

## SUMUDU TRANSFORM ITERATION METHOD FOR FRACTIONAL DIFFUSION-WAVE EQUATIONS

K. SAYEVAND\* AND K. PICHAGHCHI

**ABSTRACT.** In this article, we have implemented Sumudu transform iteration method as a new approximate analytical technique for solving fractional diffusion-wave equations. The fractional derivative is described in the Caputo sense. The solution existence, uniqueness, stability and convergence of the proposed scheme is discussed. Finally, the validity and applicability of our approach is examined with the use of a solvable model method. The results presented here are in compact and elegant expressed in term of Mittag-Leffler function which are suitable for numerical computation.

**Key Words:** Sumudu transform, Diffusion-wave equation, Caputo fractional derivative, Mittag-Leffler function.

**2010 Mathematics Subject Classification:** Primary: 34A08; Secondary: 35L05.

### 1. INTRODUCTION

In last decades, fractional calculus have been the focus of many studies due to their frequent appearance in various applications in physics, science, engineering, economics and so on. For instance, see [4, 5, 9–13] and references therein. Consequently, many problems in differential equations have presented and studied in the form of fractional order derivatives in the literature. In fractional calculus, the Mittag-Leffler function, as an analogous to exponential naturally function, plays an important role. In fact, the exponential function itself is a special form

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of these general functions. A considerable amount of literature is dedicated to the applications of the Mittag-Leffler function as solutions of fractional order models [7,8]. Our results in this article are based mainly on the Mittag-Leffler function. Actually, we consider the diffusion-wave equations presented in the scope of fractional calculus in the following form:

$$(1.1) \quad \frac{\partial^\alpha \xi(x, t)}{\partial t^\alpha} = \lambda \frac{\partial^\beta \xi(x, t)}{\partial x^\beta},$$

$$(1.2) \quad \xi(x, 0) = u(x), \quad \frac{\partial}{\partial t} \xi(x, 0) = \nu(x),$$

where  $0 < \alpha < 2$ ,  $m < \beta \leq m + 1$ ,  $m \in \mathbb{N}$ ,  $\lambda \in \mathbb{R}$ . Notice that, the second initial condition is for  $\alpha > 1$  only and the operators  $\frac{\partial^\alpha}{\partial t^\alpha}$  and  $\frac{\partial^\beta}{\partial x^\beta}$  are the Caputo fractional derivatives defined as

$$(1.3) \quad \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} [I_t^{m-\alpha} f^{(m)}(t)], & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \frac{d^m}{dt^m} f(t), & \alpha = m, \end{cases}$$

where  $I_t^\alpha$  is the Riemann-Liouville integral operator of order  $\alpha$  defined as

$$(1.4) \quad I_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & \alpha > 0, \quad t > 0, \\ f(t), & \alpha = 0. \end{cases}$$

In our attempt, we have used the Sumudu transform iteration method (STIM) to obtain relatively new analytic and approximate solution for fractional diffusion-wave equations. The importance of this equation usually arises in the wave propagation in beams and modeling formation of grooves on a flat surface. Additional background and application of Eq. (1.1)-(1.2) in science, engineering and mathematics can be found in [1]. Finally, convergence and stability of the proposed approach for these types of problems will be considered.

First we recall some preliminaries and notations regarding Sumudu transform.

**Definition 1.1.** The Sumudu transform is defined over the set of functions:

$$(1.5) \quad A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

by the following formula [2]

$$(1.6) \quad F(u) = \mathbf{S}[f(t); u] = \int_0^\infty f(ut)e^{-t} dt, \quad u \in (-\tau, \tau).$$

The Sumudu transform of the Caputo fractional derivative is defined as follows [3]

$$(1.7) \quad \mathbf{S}[D_t^\alpha f(t); u] = u^{-\alpha} \mathbf{S}[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0), \quad m-1 < \alpha \leq m,$$

$$(1.8) \quad \mathbf{S}[D_t^\alpha f(x, t); u] = u^{-\alpha} \mathbf{S}[f(x, t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(x, 0), \quad m-1 < \alpha \leq m.$$

**Theorem 1.2.** *Assuming  $H(u) = \mathbf{S}[h(t)]$  and  $G(u) = \mathbf{S}[g(t)]$ , the Sumudu convolution theorem states that the transform of*

$$(1.9) \quad h(t) * g(t) = \int_0^t h(t-\tau)g(\tau)d\tau,$$

is given by

$$(1.10) \quad uH(u)G(u) = \mathbf{S}[h(t) * g(t)].$$

*Proof.* See [2]. □

Some fundamental further established properties of Sumudu transform can be found in [2].

