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ON QUOTIENT CLEAN HYPERRING

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Abstract. In this paper, we introduce the notion of quotient Krasner hyperrings and prove that if I is a normal ideal of Krasner hyperring $(R, +, \cdot)$, then quotient clean Krasner hyperring considered in [1] by Talebi et. al are just clean rings.

Key Words: Krasner hyperring, Clean hyperring, Strong regular relation.
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1. Introduction and basic definitions

The concept of clean rings were introduced by Nicholas [14] in his study of lifting idempotents and exchange rings. Moreover, he proved a ring R is an exchange ring if and only if idempotents can be lifted modulo every left (respectively right) ideal. For a more general introduction to clean rings, see [15, 17].

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [12], at the 8^{th} Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [6, 7, 8].

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The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation, then we say that R is an *additive hyperring*. A special case of this type is the hyperring introduced by Krasner[11]. The concept of hypergroup over a Krasner hyperring has been introduced and investigated by Massouros [13] and this concept has been studied in depth by many authors, for example, see [2, 4, 16]. Recently, the notion of Γ-hyperstructure introduced and studied by many researcher and represent an intensively studied field of research, for example, see[3, 9, 10].

In this section we present some notion. These definitions and results are necessary for the next section.

Let H be a non-empty set and $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \ A \circ \{x\} = A \circ x, \ \{x\} \circ B = x \circ B.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$. A hypergroupoid $(H, \circ)$ is called a *quasi-hypergroup* if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the *reproduction axiom*. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

A special case of this type is the *hyperring* introduced by Krasner [11]. Also, Krasner introduced a new class of hyperrings and hyperfields.
A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1. $(R, +)$ is a canonical hypergroup, i.e.,
   - (i) for every $x, y, z \in R$; $(x + y) + z = x + (y + z)$,
   - (ii) for every $x, y \in R$; $x + y = y + x$,
   - (iii) there exists $0 \in R$ such that $x = x + 0$, for all $x \in R$,
   - (iv) for every $x \in R$ there exists a unique element $x^{'}$ such that $0 \in x + x^{'}$ (we shall write $-x$ for $x^{'}$ and we call it the opposite of x),
   - (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$.
2. Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element.

3. The multiplication is distributive with respect to the hyperoperation $+$.

Let $(R, +, \cdot)$ be a hyperring and I be a non-empty subset of R. Then, I is said to be a *subhyperring* of R if $(I, +, \cdot)$ is itself a hyperring. A subhyperring I of a hyperring R is a *left (right) hyperideal* of R if $r \cdot a \subseteq I$ $(a \cdot r \subseteq I)$ for all $r \in R$, $a \in A$. I is called a *hyperideal* if I is both a left and a right hyperideal. An ideal I of hyperring R is called *normal* if $x + I - x \subseteq I$, for every $x \in R$.

Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H. If A and B are non-empty subsets of H, then $A\overline{\rho}B$ means that for every $a \in A$, there is $b \in B$ such that $\rho(a) = \rho(b)$ and for every $b \in B$ there is $a \in A$ such that $\rho(a) = \rho(b)$, and $A\overline{\overline{\rho}}B$, means that for every $a \in A$ and $b \in B$, we have $\rho(a) = \rho(b)$. The equivalence relation ρ is called *regular on the right* (on the left) if for all x of H, from $a\rho b$, it follows that $(a \circ x)\overline{\rho}(b \circ x)$ $((x \circ a)\overline{\rho}(x \circ b)$ respectively) and ρ is called *strongly regular on the right* (on the left) if for all a, b of H, from $a\rho b$, it follows that $(a \circ x)\overline{\overline{\rho}}(b \circ x)$ $((x \circ a)\overline{\overline{\rho}}(x \circ b)$ respectively), and ρ is called *regular (strongly regular)* if it is regular (strongly regular) on the right and on the left. Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H. If ρ is regular, then $H/\rho = \{\rho(a) : a \in H\}$ is a semihypergroup, with respect to the hyperoperation $\rho(a) \odot \rho(b) = \{\rho(c) : c \in a \circ b\}$ and if this hyperoperation is well defined on H/ρ, then ρ is regular (see Theorem 2.5.2 in [8]). Moreover, if $(H, \circ)$ is a hypergroup and ρ is an equivalence relation on H, then ρ is strongly regular if and only if $(H/\rho, \circ)$, is a group (see Corollary 2.5.6 in [8]).

