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FUZZY SOFT $k-$IDEALS OVER SEMIRING AND FUZZY SOFT SEMIRING HOMOMORPHISM

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Abstract. In this paper, we introduce the notion of fuzzy soft semirings, fuzzy soft ideals, fuzzy soft $k-$ ideals , $k-$fuzzy soft ideals over semirings and fuzzy soft semiring homomorphism. We study some of their algebraical properties and properties of homomorphic image of fuzzy soft semiring.

Key Words: fuzzy soft semiring, fuzzy soft ideal, fuzzy soft $k-$ideal over semiring, fuzzy soft semiring homomorphism.

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1. Introduction

Notion of a semiring was introduced by Vandiver [10] in 1934. Semiring is a well known universal algebra. If in a ring, we do away with the requirement of having additive inverse of each element then the resulting algebraic structure becomes semiring. An universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, $i.e., a(b + c) = ab + ac,\ (a + b)c = ac + bc,$ for all $a, b, c \in S$. Though semiring is a generalization of ring, ideals of semiring do not coincide with ring ideals. For example an ideal of semiring need not be the kernel of some semiring homomorphism. To solve this problem Henriksen [5] defined $k-$ideals and Iizuka [6] defined $h-$ideals in semirings to obtain analogues of ring results for semirings.

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The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. Semiring is very useful for solving problems in applied mathematics and information science because semiring provides an algebraic frame work for modeling. Semirings play an important role in studying matrices and determinants.

Problems which one come face to face within life cannot be solved using classical mathematical methods. There are well known theories have their inherent difficulties. Molodtsov [8] introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties. The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty and was first introduced by Zadeh [11]. The concept of fuzzy subgroup was introduced by Rosenfeld [9]. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. Then Maji et al. [7] extended soft set theory to fuzzy soft set theory. Aktas and Cagman [2] defined the notion of soft groups. Feng et al. [3] initiated the study of soft semirings. Soft rings are defined by Acar et al. [1] and Jayanth Ghosh et al. [4] initiated the study of fuzzy soft rings and fuzzy soft ideals. In this paper we introduce the notion of fuzzy soft semirings, fuzzy soft ideals, fuzzy soft k-ideals and $k-$fuzzy soft ideals over semiring and study some of their algebraic properties. We introduce the notion of fuzzy soft semiring homomorphism and study some properties of homomorphic image of fuzzy soft semirings and fuzzy soft semiring homomorphism.

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 1.1. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

A function $f : R \to S$, where R and S are semirings is said to be semiring homomorphism if $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$. Let S be a non-empty set. A mapping $f : S \to [0,1]$ is called a fuzzy subset of S. Let f be a fuzzy subset of a non-empty set

S, for $t \in [0,1]$ the set $f_t = \{x \in S \mid f(x) \geq t\}$ is called a level subset of S with respect to f.

Definition 1.2. Let S be a semiring . A fuzzy subset μ of S is said to be fuzzy subsemiring of S if it satisfies the following conditions

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$.

Definition 1.3. A fuzzy subset μ of semiring S is called a fuzzy left(right) ideal of S if for all $x, y \in S$

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq \mu(y)(\mu(x))$

Definition 1.4. A fuzzy subset μ of semiring S is called a fuzzy ideal of S if for all $x, y \in S$

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq \max\{\mu(x), (\mu(y)\}$

An ideal I of semiring S is called a $k-$ideal if for $x, y \in S,\ x+y \in I,\ y \in I \Rightarrow x \in I$. A fuzzy ideal f of semiring S is said to be $k-$fuzzy ideal of S if $f(x+y) = f(0)$ and $f(y) = f(0)$ then $f(x) = f(0)$ for all $x, y \in S$. A fuzzy ideal μ of semiring S is said to be fuzzy $k-$ideal of S if $\mu(x) \geq \min\{\mu(x+y), \mu(y)\}$ for all $x, y \in S$. Let f and g be fuzzy subsets of S. Then $f \cup g, f \cap g$ are fuzzy subsets of S defined by $f \cup g(x) = \max\{f(x), g(x)\},\ f \cap g(x) = \min\{f(x), g(x)\}$ for all $x \in S$. And $f \circ g$ is defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy,\ x,y \in S} & \{\min\{f(x), g(y)\}\}, \\ 0 & \text{otherwise} \end{cases},\ \text{for all } z \in S.$$

A fuzzy subset μ of semiring S is said to be normal fuzzy if μ is a fuzzy subsemiring of S and $\mu(0) = 1$. For any two fuzzy subsets λ and μ of $S,\ \lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in S$.