## 2. SUMUDU TRANSFORM ITERATION METHOD IMPLEMENTATION TO FRACTIONAL DIFFUSION-WAVE EQUATIONS

Now we are ready to implement STIM for obtaining analytical and numerical solutions of Eq. (1.1)-(1.2). By taking Sumudu transform from the Eq. (1.1)-(1.2), with respect to the independent variable  $t$  and using the initial conditions, we get

$$(2.1) \quad u^{-\alpha} \bar{\xi}(x, u) - u^{-\alpha} \xi(x, 0) - u^{1-\alpha} \frac{\partial}{\partial t} \xi(x, 0) = \lambda \frac{\partial^\beta \bar{\xi}(x, u)}{\partial x^\beta}.$$

Solving for  $\bar{\xi}(x, u)$ , we get

$$(2.2) \quad \bar{\xi}(x, u) = \xi(x, 0) + u \frac{\partial}{\partial t} \xi(x, 0) + \lambda u^\alpha \frac{\partial^\beta \bar{\xi}(x, u)}{\partial x^\beta}.$$

By notice the following Sumudu transform pair

$$(2.3) \quad \frac{t^k}{\Gamma(k+1)} \xleftrightarrow{\mathbf{S}} u^k,$$

and by applying the inverse Sumudu transform to both sides of Eq. (2.2), and using the convolution Theorem 1.2, we get

$$(2.4) \quad \begin{aligned} \xi(x, t) &= \xi(x, 0) + t \frac{\partial}{\partial t} \xi(x, 0) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} * \lambda \frac{\partial^\beta \bar{\xi}(x, u)}{\partial x^\beta} \\ &= \xi(x, 0) + t \frac{\partial}{\partial t} \xi(x, 0) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^\beta \xi(x, \tau)}{\partial x^\beta} d\tau. \end{aligned}$$

According to [6], by imposing the initial conditions to obtain the solution of Eq. (1.1)-(1.2), we can construct an iteration formula as follows

$$(2.5) \quad \xi_0(x, t) = \xi(x, 0) + t \frac{\partial}{\partial t} \xi(x, 0),$$

$$(2.6) \quad \xi_{j+1}(x, t) = \xi_0(x, t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^\beta \xi_j(x, \tau)}{\partial x^\beta} d\tau, j = 0, 1, \dots$$

By the above iteration each term will be determined by the previous terms and the approximation of iteration formula can be entirely evaluated. Consequently, the solution may be written as

$$(2.7) \quad \xi(x, t) = \lim_{j \rightarrow \infty} \xi_j(x, t).$$

### 3. EXISTENCE AND UNIQUENESS OF SOLUTION

**Theorem 3.1.** *Suppose that  $\lambda \frac{\partial^\beta \xi(x, t)}{\partial x^\beta} \in L_1(0, X) \times (0, T)$  and the STIM is applied to Eq. (1.1)-(1.2). Then, this equation has a unique solution  $\xi(x, t) \in L_1(0, X) \times (0, T)$ .*

*Proof.* As before, applying the Sumudu transform with respect to time variable  $t$  from (1.1)-(1.2) and application of the inverse Sumudu transform, we have

$$(3.1) \quad \xi(x, t) = \xi(x, 0) + t \frac{\partial}{\partial t} \xi(x, 0) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^\beta \xi(x, \tau)}{\partial x^\beta} d\tau.$$

Using the rule for the Caputo fractional differentiation of the power function, we easily obtain that

$$(3.2) \quad \frac{\partial^\alpha}{\partial t^\alpha} (t^k) = 0, \quad \alpha > k.$$

Using (3.2), the direct substitution of the function defined by the expression (3.1) in the Eq. (1.1), we have

$$(3.3) \quad \begin{aligned} & \frac{\partial^\alpha}{\partial t^\alpha} \left( \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^\beta \xi(x, \tau)}{\partial x^\beta} d\tau \right) \\ &= \frac{\partial^\alpha}{\partial t^\alpha} I_t^\alpha \left( \lambda \frac{\partial^\beta \xi(x, t)}{\partial x^\beta} \right) = \lambda \frac{\partial^\beta \xi(x, t)}{\partial x^\beta}. \end{aligned}$$

Therefore,  $\xi(x, t)$  satisfies (1.1)-(1.2) and the existence of the solution is proved. Furthermore, it follows from (3.1) that  $\xi(x, t) \in L_1(0, X) \times (0, T)$ .