In this paper, the notion of quotient krasner hyperrings are studied. Let $(R, +, \cdot)$ be a Krasner hyperring and I be a hyperideal of R. Then, the quotient $[R : I^*]$ is a Krasner hyperring. But if I is a normal hyperideal , then the quotient $[R : I^*]$ is a ring and the relation I^* is a strong regular relation. Then the all quotient hypering considered in [1] are just rings.

2. Quotient Hypergroup

There are many classes of hypergroups, which have aroused a major interest such as regular hypergroups, regular reversible hypergroups, canonical hypergroups, join spaces, polygroups, complete hyper-groups, cambiste hypergroups, cogroups, associativity hypergroups, cyclic hypergroups, P-hypergroups, 1-hypergroups and others. Canonical hypergroups are a particular case of join spaces. The structure of canonical hypergroups was individualized for the first time by M. Krasner as the additive structure of hyperfields.

Let N be a subhypergroup of a canonical hypergroup G. In this section, we construct quotient canonical hypergroup $[G : N^*]$ and prove that when N is normal, $[G : N^*]$ is an abelian group.

Let $(G, +)$ be a semihypergroup and ρ be an equivalence relation on G. If A and B are nonempty subsets of G, then

If N is a subhypergroup of a canonical hypergroup G, then we define the relation

$$g_1 \equiv g_2 \Leftrightarrow g_1 \in g_2 + N,$$

for every $g_1, g_2 \in G$. This relation is denoted by $g_1 N^* g_2$.

Proposition 2.1. *Let N be a subhypergroup of canonical hypergroup G. Then, $[G : N^*]$ is a canonical hypergroup with the following hyper operation:*

$$N^*(x) \oplus N^*(y) = \{N^*(z) : z \in x + y\}.$$

Proof. Suppose that $N^*(x_1) = N^*(x_2)$ and $N^*(y_1) = N^*(y_2)$. Hence $x_2 \in x_1 + N, y_2 \in y_1 + N$. This implies that

$$x_2 + y_2 \subseteq (x_1 + N) + (y_1 + N) = (x_1 + y_1) + N.$$

Hence for every $z_2 \in x_2 + y_2$, there is $z_1 \in x_1 + y_1$ and $n \in N$ such that $z_2 \in z_1 + n$. Then

$$N^*(x_2) \oplus N^*(y_2) \subseteq N^*(x_1) \oplus N^*(y_1).$$

By a similar argument, we get

$$N^*(x_1) \oplus N^*(y_1) \subseteq N^*(x_2) \oplus N^*(y_2).$$

Thus, hyperaddition $\oplus$ is well defined. Let $N^*(x_1), N^*(x_2), N^*(x_3) \in [G : N^*]$ and $N^*(x) \in (N^*(x_1) \oplus N^*(x_2)) \oplus N^*(x_3)$. Then for some $N^*(a) \in N^*(x_1) \oplus N^*(x_2)$, we have $N^*(x) \in N^*(a) \oplus N^*(x_3)$ and $N^*(a) = N^*(b)$ that is $b \in x_1 + x_2$. This means that $x \in a + x_3 \subseteq b + N + x_3 \subseteq (x_1 + x_2) + x_3 + N = x_1 + (x_2 + x_3) + N$. Hence $x \in x_1 + c + N$ where $c \in x_2 + x_3$. Then $N^*(x) \in N^*(x_1) \oplus (N^*(x_2) \oplus N^*(x_3))$. This means that $(N^*(x_1) \oplus N^*(x_2)) \oplus N^*(x_3) \subseteq N^*(x_1) \oplus (N^*(x_2) \oplus N^*(x_3))$. Similarly, we get $N^*(x_1) \oplus (N^*(x_2) \oplus N^*(x_3) \subseteq (N^*(x_1) \oplus N^*(x_2)) \oplus N^*(x_3)$. Thus, the hyperoperation $\oplus$ is associative. Let $N^*(0) \in [G : N^*]$. Then, for any $N^*(x) \in [G : N^*]$, we have

$$N^*(x) \oplus N^*(0) = \{N^*(y) : y \in x + 0\} = N^*(x).$$

Similarly, $N^*(0) \oplus N^*(x) = N^*(x)$. Let $x \in G$. Then,

$$N^*(x) \oplus N^*(-x) = \{N^*(y) : y \in x + (-x)\}$$

Since, $0 \in x - x$, we have $N^*(0) \in N^*(x) \oplus N^*(-x)$. Let $N^*(a)$ be another inverse element of $N^*(x)$. Then, $0 \in a + x$ implies that $a \in 0 - x \in N - x$. Hence $N^*(-x) = N^*(a)$.Thus, the element $N^*(x)$ has a unique inverse $N^*(-x)$.

