \int_\Omega \lambda |u_n|^p dx - p\int_\Omega F(x, u_n)dx \leq pM_1. \tag{3.3}$$

There are two cases.
Case 1. $\lambda > 0$. From the definition of λ_*,

$$\lambda_* \int_\Omega |u_n|^p dx \leq \int_\Omega m_1 |\nabla u_n|^p dx, \quad n \in \mathbb{N}. \tag{3.4}$$

Using (3.4) in (3.2),

$$(m_0 - \frac{m_1\lambda}{\lambda_*})\int_\Omega |\nabla u_n|^p dx - \int_\Omega F^\circ(x, u_n; u_n)dx \leq \epsilon_n. \tag{3.5}$$

According to proposition 2.2 and dividing the (3.5), by $\|u_n\|^p$,

$$(m_0 - \frac{m_1\lambda}{\lambda_*})\int_\Omega |\nabla z_n|^p dx \leq \frac{\epsilon_n}{\|u_n\|^p} + \frac{L|\Omega|}{\|u_n\|^p}. \tag{3.6}$$

Since $(m_0 - \frac{m_1\lambda}{\lambda_*}) > 0$, as $n \to \infty$ in (3.6), then

$$\nabla z_n \to 0 \quad \text{in} \quad L^p(\Omega). \tag{3.7}$$

Case 2. $\lambda \leq 0$. Using (3.1),

$$m_0 \int_\Omega |\nabla u_n|^p dx - \int_\Omega F^\circ(x, u_n; u_n)dx \leq \epsilon_n. \tag{3.8}$$

Dividing the inequality by $\|u_n\|^p$,

$$m_0 \int_\Omega |\nabla z_n|^p dx \leq \frac{\epsilon_n}{\|u_n\|^p} + \frac{L|\Omega|}{\|u_n\|^p}. \tag{3.9}$$

Since $m_0 > 0$, so

$$\nabla z_n \to 0 \quad \text{in} \quad L^p(\Omega). \tag{3.10}$$

Using (3.4) in (3.2) in another way,

$$(\frac{m_0\lambda_*}{m_1} - \lambda)\int_\Omega |u_n|^p dx - \int_\Omega F^\circ(x, u_n; u_n)dx \leq \epsilon_n. \tag{3.11}$$

Similarly, considering in two cases

$$z_n \to 0 \quad \text{in} \quad L^p(\Omega). \tag{3.12}$$

From (3.7), (3.10) and (3.12),

$$z_n \to 0 \quad \text{in} \quad X. \tag{3.13}$$

Denote by $z_n = \frac{u_n}{\|u_n\|}$ then $\|z_n\| = 1$ for all $n \geq 1$ which represents a contradiction with (3.13). So $\{u_n\}_{n\geq 1} \subseteq X$ is bounded.

□

Remark 3.3. From (M_1) and proposition (2.4) we can easily see that Φ' is of (S_+) type.

Lemma 3.4. *Suppose that conditions M_1, M_2 are satisfied and $\lambda \in (-\infty, \frac{m_0\lambda_*}{m_1})$. Then $\mathcal{I}$ satisfies the compactness condition(C).*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a Cerami sequence. Then, by Lemma (3.2), the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad X,$$
$$u_n \to u \quad \text{in} \quad L^p(\Omega).$$

Since $\partial\mathcal{I}(u_n) \subseteq X^\star$ is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can assume that $u_n^* \in \partial\mathcal{I}(u_n)$ such that

$$\|u_n^*\|_* = m(u_n), \quad \text{for} \quad n \geq 1.$$

Then, for every $n \geq 1$,

$$u_n^* = \Phi' u_n - \lambda |u_n|^{p-2} u_n - v_n^*, \tag{3.14}$$

where $v_n^* \in \partial\mathcal{F}(u_n) \subseteq L^{p'}(\Omega)$ for $n \geq 1$. Choosing the sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$,

$$|\langle u_n^*, \nu\rangle| \leq \epsilon_n \|\nu\| \quad \text{for all} \quad \nu \in W_0^{1,p}(\Omega), \tag{3.15}$$

where $\epsilon_n \searrow 0$.
Putting $\nu = u_n - u$ in (3.15) and using (3.14),

$$\langle \Phi' u_n, u_n - u\rangle - \lambda \int_\Omega |u_n|^{p-2} u_n (u_n - u)(x)dx - \int_\Omega v_n^*(x)(u_n - u)(x)dx$$
$$\leq \epsilon_n \|u_n - u\|. \tag{3.16}$$

Since $\{u_n\}_{n\geq 1} \subseteq L^p(\Omega)$ is bounded, Hölder inequality implies that

$$\lambda \int_\Omega |u_n(x)|^{p-2} u_n(x)(u_n - u)(x)dx \leq \lambda(\frac{1}{p^-} + \frac{1}{p'^-})\| |u_n|^{p-1}\|_{p'} \|u_n - u\|_p \to 0$$