The uniqueness follows from the linearity of fractional differentiation and the properties of the Sumudu transform. Indeed, if there exist two solutions  $\xi(x, t)$  and  $\hat{\xi}(x, t)$ , then  $\zeta(x, t) = \xi(x, t) - \hat{\xi}(x, t)$  must satisfy  $\frac{\partial^\alpha}{\partial t^\alpha} \zeta(x, t) = 0$ . Then, the Sumudu transform  $\bar{\zeta}(x, u) = 0$ , and therefore  $\zeta(x, t) = 0$  almost everywhere in the considered interval, which proves that the solution in  $L_1(0, X) \times (0, T)$  is unique.  $\square$

#### 4. CONVERGENCE ANALYSIS

**Theorem 4.1.** *Under the assumption of boundedness the initial conditions (1.2), The STIM to approximate the solution of Eq. (1.1)-(1.2) is convergent.*

*Proof.* Applying the STIM method to Eq. (1.1)-(1.2), we obtain the following recursive formula

$$(4.1) \quad \xi_0(x, t) = \xi(x, 0) + t \frac{\partial}{\partial t} \xi(x, 0),$$

$$(4.2) \quad \xi_{j+1}(x, t) = \xi_0(x, t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^\beta \xi_j(x, \tau)}{\partial x^\beta} d\tau.$$

We can write (4.1) as

$$(4.3) \quad \xi_{j+1}(x, t) = \xi_0(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} K(x, \tau) \xi_j(x, \tau) d\tau,$$

therefore

$$(4.4) \quad |\xi_1(x, t) - \xi_0(x, t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |K(x, \tau)| |\xi_0(x, \tau)| d\tau.$$

By assumption the boundedness of initial conditions, we have

$$(4.5) \quad K_1 = \|\xi(x, 0)\|_\infty < \infty, \quad K_2 = \|K(x, t)\|_\infty < \infty.$$

Therefore

$$(4.6) \quad |\xi_1(x, t) - \xi_0(x, t)| \leq K_1 K_2 \frac{t^\alpha}{\gamma(\alpha + 1)}.$$

In the same manner

$$(4.7) \quad |\xi_2(x, t) - \xi_1(x, t)| \leq K_1 K_2^2 \frac{t^{2\alpha}}{\gamma(2\alpha + 1)},$$

and finally

$$(4.8) \quad |\xi_j(x, t) - \xi_{j-1}(x, t)| \leq K_1 K_2^j \frac{t^{j\alpha}}{\gamma(j\alpha + 1)}.$$

As we know, the series of  $\sum_{j=0}^{\infty} K_1 K_2^j \frac{t^{j\alpha}}{\gamma(j\alpha + 1)} = K_1 E_\alpha(K_2 t^\alpha)$ , where  $E_\alpha(z)$  is the one-parametric Mittag-Leffler function defined as follows:

$$(4.9) \quad E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \quad z \in \mathbb{C},$$

is convergent in the whole real line, therefore the series of

$$(4.10) \quad (\xi_0(x, t) + [\xi_1(x, t) - \xi_0(x, t)] + \cdots + [\xi_j(x, t) - \xi_{j-1}(x, t)] + \cdots),$$

is absolutely convergent, i.e., the sequence  $\{\xi_j(x, t)\}_{j=1}^{\infty}$  is convergent for  $(x, t) \in (0, X) \times (0, T)$ .  $\square$

## 5. STABILITY ANALYSIS

**Theorem 5.1.** *Under the assumptions of Theorem 4.1, the Eq. (1.1)-(1.2) has a solution  $\xi(x, t)$  which is asymptotically Mittag-Leffler stable.*