Let $N^*(x) \in N^*(y) \oplus N^*(z)$. Then, there is $a \in y + z$ such that $N^*(x) = N^*(a)$. This implies that $y \in a - z \subseteq x + N - z$. Hence there is $b \in x - z$ such that $y \in b + N$. Thus, $N^*(y) = N^*(b) \in N^*(x) \oplus N^*(-z)$. Similarly, we can see $N^*(z) \in N^*(x) \oplus N^*(-y)$. G is commutative, it is obvious that $[G : N^*]$ is commutative. Therefore, $[G : N^*]$ is commutative. $\square$

Proposition 2.2. *Let N be a normal canonical subhypergroup of hypergroup G. Then, for every $x_1, x_2 \in G$ the following are equivalent:*

(i) $x_2 \in x_1 + N$,
(ii) $x_1 - x_2 \subseteq N$,
(iii) $(x_1 - x_2) \cap N \neq \emptyset$.

Proof. Suppose that $(x_1 - x_2) \cap N \neq \emptyset$. Then there exists $x \in (x_1 - x_2) \cap N$. So $-x_2 + x_1 \subseteq -x_2 + x + x_2 \subseteq N$. If $x \in -x_2 + x_1$, then $x \in N$. Hence $-x_2 \in x - x_1$ and $x_2 \in x_1 - x \subseteq x_1 + N$. Therefore, (iii) implies (i). It is easy to see that (i) implies (ii) and (ii) implies (iii). $\square$

Remark 2.3. Let G be a canonical hypergroup and N be a normal canonical subhypergroup of G. Then, the equivalence relation $\equiv$ coincide with the equivalence relation defined by Talebi et. al in [1].

Definition 2.4. Let G be a canonical hypergroup and N be a subhypergroup of G. We denote $\Omega(N) = \{x \in G : x - x \subseteq N\}$.

Proposition 2.5. *Let G be a canonical hypergroup and N be a subhypergroup of G. Then, $\Omega(N)$ is a subhypergroup of G and $N \subseteq \Omega(N)$.*

Proof. Since $N \neq \emptyset$, the set $\Omega(N)$ is non-empty. Let $x_1, x_2 \in \Omega(N)$, $x \in x_1 - x_2$. Then,

$$x - x \subseteq (x_1 - x_2) - (x_1 - x_2) = (x_1 - x_1) + (x_2 - x_2) \subseteq N + N = N.$$

Hence $x_1 - x_2 \subseteq \Omega(N)$. Moreover, for every $x \in N$, since N is a subhypergroup of G, $x - x \subseteq N$. Therefore, $\Omega(N)$ is a subhypergroup of G containing N. $\square$

Proposition 2.6. *Let G be a canonical hypergroup and $x_1, x_2 \in \Omega(\{0\})$. Then, $x_1 + x_2$ is a singleton set.*

Proof. The proof is straightforward. $\square$

Proposition 2.7. *Let G be a hypergroup. Then, $\Omega(\{0\})$ is an abelian group and for every subgroup M_1 of G, $M_1 \subseteq \Omega(\{0\})$.*

Proof. Suppose that $x_1, x_2 \in \Omega(\{0\})$ and $x, y \in x_1 + x_2$. Then

$$x - y \subseteq (x_1 + x_2) - (x_1 + x_2) = (x_1 - x_2) - (x_1 - x_2) = 0.$$

This implies that $x_1 + x_2$ is a singleton and $\Omega(\{0\})$ is a subgroup. Let M_1 be any subgroup of G and $x \in M_1$. Then, $x - x = \{0\}$. Hence $x \in \Omega(\{0\})$ and $M_1 \subseteq \Omega(\{0\})$. This completes the proof. $\square$

