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WEAKLY LINEAR HOMOMORPHISMS IN SKEW BOOLEAN MODULES

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Abstract. In this paper we introduce the notion of linear, weakly linear and strongly linear homomorphisms between two Skew Boolean modules and obtain various properties. We also introduce a scalar multiplication on the set of all weakly linear homomorphisms $(wHom(V, W))$ and prove that $(wHom(V, W))$ is again a Skew Boolean module . Further We show that the set of all strongly linear homomorphisms $(stHom(V, W))$ is a sub Skew Boolean module of $wHom(V, W)$.

Key Words: Boolean like rings, Skew Boolean module, sub Skew Boolean module, Linear homomorphisms.
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1. Introduction

The present authors have introduced the notion of Skew Boolean modules in [2] by generalizing the notion of Boolean vector spaces (Vector spaces over Boolean algebras) of Subrahmanyam([5]). Jagannadham in [3] studied the notion of linear homomorphism in Boolean vector spaces. This paper is a continuation of the study of Skew Boolean modules. In this paper, we introduce the concept linear, weakly linear and strongly linear homomorphisms between two Skew Boolean modules (see defintion 2.1)and study its properties. This paper is divided in to 3 sections. In section 1, we give definition and certain preliminary results on

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Boolean like rings and Skew Boolean modules. In section 2, we introduce the concept of linear, weakly linear and strongly linear homomorphism and study certain results. In section 3, we further study on the quotient Skew Boolean modules.

2. Preliminaries

In this section, we collect certain important definitions on Boolean like rings of A.L Foster ([1]) and Swaminathan ([7]), Skew Boolean Modules of Dawit C and K. Venkateswarlu ([2]).
Throughout this paper, R denotes a Boolean like ring and R_B denotes the Boolean subring of idempotents of R.

Definition 2.1. A Boolean like ring R (for short, a BLR) is a commutative ring with unity in which for all elements $a, b, \in R$, $a + a = 0$, $ab(1 + a)(1 + b) = 0$.

Lemma 2.2. *For all $a \in R$, $a^4 = a^2$(weak idempotent law)*

Definition 2.3. Let R be a BLR and R_B its Boolean subring of all the idempotents of R. Let $a, b \in R$. Define $a < b$ if and only if there exist $x \in R_B$ such that $bx = a$.

Theorem 2.4. *$(R, <)$ is a partially ordered set and the partial ordering $<$ coincides with the natural partial ordering in R_B.*

Definition 2.5. An abelian group $V = (V, +)$ is said to be a Skew Boolean module if and only if there exist a mapping: $R \times V \to V$ (the image of (a, x) will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$;

(1) $a^2(x + y) = a^2x + a^2y$;
(2) $(ab)x = a(bx)$ if $a^2 = a$;
(3) $1x = x$;
(4) $(a + b)x = ax + bx$ if $ab = 0$.

The elements of V shall be called Boolean vectors and that of R as scalars and the operation :$(a, x) \to ax$, a scalar multiplication.

Lemma 2.6. *For $x \in V$ and $c \in R_B$, $(1 - c)x = x - cx$.*

Definition 2.7. A Skew Boolean module over R is said to be "normed" (or R_B normed) if and only if there exists a mapping $|.| : V \to R_B$ such that

(1) $|x| = 0$ if and only if $x = 0$
(2) $|ax| = a\,|x|$ for all $a \in R_B$ and $x \in V$.

Example 2.8. Let R be the Boolean Like ring $H_4 = \{0, 1, p, q\}$ as in [2] and let V be the group $\{0, 1, 2, 3, 4, 5\}$ of addition modulo 6. Define the scalar multiplication by putting $0x = x$, $1x = x$, $px = x$ and $qx = x + 3$. Then it can be easily verified that V is a Skew Boolean module over R. For each $x \in V$, define $|0| = 0$ and $|x| = 1$ if $x \neq 0$, then $|.|$ defines a norm on V.

Lemma 2.9. *If V is a normed Skew Boolean module over R, then* $|x| = |-x|$ *for all* $x \in V$.