Let U be an initial universe set, E be the set of parameters and $\mathbb{P}(U)$ denotes the power set of U. A pair (f, E) is called a soft set over U where f is a mapping given by $f : E \to \mathbb{P}(U)$. For a soft set (f, A), the set $\{x \in A \mid f(x) \neq \emptyset\}$ is called a support of (f, A) and it is denoted by Supp (f, A). If Supp$(f, A) \neq \emptyset$ then (f, A) is called a non null soft set. Let $(f, A), (g, B)$ be two soft sets over U. Then (f, A) is said to be soft subset of (g, B) denoted by $(f, A) \subseteq (g, B)$ if $A \subseteq B$ and $f(a) \subseteq g(a)$ for all $a \in A$. Let $A \subseteq E$. A pair (f, A) is called a fuzzy soft set over U where f is a mapping given by $f : A \to I^U$ where I^U denotes the collection of all fuzzy subsets of U. Let X be a group and (f, A) be a soft set over X. Then (f, A) is said to be soft group over X if and only if $f(a)$ is a subgroup of X for each $a \in A$. Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset f of S is called a ϕ invariant if $\phi(x) = \phi(y) \Rightarrow f(x) = f(y)$.

Definition 1.5. Let X be a group and (f, A) be a fuzzy soft set over X. Then (f, A) is said to be fuzzy soft group over X if and only if for each $a \in A,\ x, y \in X$.

(i) $f_a(x * y) \geq f_a(x) * f_a(y)$
(ii) $f_a(x^{-1}) \geq f_a(x)$

where f_a is the fuzzy subset of X corresponding to the parameter $a \in A$.

Definition 1.6. Let $(f, A), (g, B)$ be fuzzy soft sets over U. The intersection of fuzzy soft sets over U. (f, A) and (g, B) is denoted by $(f, A) \cap (g, B) = (h, C)$ where $C = A \cap B$ is defined as

$$h_c = f_c \cap g_c, \quad \text{if} \ \ c \in A \cap B.$$

Definition 1.7. Let $(f, A), (g, B)$ be fuzzy soft sets over U. The union of soft sets (f, A) and (g, B) is denoted by $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$ and it is defined as

$$h_c = \begin{cases} f_c & \text{if } c \in A \setminus B; \\ g_c & \text{if } c \in B \setminus A; \\ f_c \cup g_c & \text{if } \ c \in A \cap B. \end{cases}$$

Definition 1.8. Let $(f, A), (g, B)$ be fuzzy softsets over U. (f, A) AND (g, B) is denoted by "$(f, A) \wedge (g, B)$"and it is defined by $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$ and $h_c(x) = \min\{f_a(x), g_b(x)\}$ for all $c = (a, b) \in A \times B$ and $x \in U$.

Definition 1.9. Let $(f, A), (g, B)$ be fuzzy softsets over U."(f, A) OR (g, B)" is denoted by $(f, A) \vee (g, B)$ and it is defined by $(f, A) \vee (g, B) = (h, C)$ where $C = A \times B$ and $h_c(x) = \max\{f_a(x), g_b(x)\}$ for all $c = (a, b) \in A \times B, x \in U$.

2. Fuzzy soft semiring and fuzzy soft ideal over semiring

In this section, the concepts of soft semiring, fuzzy soft semiring and fuzzy soft ideal over semiring are introduced and study the properties related to these notions.

Definition 2.1. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to \mathbb{P}(S)$ where $\mathbb{P}(S)$ is the power set

of S. Then (f, A) is called a soft semiring over S if and only if for each $a \in A, f(a)$ is subsemiring of S. i.e.
$(i) x, y \in S \Rightarrow x + y \in f(a)$ $(ii) x, y \in S \Rightarrow x \cdot y \in f(a)$.

Definition 2.2. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft semiring over S if and only if for each $a \in A, f(a) = f_a$ is the fuzzy sub semiring of S. i.e.,(i) $f_a(x+y) \geq min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq min\{f_a(x), f_a(y)\}$ for all $x, y \in S$.

Definition 2.3. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to \mathbb{P}(S)$. Then (f, A) is called a soft left(right) ideal over S if and only if for each $a \in A, f(a)$ is a left(right) ideal of S. i.e.,
(i) $x, y \in f(a) \Rightarrow x + y \in f(a)$ (ii) $x, y \in f(a), r \in S \Rightarrow rx(xr) \in f(a)$.