Corollary 2.8. *Let G be a canonical hypergroup and N be a subhypergroup of G. Then, $(M, +)$ is abelian group if and only if $\Omega(\{0\}) = M$.*

Proposition 2.9. *Let G be a canonical hypergroup and N be a subhypergroup of G. Then, N is normal if and only if $\Omega(N) = G$.*

Proof. Suppose that N be a subhypergroup and $\Omega(N) = G$. Then for every $x \in G$ and $n \in N$ we have

$$x + n - x = x - x + n \subseteq N + n \subseteq N + N = N.$$

Hence N is normal. Let N be a normal subhypergroup and $x \in G$. This implies that $x + 0 - x \subseteq x + N - x \subseteq N$. Hence $x - x \subseteq N$, for every $x \in G$. Therefore, $G \subseteq \Omega(N)$. This completes the proof. $\square$

Corollary 2.10. *Let N_1 and N_2 be subhypergroups of G such that $N_1 \subseteq N_2$ and N_1 be normal subhypergroup. Then, N_2 is also normal.*

Corollary 2.11. *Let G be a canonical hypergroup such that $\{0\}$ is normal. Then, all subhypergroups of G are normal.*

Theorem 2.12. *Let G be a canonical hypergroup. Then, $(G, +)$ is abelian group if and only if $\{0\}$ is a normal canonical subhypergroup.*

Proof. We know that $(G, +)$ is abelian group if and only if $\Omega(\{0\}) = G$. Moreover, $\Omega(\{0\}) = G$ if and only if $\{0\}$ is a normal subhypergroup. Hence, $(G, +)$ is an abelian group if and only if $\{0\}$ is a normal subhypergroup of G. This completes the proof. $\square$

Definition 2.13. Let G be a canonical hypergroup. Then, we define

$$S(G) = \left\{ x \in G : \exists\ 1 \leq i \leq n,\ x_i \in G,\ x \in \sum_{i=1}^{n} (x_i - x_i) \right\}.$$

Proposition 2.14. *Let G be a canonical hypergroup. Then, $S(G)$ is a smallest normal canonical subhypergroup of G.*

Proof. Suppose that $x, y \in S(G)$. Then,

$$x \in \sum_{i=1}^{n}(x_i - x_i),\ y \in \sum_{j=1}^{m}(y_j - y_j),$$

where $x_i, y_j \in G$. This implies that $x - y \in S(G)$ and $S(G)$ is a canonical subhypergroup of G. Now, for $a \in S(G)$ there exists $n \in \mathbb{N}$ and $x_i \in G$ such that $a \in \sum_{i=1}^{n}(x_i - x_i)$. Thus, for every $x \in G$,

$$x + a - x \subseteq x + \sum_{i=1}^{n}(x_i - x_i) - x = (x - x) + \sum_{i=1}^{n}(x_i - x_i) \in S(G).$$

Hence $S(G)$ is a normal canonical subhypergroup of G.

Assume that N is a normal canonical subhypergroup of G. Then, for every $x \in G$,

$$x - x = x + 0 - x \subseteq x + N - x \subseteq N.$$

Since N is a canonical subhypergroup of G, for every $x_i \in G$, $\sum_{i=1}^{n}(x_i - x_i) \subseteq N$. This implies that $S(G) \subseteq N$. Therefore, $S(G)$ smallest normal canonical subhypergroup of G. □

Corollary 2.15. *Let G be a canonical hypergroup. Then, G ia an abelian group if and only if $S(G) =< 0 >$.*

Theorem 2.16. *Let G be a canonical hypergroup and N be a normal canonical subhypergroup of G. Then, $[G : N^*]$ is an abelian group.*

Proof. Suppose that N is a normal canonical subhypergroup of G. Since

$$N^*(x) \oplus N^*(0) \oplus N^*(-x) = (x + N - x) + N \subseteq N + N = N.$$

This implies that $\{N\}$ is a normal canonical subhypergroup in $[G : N^*]$. By Corollary 2.12, $[G : N^*]$ is an abelian group. □

Remark 2.17. Suppose that $(R, +, \cdot)$ is a krasner hyperring and I is a hyperideal of R. Hence $[R : I^*]$ is a hyperring. Moreover when I is a normal ideal of R, then the quotient hypering considered in [1] are just rings.

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