*Proof.* Consider the fractional diffusion-wave equations in the form (1.1)-(1.2). Assume now that  $\xi(x, t)$  and  $\zeta(x, t)$  be the solutions of Eq. (1.1)-(1.2) and the following fractional diffusion-wave equations, respectively

$$(5.1) \quad \frac{\partial^\alpha \zeta(x, t)}{\partial t^\alpha} = \lambda \frac{\partial^\beta \zeta(x, t)}{\partial x^\beta},$$

$$(5.2) \quad \zeta(x, 0) = u(x), \quad \frac{\partial}{\partial t} \zeta(x, 0) = \chi(x),$$

where  $|\xi(x, 0) - \zeta(x, 0)| \leq \epsilon_0$  and  $|\frac{\partial}{\partial t} \xi(x, 0) - \frac{\partial}{\partial t} \zeta(x, 0)| \leq \epsilon_1$ . From (4.3), we have

$$(5.3) \quad \xi_{j+1}(x, t) = \xi_0(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} K(x, \tau) \xi_j(x, \tau) d\tau,$$

and suppose that

$$(5.4) \quad |K(x, t)| \leq L, \quad \forall (x, t) \in (0, X) \times (0, T).$$

Therefore

$$(5.5) \quad |\xi_0(x, t) - \zeta_0(x, t)| \leq \sum_{k=0}^1 |\epsilon_k| \frac{t^k}{k!},$$

$$(5.6) \quad \begin{aligned} & |\xi_1(x, t) - \zeta_1(x, t)| \leq |\xi_0(x, t) - \zeta_0(x, t)| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} |K(x, \tau)| |\xi_0(x, \tau) - \zeta_0(x, \tau)| d\tau \\ & \leq \sum_{k=0}^1 |\epsilon_k| \frac{t^k}{k!} + \sum_{k=0}^1 |\epsilon_k| L \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{t^k}{k!} \\ & \leq \sum_{k=0}^1 |\epsilon_k| \left( \frac{t^k}{k!} + L \frac{t^{\alpha+k}}{\Gamma(\alpha + k + 1)} \right) \\ & \leq \sum_{k=0}^1 |\epsilon_k| \left( \sum_{m=0}^1 L^m \frac{t^{m\alpha+k}}{\Gamma(m\alpha + k + 1)} \right). \end{aligned}$$

Similarly, we have

$$(5.7) \quad |\xi_2(x, t) - \zeta_2(x, t)| \leq \sum_{k=0}^1 |\epsilon_k| \left( \sum_{m=0}^2 L^m \frac{t^{m\alpha+k}}{\Gamma(m\alpha + k + 1)} \right),$$

and by induction

$$(5.8) \quad |\xi_j(x, t) - \zeta_j(x, t)| \leq \sum_{k=0}^1 |\epsilon_k| \left( \sum_{m=0}^j L^m \frac{t^{m\alpha+k}}{\Gamma(m\alpha + k + 1)} \right).$$

Taking the limit of (5.8) as  $j \rightarrow \infty$  and by the definition of Mittag-Leffler function, we have

$$(5.9) \quad \begin{aligned} & |\xi(x, t) - \zeta(x, t)| \leq \sum_{k=0}^1 |\epsilon_k| \left( \sum_{m=0}^{\infty} L^m \frac{t^{m\alpha+k}}{\Gamma(m\alpha + k + 1)} \right) \\ & \leq \sum_{k=0}^1 |\epsilon_k| t^k E_{\alpha, k+1}(Lt^\alpha), \end{aligned}$$

where  $E_{\alpha,\beta}(z)$  is the two-parametric Mittag-Leffler function defined by the following series representation, valid in the whole complex plane

$$(5.10) \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad z \in \mathbb{C},$$

and this completes the proof.

It concludes from this theorem that small changes in initial conditions cause only small changes of the obtained solution.  $\square$

## 6. ERROR ANALYSIS

**Theorem 6.1.** *By application the STIM on Eq. (1.1)-(1.2), the upper bound of error can be estimated as follows:*

$$(6.1) \quad e_j = |\xi(x, t) - \xi_{j-1}(x, t)| \leq \frac{ML^{(j-1)\alpha}\delta^{j\alpha}}{\Gamma(1 + j\alpha)},$$

where  $L$  is an upper bound of  $K(x, t)$ ,  $|t| < \delta$  and  $M = \max |\lambda \frac{\partial^\beta \xi_0(x, t)}{\partial t^\beta}|$ .