Definition 2.4. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to \mathbb{P}(S)$. Then (f, A) is called a soft ideal over S if and only if for each $a \in A, f(a)$ is an ideal of S. i.e.,
(i) $x, y \in f(a) \Rightarrow x + y \in f(a)$ (ii) $x \in f(a), r \in S \Rightarrow r \cdot x \in f(a)$ and $x \cdot r \in f(a)$.

Definition 2.5. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft left(right) ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \to [0, 1]$ is a fuzzy left(right) ideal of S. i.e.,
(i) $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq f_a(y)\big(f_a(x)\big)$ for all $x, y \in S$.

Definition 2.6. Let S be a semiring, E be a parameter set, $A \subseteq E$ and f be a mapping given by $f : A \to [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \to [0, 1]$ is a fuzzy ideal of S. i.e.,
(i) $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$ (ii) $f_a(x \cdot y) \geq max\{f_a(x), f_a(y)\}$ for all $x, y \in S$.

Definition 2.7. Let $(f, A), (g, B)$ be fuzzy soft ideals over semiring S. The product (f, A) and (g, B) is defined as $((fog), C)$ where $C = A \cup B$

and

$$f \circ g(x) = \begin{cases} f_c(x), & \text{if } c \in A \setminus B; \\ g_c(x), & \text{if } c \in B \setminus A; \\ \sup\limits_{x=ab}\{\min\{f_c(a), g_c(b)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

for all $c \in A \cup B$ and $x \in S$.

Theorem 2.8. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring$S$. Then $(f, A) \cup (g, B)$ is a fuzzy soft semiring over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft semirings over semiring S and $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$.
Case (i) : Suppose $A \cap B = \emptyset$ and $c \in C = A \cup B$. Then $c \in A$ or $c \in B$. If $c \in A$ then $h_c = f_c$ and if $c \in B$ then $h_c = g_c$, since (f, A) and (g, B) are fuzzy soft semirings over semiring S, in both cases h_c is a fuzzy subsemiring of S.
Case(ii) : If $A \cap B \neq \emptyset$.
If $c \in A \setminus B$ then $h_c = f_c, h_c$ is a fuzzy subsemiring.
If $c \in B \setminus A$ then $h_c = g_c, h_c$ is a fuzzy subsemiring.
If $c \in A \cap B$ then $h_c = f_c \cup g_c$. Let $x, y \in S$. Then

$$\begin{aligned} h_c(x + y) =& f_c \cup g_c(x + y) \\ =& max\{f_c(x + y), g_c(x + y)\} \\ \geq& max\{min\{f_c(x), fc(y)\}, min\{g_c(x), g_c(y)\}\} \\ =& min\{max\{f_c(x), g_c(x)\}, max\{f_c(y), g_c(y)\}\} \\ =& min\{f_c \cup g_c(x), f_c \cup g_c(y)\} \\ h_c(xy) =& fc \cup g_c(xy) \\ =& max\{f_c(xy), g_c(xy)\} \\ \geq& max\{min\{f_c(x), f_c(y\}, min\{g_c(x), g_c(y)\}\} \\ =& min\{max\{f_c(x), g_c(x)\}, max\{f_c(y), g_c(y)\}\} \\ =& min\{f_c \cup g_c(x), f_c \cup g_c(y)\} \end{aligned}$$

Hence h_c is a fuzzy subsemiring of S. Therefore $(f, A) \cup (g, B)$ is a fuzzy soft semiring over semiring S. □

Theorem 2.9. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring S. Then $(f, A) \cap (g, B)$ is a fuzzy soft semiring over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft semirings over semiring S and $(f, A) \cap (g, B) = (h, C)$ where $C = A \cap B$ and

$$h_c(x) = f_c \cap g_c(x),$$

for all $c \in C = A \cap B$, for all $x \in S$.

If $c \in A \cap B$ then $h_c(x) = f_c \cap g_c(x)$. Let $x, y \in S$. Then

$$\begin{aligned} h_c(x+y) =& (f_c \cap g_c)(x+y) \\ =& min\{f_c(x+y), g_c(x+y)\} \\ \geq& min\{min\{f_c(x), f_c(y)\}, min\{g_c(x), g_c(y)\}\} \\ =& min\{min\{f_c(x), g_c(x)\}, min\{f_c(y), g_c(y)\}\} \\ =& min\{(f_c \cap g_c)(x), (f_c \cap g_c)(y)\} \\ h_c(xy) =& (f_c \cap g_c)(xy) \\ =& min\{f_c(xy), g_c(xy)\} \\ \geq& min\{min\{f_c(x), f_c(y)\}, min\{g_c(x), g_c(y)\}\} \\ =& min\{min\{f_c(x), g_c(x)\}, min\{f_c(y), g_c(y)\}\} \\ =& min\{(f_c \cap g_c)(x), (f_c \cap g_c)(y)\}. \end{aligned}$$