Lemma 2.10. *If V is a normed Skew Boolean module, then for any* $x \in V$, $|x|\,x = x$.

Definition 2.11. Let V be a Skew Boolean module over a BLR R. A non empty subset W of V is called a sub Skew Boolean module of V provided

(1) W is an additive subgroup of V
(2) If $x \in W$ and $a \in R$, then $a^2x \in W$.

Remark 2.12. Let V be a Skew Boolean module over R. The lattice of all sub Skew Boolean modules will be denoted by $L(V)$. It is easily verified that the lattice sum ($\vee$) and the lattice product ($\wedge$) of sub Skew Boolean modules in $L(V)$ are given, respectively by :

(1) $W_1 \vee W_2 = \{x + y, x \in W_1, y \in W_2\}$
(2) $W_1 \wedge W_2 = W_1 \cap W_2$

It is easy to see that $W_1 \vee W_2$ and $W_1 \wedge W_2$ are sub Skew Boolean modules of V.

Definition 2.13. Let V be a normed Skew Boolean module. Define

(1) $[V] = \{|x| : x \in V\}$
(2) $V_a = \{x : |x| < a^2\}$

Lemma 2.14. *For all* $a, b \in R$

(1) $V_{a^2} \vee V_{b^2} = V_{a^2+b^2+a^2b^2}$
(2) $V_{a^2} \wedge V_{b^2} = V_{a^2b^2}$
(3) *If* $a \in R_B$, $V_{a'} = (V_a)'$

For the literature on Skew Boolean modules, readers are advised to refer 2.

3. LINEAR HOMOMORPHISMS

Definition 3.1. Let V and W be two Skew Boolean modules.

(1) A mapping $T : V \to W$ is said to be a linear homomorphism of V into W if $for\ all\ x, y \in V$ and $a, b \in R$, $T(ax + by) = aT(x) + bT(y)$ whenever $ab = 0$
(2) A mapping $T : V \to W$ is said to be a weakly linear homomorphism of V into W if $for\ all\ x, y \in V$ and $a, b \in R$, $T(a^2x + b^2y) = a^2T(x) + b^2T(y)$ whenever $ab = 0$.
(3) A mapping $T : V \to W$ is called strongly linear homomorphism if $T(a^2x + b^2y) = a^2T(x) + b^2T(y)\ \forall x, y \in V,\ for\ all\ a, b \in R$.

Notation: We denote certain different classes of linear homomorphism as follows.

(1) $Hom(V, W) = \{T : V \to W;$ T is a linear homomorphism$\}$
(2) $wHom(V, W) = \{T : V \to W;$ T is a weakly linear homomorphism$\}$
(3) $stHom'(V, W) = \{T : V \to W;$ T is a strongly linear homomorphism$\}$

Remark 3.2. From the definition 3.1, it is clear that

$$stHom(V, W) \subset wHom(V, W) \subset Hom(V, W)$$

.

We emphasize on the class of weakly linear homomorphisms in the rest of the paper. Now we furnish the following

Example 3.3. Let V be the Skew Boolean module as in example 2.8. Define a mapping $T : V \to V$ by putting $T(0) = 0$, $T(1) = 5$, $T(2) = 1$, $T(3) = 3$, $T(4) = 2$, $T(5) = 4$, then T is a weakly linear homomorphism.

Lemma 3.4. *Let V and W be Skew Boolean modules over R, then* $T \in wHom(V, W)$ *if and only if* $T(a^2x) = a^2T(x)$ *for all* $x \in V$ *and* $a \in R$.

Proof. ($\Rightarrow$) Let $T \in wHom(V, W)$, then putting $b = 0$ in definition 3.1 we willl get the desired result.