*Proof.* The proof is by induction. For  $j = 1$  we have

$$(6.2) \quad \begin{aligned} e_1 &= |\xi_1(x, t) - \xi_0(x, t)| = \left| \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{\partial^\beta \xi_0(x, t)}{\partial t^\beta} d\tau \right| \\ &\leq \frac{M|t|^\alpha}{\Gamma(1 + \alpha)} \leq \frac{M\delta^\alpha}{\Gamma(1 + \alpha)}. \end{aligned}$$

Now suppose that (6.1) holds for  $j$ . We prove that it holds for  $j + 1$ :

$$(6.3) \quad \begin{aligned} e_{j+1} &= |\xi_{j+1}(x, t) - \xi_j(x, t)| \\ &= \left| \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} K(x, \tau) \{ \xi_j(x, \tau) - \xi_{j-1}(x, \tau) \} d\tau \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} L |\xi_j(x, \tau) - \xi_{j-1}(x, \tau)| d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} L |e_j(x, \tau)| d\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{ML^{(j-1)\alpha}\delta^{j\alpha}}{\Gamma(1 + j\alpha)} d\tau \\ &\leq \frac{ML^{j\alpha}\delta^{(j+1)\alpha}}{\Gamma(1 + (j + 1)\alpha)}. \end{aligned}$$

But, from Theorem 4.1  $\xi(x, t) = \lim_{j \rightarrow \infty} \xi_j(x, t)$  and thus

$$(6.4) \quad e_j = |\xi(x, t) - \xi_{j-1}(x, t)| \leq \frac{ML^{(j-1)\alpha} \delta^{j\alpha}}{\Gamma(1 + j\alpha)}.$$

□

## 7. TEST EXAMPLES

In this section, we apply the proposed method to some fractional diffusion-wave models. As we will see, the Mittag-Leffer function appears in the solutions of these examples.

*Example 7.1.* Consider the following fractional diffusion-wave equation

$$(7.1) \quad \frac{\partial^\alpha \xi(x, t)}{\partial t^\alpha} = \lambda \frac{\partial^{2\alpha} \xi(x, t)}{\partial x^{2\alpha}}, \quad 0 < \alpha < 2,$$

$$(7.2) \quad \xi(x, 0) = E_\alpha(x^\alpha), \quad \frac{\partial}{\partial t} \xi(x, 0) = 0.$$

The second initial condition is for  $\alpha > 1$  only. From (2.5)-(2.6) and by imposing the initial condition, we have

$$(7.3) \quad \xi_{j+1}(x, t) = E_\alpha(x^\alpha) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{\partial^{2\alpha} \xi_j(x, \tau)}{\partial x^{2\alpha}} d\tau.$$

Hence

$$(7.4) \quad \xi_j(x, t) = E_\alpha(x^\alpha) \sum_{k=0}^j \frac{\lambda^k t^{k\alpha}}{\Gamma(1 + k\alpha)},$$

and therefore

$$(7.5) \quad \xi(x, t) = \lim_{j \rightarrow \infty} \xi_j(x, t) = E_\alpha(x^\alpha) E_\alpha(\lambda t^\alpha).$$

We see that our approximate solution is in coincide with the exact solution  $\xi(x, t) = E_\alpha(x^\alpha) E_\alpha(\lambda t^\alpha)$ . The Eq. (7.1)-(7.2) is calculated numerically for different values of  $\alpha$  with  $\lambda = 1$ . The results of  $\xi(x, t)$  are presented in Figs. 1, 2, 3.

*Example 7.2.* Consider the following fractional diffusion-wave equation

$$(7.6) \quad \frac{\partial^\alpha \xi(x, t)}{\partial t^\alpha} = \lambda \frac{\partial^{4\alpha} \xi(x, t)}{\partial x^{4\alpha}}, \quad 0 < \alpha < 2,$$

$$(7.7) \quad \xi(x, 0) = \cos_\alpha(x^\alpha), \quad \frac{\partial}{\partial t} \xi(x, 0) = 0,$$

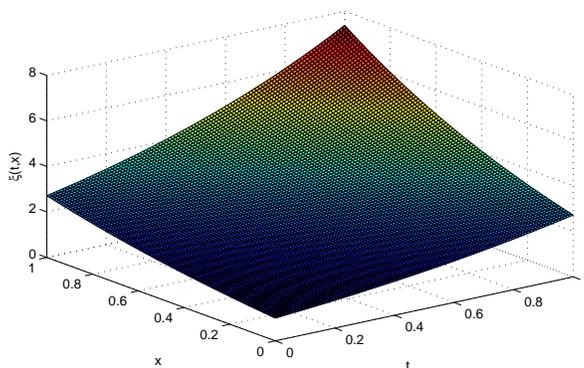


FIGURE 1. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 1$  and  $\lambda = 1$ .