Hence h_c is a fuzzy subsemiring of S. Therefore $(f, A) \cap (g, B)$ is a fuzzy soft semiring over semiring S. □

Theorem 2.10. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring S. Then $(f, A) \wedge (g, B)$ is a fuzzy soft semiring over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft semirings over semiring S. By Definition 1.8, $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$. Let $c = (a, b) \in C = A \times B$ and $x, y \in S$. Then

$$\begin{aligned} h_c(x+y) =& min\{f_a(x+y), g_b(x+y)\} \\ \geq& min\{min\{f_a(x), f_a(y)\}, min\{g_b(x), g_c(y)\}\} \\ =& min\{min\{f_a(x), g_b(x)\}, min\{f_a(y), g_b(y)\}\} \\ =& min\{h_c(x), h_c(y)\} \\ h_c(xy) =& min\{f_a(xy), g_b(xy)\} \\ \geq& min\{min\{f_a(x), f_a(y)\}, min\{g_b(x), g_b(y)\}\} \\ =& min\{min\{f_a(x), g_b(x)\}, min\{f_a(y), g_b(y)\}\} \\ =& min\{h_c(x), h_c(y)\}. \end{aligned}$$

Hence h_c is a fuzzy subsemiring of S. Therefore $(f, A) \wedge (g, B)$ is a fuzzy soft semiring over semiring S. □

The following theorem can be proved easily.

Theorem 2.11. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring S. Then $(f, A) \vee (g, B)$ is a fuzzy soft semiring over semiring semiring S.*

Theorem 2.12. *Let S be a semiring , E be a parameter set and $A \subseteq E$. Then (f, A) is a fuzzy soft left ideal over semiring S if and only if for each $a \in A, (f_a)_t (t \in Im(f_a))$ is a left ideal of S where f_a is the fuzzy subset of S.*

Proof. Let S be a semiring , E be a parameter set and $A \subseteq E$. Suppose (f, A) is a fuzzy soft left ideal over S. Let $a \in A$ and $t \in Im(f_a)$ and $x, y \in (f_a)_t, r \in S$. Then $f_a(x + y) \geq min\{f_a(x), f_a(y)\} \geq min\{t, t\} = t$ and

$f_a(rx) \geq f_a(x) \geq t$ which implies that $x + y, rx \in (f_a)_t$. Therefore for each $t \in Im(f_a)$, $(f_a)_t$ is a left ideal of S. Conversely, suppose that $(f_a)_t$ is a left ideal of S for each $t \in Im(f_a)$ and corresponding to each $a \in A$ and $x, y \in S$. Suppose $f_a(x + y) < min\{f_a(x), f_a(y)\} = t_1$ (say). Then $x, y \in (f_a)_{t_1}1$, but $x + y \neq (f_a)_{t_1}$ which is a contradiction. So $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$. Suppose $f_a(xy) < f_a(y) = t_2$ (say). Then $y \in (f_a) - t_2$, but $xy \neq (f_a)_{t_2}$ which is a contradiction. Therefore $f_a(xy) \geq f_a(y)$. Hence f_a is a fuzzy left ideal of S. Therefore (f, A) is a fuzzy soft left ideal over semiring semiring S. □

The following corollary follows from above theorem

Corollary 2.13. *Let μ be a fuzzy set in a semiring S and $A = [0, 1]$. Then (μ, A) is a fuzzy soft left ideal over S if and only if μ_t is a fuzzy left ideal of S.*

Theorem 2.14. *Let S be a semiring, E be a parameter set and $A \subseteq E$. Then (f, A) is a fuzzy soft left(right) ideal over S if and only if for each $a \in A$, the corresponding fuzzy set f_a of S satisfies the following conditions*
(1) $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$
(2) $\chi_S o f_a \subseteq f_a (f_a o \chi_S \subseteq f_a)$
where χ_S stands for characteristic function of S.