($\Leftarrow$) Conversely, Let $T(a^2x) = a^2T(x)$ and $ab = 0$

$$\begin{aligned}
T(a^2x + b^2y) &= (1 + a^2 + a^2)T(a^2x + b^2y) \\
&= (1 + a^2)T(a^2x + b^2y) + a^2T(a^2x + b^2y) \\
&= T((1 + a^2)(a^2x + b^2y)) + T(a^2(a^2x + b^2y)) \\
&= T((1 + a^2)a^2x + (1 + a^2)b^2y) + T(a^2x + a^2b^2y) \\
&= T((1 + a^2)a^2)x + ((1 + a^2)b^2)y) + T(a^2x + a^2b^2y) \\
&= T(b^2y) + T(a^2x) \\
&\Rightarrow T \in wHom(V, W)
\end{aligned}$$

□

Theorem 3.5. *Let V and W be normed Skew Boolean modules over R. Then $T \in wHom(V, W)$ if and only if*

(1) $T(0) = 0$
(2) $|T(x) - T(y)| < |x - y|$ *for all* $x, y \in V$.

Proof. ($\Rightarrow$) Let $T \in wHom(V, W)$, then $T(0) = T(0^2 0) = 0^2T(0) = 0$. Let $x, y \in V$, then

$$\begin{aligned}
|(1 - |x - y|)x - (1 - |x - y|)y| &= |(1 - |x - y|)(x - y)| \\
&= (1 - |x - y|)\,|x - y| \\
&= 0 \\
&\Rightarrow (1 - |x - y|)x = (1 - |x - y|)y
\end{aligned}$$

Thus,

$$\begin{aligned}
(1 - |x - y|)\,|T(x) - T(y)| &= |(1 - |x - y|)(T(x) - T(y))| \\
&= |(1 - |x - y|)T(x) - (1 - |x - y|)T(y)| \\
&= |T((1 - |x - y|)x) - T((1 - |x - y|)y)| \\
&= 0 \\
&\Rightarrow |T(x) - T(y)| = |T(x) - T(y)|\,|x - y| \\
&\Rightarrow |T(x) - T(y)| < |x - y|
\end{aligned}$$

Conversely, Let $x \in V$ and $a \in R$, then $(1 - a^2)x = x - a^2x$ (by lemma 2.6) and also $|-(1 - a^2)x| = |(1 - a^2)x|$ (by lemma 2.9). Now,

$$|T(a^2x) - T(x) < |a^2x - x| = |x - a^2x| = |(1 - a^2)x| = (1 - a^2)|x|$$

and

$$\begin{aligned}|a^2T(a^2x) - a^2T(x)| &= |a^2(T(a^2x) - T(x))\\ &= a^2|T(a^2x) - T(x)|\\ &< a^2|a^2x - x|\\ &= a^2(1 - a^2)|x|\\ &= 0\end{aligned}$$

$\Rightarrow |a^2T(a^2x) - a^2T(x)| = 0 \Rightarrow a^2T(a^2x) = a^2T(x)$
Moreover,
$|T(a^2x)| = |T(a^2x) - 0| = |T(a^2x) - T0| < |a^2x - 0| = |a^2x|$
$\Rightarrow |T(a^2x)||a^2x| = |T(a^2x)|$. Thus,
$|(1 - a^2)T(a^2x)| = (1 - a^2)|T(a^2x)| = (1 - a^2)|a^2x||T(a^2x)| = (1 - a^2)a^2|x||T(a^2x)| = 0 \Rightarrow (1 - a^2)T(a^2x) = 0$
Since $a^2T(a^2x) = a^2T(x)$ and $(1 - a^2)T(a^2x) = 0$, we have

$$\begin{aligned}T(a^2x) &= (1 + a^2 + a^2)T(a^2x)\\ &= (1 + a^2)T(a^2x) + a^2T(a^2x)\\ &= a^2T(a^2x)\\ &= a^2T(x)\end{aligned}$$

Thus, by lemma 3.4, $T \in wHom(V, W)$. □

Definition 3.6. Let V and W be Skew Boolean modules over R. We define addition and scalar multiplication on $wHom(V, W)$ as follows; If $T, S \in wHom(V, W)$ and $a \in R$;

(1) $(T + S)(x) = T(x) + S(x)$ and ;
(2) $(aT)(x) = a^2T(x)$ for each $x \in V$.