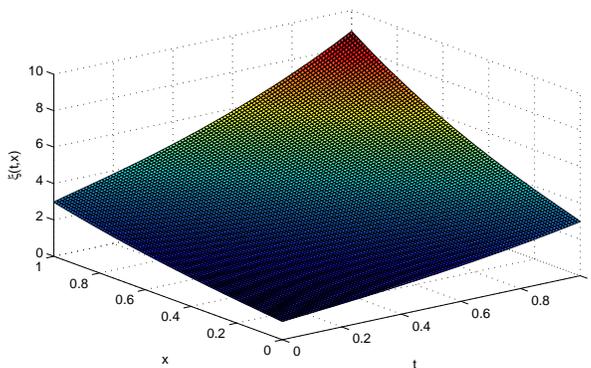


FIGURE 2. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 0.9$  and  $\lambda = 1$ .

where the cos function in fractional sense is defined as

$$(7.8) \quad \cos_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2\alpha k}}{\Gamma(1 + 2\alpha k)}.$$

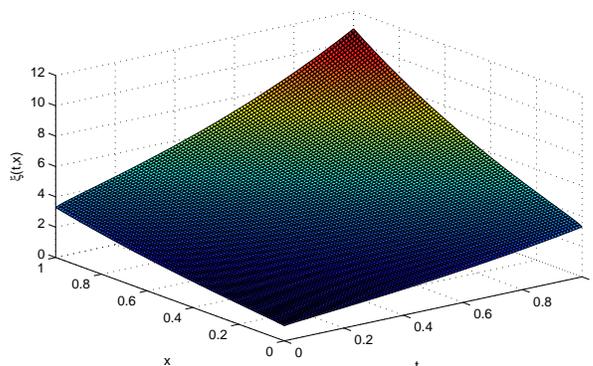


FIGURE 3. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 0.8$  and  $\lambda = 1$ .

As before, the second initial condition is for  $\alpha > 1$  only. From (2.5)-(2.6) and by imposing the initial conditions, we have

$$(7.9) \quad \xi_{j+1}(x, t) = \cos_\alpha(x^\alpha) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{\partial^{4\alpha} \xi_j(x, \tau)}{\partial x^{4\alpha}} d\tau.$$

Hence

$$(7.10) \quad \xi_j(x, t) = \cos_\alpha(x^\alpha) \sum_{k=0}^j \frac{\lambda^k t^{k\alpha}}{\gamma(1 + k\alpha)},$$

and therefore

$$(7.11) \quad \xi(x, t) = \lim_{j \rightarrow \infty} \xi_j(x, t) = \cos_\alpha(x^\alpha) E_\alpha(\lambda t^\alpha),$$

which is an exact solution of the given fractional diffusion wave Eq. (7.6)-(7.7). This equation is calculated numerically for different values of  $\alpha$  with  $\lambda = 1$ . The results of  $\xi(x, t)$  are presented in Figs. 4, 5, 6.

## 8. CONCLUSIONS

In this manuscript, the STIM has been successfully applied to compute the approximate analytical solution of the fractional diffusion-wave equations. The convergence and stability of the method, as applied to this equation have been thoroughly investigated. The obtained results show that, our approach provides the solution in terms of convergent series with easily computable components in a direct way and is very

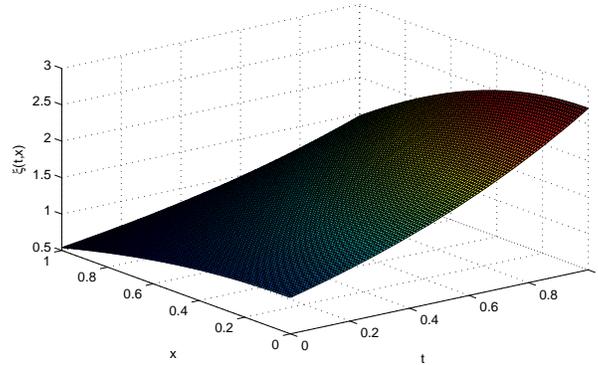


FIGURE 4. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 1$  and  $\lambda = 1$ .