(as $\quad n \to \infty$) where $\frac{1}{p} + \frac{1}{p'} = 1$.. The following relation

$$\int_\Omega v_n^*(x)(u_n - u)(x)dx \to 0 \quad \text{as} \quad n \to \infty,$$

holds true.
If we pass to the limit in (3.16) as $n \to \infty$

$$\limsup_{n\to\infty} \langle \Phi' u_n, u_n - u\rangle \leq 0. \tag{3.17}$$

Taking into account, the operator Φ' has the (S_+) property and

$$u_n \to u \quad \text{in} \quad X.$$

This proves that $\mathcal{I}$ satisfies the cerami condition. $\square$

Lemma 3.5. *If hypotheses M_1, M_2 and F_1, F_2 hold, then there exist $r, \alpha > 0$ such that $\mathcal{I}(u) \geq \alpha$ for every $u \in W_0^{1,p}(\Omega)$, with $\|u\| = r$.*

Proof. Due to F_2, let $\epsilon \in (0, \gamma + \lambda)$. There is $\delta > 0$ such that for $x \in \Omega$ and all s such that $|s| \leq \delta$,

$$F(x,s) \leq \frac{1}{p}(-\gamma + \epsilon)|s|^p. \tag{3.18}$$

F_1 and (3.18), imply that

$$\begin{aligned}
\mathcal{I}(u) &\geq m_0 \int_\Omega [\frac{|\nabla u(x)|^p}{p} - \frac{\lambda |u(x)|^p}{p} - F(x,u(x))]dx \\
&\geq \frac{m_0}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx + \frac{1}{p}(\epsilon - \gamma)\int_\Omega |u|^p dx - c_\epsilon \int_\Omega |u|^\theta dx \\
&\geq \frac{m_0}{p}\int_\Omega |\nabla u|^p dx - (\frac{\lambda}{p} + \frac{1}{p}(\gamma - \epsilon))\int_\Omega |u|^p dx - c_\epsilon \int_\Omega |u|^\theta dx \\
&\geq \kappa_1 [\int_\Omega |\nabla u|^p - |u|^p dx] - c_\epsilon \|u\|_\theta^\theta,
\end{aligned}$$

where $\kappa_1 := \min\{\frac{m_0}{p}, \frac{\lambda}{p} + \frac{1}{p}(\gamma - \epsilon)\}$. Then,

$$\mathcal{I}(u) \geq \kappa_1 \|u\|^p - c_\epsilon \|u\|^\theta.$$

Since $p < \theta$, if we choose $r > 0$ small enough, $\mathcal{I}(u) \geq L > 0$ for all $u \in W_0^{1,p}(\Omega)$, with $\|u\| = r$ and some $L > 0$. $\square$

Theorem 3.6. *If hypotheses M_1, M_2 and F_3 hold, then the problem (1.1) has a non-trivial solution.*

Proof. To apply the mountain pass theorem, it remains to show that there exists an $e \in X$ with $\|e\| > \rho$ and $\mathcal{I}(e) \leq 0$. Let us fix $\bar{u} \in X$

$$\begin{aligned}
\mathcal{I}(\bar{u}) &= \widehat{M}(\int_\Omega \frac{1}{p}|\nabla \bar{u}|^p dx) - \int_\Omega \frac{\lambda |\bar{u}(x)|^p}{p} dx - \int_\Omega F(x, \bar{u}(x))dx \\
&\leq \frac{m_1}{p}\int_\Omega |\nabla \bar{u}|^p dx - \int_\Omega \frac{\lambda}{p}|\bar{u}(x)|^p dx - \int_\Omega F(x,\bar{u}(x))dx \\
&\leq \upsilon(\int_\Omega |\nabla \bar{u}|^p - \int_\Omega |\bar{u}(x)|^p) dx - \int_\Omega F(x, \bar{u}(x))dx,
\end{aligned}$$

where $\upsilon = \max\{\frac{m_1}{p}, \frac{\lambda}{p}\}$. Using hypothesis F_3, $\mathcal{I}(\bar{u}) \leq 0$. Now, according to the theorem 3.1, $u \in W_0^{1,p}(\Omega)$ such that $\mathcal{I}(u) > 0 \geq \mathcal{I}(0)$ and $0 \in \partial\mathcal{I}(u)$.
Therefore $\mathcal{I}$ has at least one non-trivial critical point, so problem (1.1) has a non-trivial solution. $\square$

We use Bartschs fountain theorem under Cerami condition to prove theorem 3.9.

Since X is a reflexive and separable Banach space, then X^* is too. There exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \qquad X^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and

$$\langle e_j, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{3.19}$$

For $k = 1, 2, ...$ denote by

$$\mathbf{X}_j = span\{e_j\}, \quad \mathbf{Y}_j = \oplus_{j=1}^{k} \mathbf{X}_j, \quad \mathbf{Z}_j = \overline{\oplus_{j=k}^{\infty} \mathbf{X}_j}. \tag{3.20}$$

Before proving the theorem, let us show the following lemma.