Lemma 3.7. *If $T, S \in wHom(V, W)$, then*

(1) *$T + S \in wHom(V, W)$.*
(2) *$aT \in wHom(V, W)$*

Proof. (1) Let $a \in R$, $x \in V$ and $T, S \in wHom(V, W)$, then $(T + S)(a^2x) = T(a^2x) + S(a^2x) = a^2T(x) + a^2S(x) = a^2(T(x) + S(x)) = a^2(T + S)(x) \Rightarrow T + S \in wHom(V, W)$ (by lemma 3.4).

(2) Let $a, b \in R$, $T \in wHom(V, W)$ and $x \in V$, then $(a^2T)(b^2x) = (a^2)^2T(b^2x) = a^2(b^2T(x)) = (a^2b^2)T(x) = b^2(a^2T(x)) = b^2(a^2T)(x) \Rightarrow a^2T \in wHom(V, W)$ (by lemma 3.4).

□

Remark 3.8. With the above definition of $'+'$ and scalar multiplication, we observed that $Hom(V, W)$ is not a Skew Boolean module over R.

Theorem 3.9. *Let V and W be Skew Boolean module over R, then $wHom(V, W)$ is a Skew Boolean module over R (with addition and scalar multiplication defined by definition 3.1).*

Proof. The proof of this theorem is a routine verification of the axioms 1 to 4 of definition 2.5. □

Theorem 3.10. *$stHom(V, W)$ is a sub Skew Boolean module of $wHom(V, W)$*

Proof. It is easy to see that the map $\theta : V \to W$ such that $\theta(x) = 0\ \forall x \in V$ belongs to $stHom(V, W)$. Now, let $T, S \in stHom(V, W)$, then

$(T - S)(x + y) = T(x + y) - S(x + y) = T(x) + T(y) - S(x) - S(y) = T(x) - S(x) + T(y) - S(y) = (T - S)(x) + (T - S)(y)$

$\Rightarrow T - S \in stHom(V, W)$. Thus, $stHom(V, W)$ is a sub group of $wHom(V, W)$. Moreover, Let $a \in R$ and $T \in stHom(V, W)$, then $(a^2T)(x + y) = (a^2)^2T(x + y) = a^2(T(x) + T(y)) = a^2T(x) + a^2T(y) = (a^2T)(x) + (a^2T)(y)$
$\Rightarrow a^2T \in stHom(V, W)$. Hence, $stHom(V, W)$ is a sub Skew Boolean module of $wHom(V, W)$. □

4. Quotient Skew Boolean modules

Recall the following from remark 2.12

(1) $W_1 \vee W_2 = \{x + y, x \in W_1, y \in W_2\}$
(2) $W_1 \wedge W_2 = W_1 \cap W_2$

<u>Notation</u>:If $W_1 \vee W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then we write $V = W_1 \oplus W_2$.

Remark 4.1. Let $a \in R$, then $V = V_{a^2} \oplus V_{1+a^2}$

Proof. By lemma 2.14, we have $V_{a^2} \vee V_{1+a^2} = V_{a^2+(1+a^2)+a^2(1+a^2)} = V_1 = V$. Moreover, Let $x \in V_{a^2} \cap V_{1+a^2}$, then $|x| < a^2$ and $|x| < 1 + a^2$ which implies $|x| = a^2|x|$ and $|x| = (1 + a^2)|x|$. These together will give us $|x| = 0$ which implies $x = 0$. Hence the $V = V_{a^2} \oplus V_{1+a^2}$. □

Theorem 4.2. *Let V be a Skew Boolean module over R and let W and Z be sub Skew Boolean modules of V. If $V = W \oplus Z$, then Z and V/W are isomorphic. Also, the mapping, $T(x) = x + W$ for all $x \in Z$, is an isomorphism of Z on to V/W.*