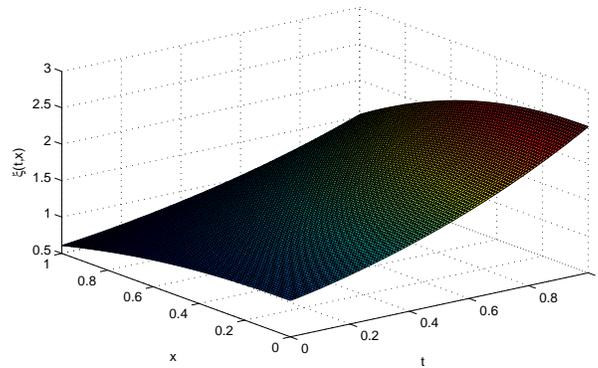


FIGURE 5. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 1.1$  and  $\lambda = 1$ .

effective and convenient for dealing with the fractional diffusion-wave equations. Since we need to find just a few terms of the series, then the rest will be determined, automatically.

#### REFERENCES

- [1] O. P. Agrawal, *A general solution for a fourth-order fractional diffusion-wave equation defined in a bounded domain*, *Computers and Structures* **79** (2001), 1497–1501.

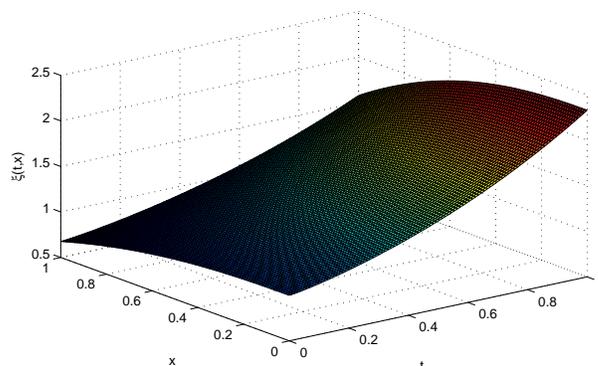


FIGURE 6. Plot of the approximation  $\xi(x, t)$  for  $\alpha = 1.2$  and  $\lambda = 1$ .

- [2] F. B. M. Belgacem, A. A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, **Journal of Applied Mathematics and Stochastic Analysis**, 2006, Article ID 91083, (2006), 1-23.
- [3] V. B. L. Chaurasia, J. Singh, *Application of Sumudu transform in Schrödinger equation occurring in quantum mechanics*, **Applied Mathematical Sciences** 4 (2010), 2843-2850.
- [4] A. Golbabai, K. Sayevand, *Fractional calculus - A new approach to the analysis of generalized fourth-order diffusion-wave equations*, **Comput. Math. Appl.**, 61 (2011), 2227–2231.
- [5] A. Golbabai, K. Sayevand, *Analytical treatment of differential equations with fractional coordinate derivatives*, **Comput. Math. Appl.**, 62 (2011), 1003–1012.
- [6] E. Hesameddini, H. Latifizadeh, *Reconstruction of variational iteration algorithms using the Laplace transform*, **Int. J. Nonlinear Sci. Numer. Simul.** 10 (2009), 1377-1382.
- [7] A. A. Kilbas, A. A. Koroleva, S.V. Rogosin, *Multi-parametric Mittag-Leer functions and their extension*, **Fract. Calc. Appl. Anal.**, 16 (2) (2013), 378-404.
- [8] V. Kiryakova, *The multi-index Mittag-Leer functions as an important class of special functions of fractional calculus*, **Comput. Math. Appl.**, 59 (2010), 1885-1895.
- [9] F. Mainardi, *The fundamental solutions for the fractional diffusion-wave equation*, **Appl. Math. Lett.**, 9 (1996), 23–28.
- [10] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
- [11] K. Sayevand, A. Golbabai, Ahmet Yildirim, *Analysis of differential equations of fractional order*, **Appl. Math. Model.**, 36 (2012), 4356–4364.

- [12] **K. Sayevand, K. Pichaghchi**, *Successive approximation: A survey on stable manifold of fractional differential systems*, **Fract. Calc. Appl. Anal.**, **18** (3) (2015), 621–641.
- [13] **J. Tenreiro Machado, V. Kiryakova, F. Mainardi**, *Recent history of fractional calculus*, **Commun. Nonlinear Sci. Numer. Simu.**, **16** (2011), 1140–1153.

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