Lemma 3.7. *If $\theta < p^*$ for all $x \in \bar{\Omega}$, then $\lim_{k \to +\infty} \mu_k = 0$, where $\mu_k = \sup\{|u|_\theta \,;\ \|u\| = 1,\ u \in \mathbf{Z}_k\}$.*

Proposition 3.8. *(Non-smooth fountain theorem (cf. [7])) Assume that* ($\mathbf{A_1}$) *X is a Banach space, $\mathcal{I} : X \to \mathbb{R}$ is an invariant locally Lipschitz functional, the subspaces $\mathbf{X}_k, \mathbf{Y}_k$ and $\mathbf{Z}_k$ are defined by (3.20).*
If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

($\mathbf{A_2}$) $\inf\{\mathcal{I}(u) : u \in \mathbf{Z}_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$;

($\mathbf{A_3}$) $\max\{\mathcal{I}(u) : u \in \mathbf{Y}_k, \|u\| = \rho_k\} \leq 0$;

($\mathbf{A_4}$) *$\mathcal{I}$ satisfies the non-smooth Cerami condition for every $c > 0$.*
Then $\mathcal{I}$ has an unbounded sequence of critical values.

Theorem 3.9. *Assume that M satisfies $(M_1), (M_2)$ and F fulfills hypotheses F_1, F_4. Then for any $\lambda \in (0, \frac{m_0 \lambda_*}{m_1})$, problem 1.1 has infinite weak solutions.*

Proof. We prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (i) and (ii) are fulfilled. Thus, the assertion of conclusion can be obtained from fountain theorem.
(i) For $u \in \mathbf{Z}_k$ such that $\|u\| = r_k$ we have by condition F_1

$$\begin{aligned}
\mathcal{I}(u) &= \widehat{M}(\int_\Omega \frac{1}{p}|\nabla u|^p dx) - \int_\Omega \frac{\lambda|u(x)|^p}{p}dx - \int_\Omega F(x,u(x))dx \\
&\geq \frac{m_0}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega |u(x)|^p)dx - \int_\Omega F(x,u(x))dx \\
&\geq (\int_\Omega |\nabla u|^p - |u(x)|^p dx) - c_0\int_\Omega |u|^\theta dx - c_1\|u\|_{L^1} \\
&\geq \frac{\kappa_2}{p}\|u\|^p - c_0(\mu_k\|u\|)^\theta - c_2,
\end{aligned}$$

where $\kappa_2 := \min\{m_0, \lambda\}$. For $r_k = (\frac{c_0\theta\mu_k^\theta)}{\kappa_2})^{\frac{1}{p-\theta}}$, it follows that

$$\mathcal{I}(u) \geq \kappa_2 r_k^p(\frac{1}{p} - \frac{1}{\theta}) - c_2.$$

Since $\mu_k \to 0$ and $p < \theta$, we have $r_k \to +\infty$ as $k \to +\infty$. Consequently,

$$\mathcal{I}(u) \to +\infty \ \ as \ \ \|u\| \to +\infty, \ u \in \mathbf{Z}_k.$$

(ii) Let $u \in \mathbf{Y}_k$ be such that $\|u\| = \rho_k > r_k > 1$. By condition F_4

$$\begin{aligned}
\mathcal{I}(u) &= \widehat{M}(\int_\Omega \frac{1}{p}|\nabla u|^p dx) - \int_\Omega \frac{\lambda|u(x)|^p}{p}dx - \int_\Omega F(x,u(x))dx \\
&\leq \frac{m_1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega \frac{\lambda}{p}|u(x)|^p dx - \int_\Omega F(x,u(x))dx \\
&\leq \upsilon\|u\|^p - \int_\Omega (d_1|u|^\mu - d_2)dx,
\end{aligned}$$

where $\upsilon = \max\{\frac{m_1}{p}, \frac{\lambda}{p}\}$. Note that the space $\mathbf{Y}_k$ has finite dimension, then all norms are equivalent and

$$\mathcal{I}(u) \leq \upsilon\|u\|^p - d_1\|u\|^\mu + C.$$

Finally,

$$\mathcal{I}(u) \to -\infty \quad \text{as} \quad \|u\| \to +\infty, \ u \in \mathbf{Y}_k,$$

because $\mu > p$. The assertion (ii) is then satisfied and the proof of theorem 3.9 is complete.

□

The next lemma points out the relationship between the critical point of $\mathcal{I}(u)$ and the solution of Problem 1.1.

Lemma 3.10. *Every critical point of $\mathcal{I}$ is a solution of problem* (1.1).

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a critical point of functional $\mathcal{I} = \Phi - \lambda\varphi - \mathcal{F}$. From definition 2.1 and proposition 2.2

$$(\Phi - \lambda\phi - \mathcal{F})^0(u; v) \geq 0, \quad \forall v \in X.$$

By using Proposition 2.5 it leads to

$$M\Big(\int_\Omega |\nabla u(x)|^p dx\Big) \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda \int_\Omega |u(x)|^{p-2}u(x)v(x)dx$$
$$- \int_\Omega F^0(x, u(x); v(x))dx \geq 0, \quad \forall v \in X.$$

for every $v \in X$, which it completes the proof. □

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