Proof. Consider the mapping T of Z in to V/W defined by $T(x) = x+W$ for all $x \in Z$. Thus T is a group isomorphism (is a standard result in the theory of groups) of the group Z on to the group V/W. Hence, it remains to show that T preserves the scalar multiplication.
If $x \in Z$ and $a \in R$, then $a^2x \in Z$, thus,
$T(a^2x) = a^2x + W = (a^2)^2x + W = a^2(x + W) = a^2T(x)$
Thus, T is an isomorphism of the Skew Boolean module Z on to the Skew Boolean module V/W. □

Subrahmanyam in [5] introduced the concept of Boolean semi rings as a system $(R, +, .)$ satisfying the following axioms

1. $(R, +)$ *is an abelian group*
2. $(R, .)$ *is a semi group*
3. $a.a = a$ *for all* $a \in R$
4. $a.(b + c) = a.b + a.c$ *for all* $a, b, c \in R$
5. $a.b.c = b.a.c$ *for all* $a, b, c \in R$.

Definition 4.3. Let V and W be Skew Boolean modules over R, and let W be normed. We define the multiplication of the elements in $wHom(V, W)$ as

$$(TS)(x) = |T(x)|S(x)$$

Now we have the following

Lemma 4.4. *Let V and W be Skew Boolean modules over R, and let W be normed. If $T, S \in wHom(V, W)$, then $TS \in wHom(V, W)$.*

Proof. Let $T, S \in Hom(V, W)$, $a \in R$ and $x \in V$, then by lemma 3.4, we have $T(a^2x) = a^2T(x)$ and $S(a^2x) = a^2S(x)$. But then,
$(TS)(a^2x) = |T(a^2x)|S(a^2x) = |a^2T(x)|(a^2S(x)) = (a^2|T(x)|)(a^2S(x)) = a^2(|T(x)|a^2)S(x) = a^2(|T(x)|S(x)) = a^2(TS)(x)$

Thus, by lemma 3.4, we have $TS \in wHom(V, W)$. □

Now we conclude this article with the following theorem.

Theorem 4.5. *If V is a Skew Boolean modules over R and W is a normed Skew Boolean module over R, then $wHom(V, W)$ is a Boolean* *semi ring with addition and multiplication given by definitions 3.1 and 4.3,respectively.*

Proof. (i) By theorem 3.9, we have $(wHom(V,W),+)$ is an abelian group.

(ii) Let $T,S,H \in wHom(V,W)$ and $x \in V$. Then, by lemma 4.4 the product of two elements $TS \in wHom(V,W)$. Thus we have
$[(TS)H](x) = |(TS)(x)|H(x) =||T(x)|S(x)|H(x)$
$=(|T(x)||S(x)H(x)) =|T(x)|(|S(x)|H(x)) = |T(x)|((SH)(x))=$
$[T(SH)](x)$

so that, $(TS)H = T(SH)$ for all $T,S,H \in wHom(V,W)$. Hence, $(wHom(V,W),.)$ is a Boolean semi ring.

(iii) Let $T \in wHom(V,W)$ and $x \in V$, then $(TT)(x) = |T(x)|T(x) = T(x)$ (by lemma 2.10) for all $x \in V$. Hence, $T.T = T$

(iv) Let $T,S,H \in wHom(V,W)$ and $x \in V$, then
$[T(S+H)](x) = |T(x)|(S+H)(x) = |T(x)|[S(x)+H(x)]=$
$|T(x)|S(x)+|T(x)|H(x) = (T.S)(x)+(T.H)(x) =(T.S+T.H)(x)$
Thus, $T.(S+H) = T.S + T.H$.

(v) Let $T,S,H \in wHom(V,W)$ and $x \in V$, then
$(T.S.H)(x) = ((T.S).H)(x) = |(T.S)(x)|H(x)$
$= ||T(x)|S(x)|H(x) = (|T(x)||S(x)|)H(x) =(|S(x)||T(x)|)H(x)=$
$|S(x)|(|T(x)|H(x)) = |S(x)|(T.H)(x) = (S.(T.H))(x)$
Thus, $T.S.H = S.T.H$ for all $T,S,H \in Hom(V,W)$.

Hence, $(wHom(V,W),+,.)$ is a Boolean semi ring. □

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