Proof. Let S be a semiring , E be a parameter set and $A \subseteq E$. Suppose (f, A) is a fuzzy soft left ideal over S. Then, for each $a \in A, f_a$ is a fuzzy left ideal of S. Let $z \in S$. Then

$$\chi_S o f_a(z) = \sup_{z=xy}\{min\{\chi_S(x), f_a(y)\}\} = \sup_{z=xy}\{f_a(y)\} \leq f_a(xy) = f_a(z)$$

If z cannot be expressed as $z = xy$ where $x, y \in S$ then $\chi_S o f_a(z) = 0 \leq f_a(z)$. Conversely, suppose that for each $a \in A, f_a$ satisfies the given two conditions. Let $x, y \in S$. Then we have

$$f_a(xy) \geq \chi_S o f_a(xy) = \sup_{xy=pq} min\{\chi_S(p), f_a(q)\} \geq min\{\chi_S(x), f_a(y)\}$$

$= f_a(y)$. This shows that for each $a \in A, f_a$ is a fuzzy left ideal of S. So (f, A) is a fuzzy soft left ideal over semiring S.
Similarly we can prove the result for a fuzzy soft right ideal over S. $\square$

Theorem 2.15. *Let S be a semiring , E be a parameter set and $A \subseteq E$. Define a fuzzy subset α_a of S corresponding to $a \in A$ by*

$$\alpha_a(x) = \begin{cases} s, & \text{if } x \in I(a); \\ t, & \text{otherwise} \end{cases}$$

for all $x \in S$ and $s, t \in [0, 1]$ with $s < t$. Then (α, A) is a fuzzy soft left ideal over S. if and only if (I, A) is a soft left ideal over semiring S.

Proof. Let S be a semiring, E be a parameter set and $A \subseteq E$. Let (α, A) be a fuzzy soft left ideal over semiring S and $x, y \in I(a)$ and $r \in S$. Then $\alpha_a(x) = s = \alpha_a(y)$ and hence

$$\alpha_a(x + y) \geq min\{\alpha_a(x), \alpha_a(y)\} = min\{s, s\} = s.$$

So $x + y \in I(a)$. Also,$\alpha_a(rx) \geq \alpha_a(x) = s$. Therefore $rx \in I(a)$. Hence $I(a)$ is a left ideal of S. Thus (I, A) is a soft left ideal over semiring S.
Conversely, (I, A) be soft left ideal over semiring S for each $a \in A, I(a)$ is a left ideal of S. Let $x, y \in S$ then the following four cases arise for consideration.
Case (i) : $x, y \in I(a)$.
Then $x + y \in I(a), x \cdot y \in I(a)$ and hence $\alpha_a(x + y) = s, \alpha_a(x) = s = \alpha_a(y) = \alpha_(x \cdot y)$. Therefore $\alpha_a(x + y) = min\{\alpha_a(x), \alpha_a(y)\} = s$ and $\alpha_a(xy) = s = \alpha_a(y)$.
Case (ii) : $x \in I(a), y \notin I(a)$.
Then $x+y \notin I(a)$ and hence $\alpha_a(x) = s, \alpha_a(y) = t$. Therefore $\alpha_a(x+y) = t$. Now

$$t = \alpha_a(x + y) \geq min\{\alpha_a(x), \alpha_a(y)\} = min\{s, t\} = s.$$

Hence $y \cdot x \in I(a)$ implies that $\alpha_a(yx) = s = \alpha_a(x)$.
Case(iii) : $x, y \notin I(a)$.
Then $x+y \notin I(a)$ and $xy \notin I(a)$ and hence $\alpha_a(x) = t = \alpha_a(y).t = \alpha_a(x+y) \geq min\{\alpha_a(x), \alpha_a(y)\} = min\{t, t\} = t$. Therefore $\alpha_a(xy) = t = \alpha_a(y)$. Hence α_a is a fuzzy left ideal of S for each $a \in A$. Therefore (α, A) is a fuzzy soft left ideal over semiring S. □

Theorem 2.16. *If (f, A) is a fuzzy soft semiring over semiring S and for each $a \in A, f_a^+$ is defined by $f_a^+(x) = f_a(x) + 1 - f_a(0)$ for all $x \in S$ then (f^+, A) is a normal fuzzy soft semiring over S,i.e., f_a^+ is a normal fuzzy semiring over S for all $a \in A$ and (f, A) is a subset of (f^+, A).*

Proof. Let (f, A) be a fuzzy soft semiring over semiring S .For each $a \in A, f_a^+$ is defined by $f_a^+(x) = f_a(x) + 1 - f_a(0)$ for all $x \in S$. Suppose $x, y \in S$ and $a \in A$. Then

$$\begin{aligned}
f_a^+(x+y) &= f_a(x+y) + 1 - f_a(0) \\
&\geq min\{f_a(x), f_a(y)\} + 1 - f_a(0) \\
&= min\{f_a(x) + 1 - f_a(0), f_a(y) + 1 - f_a(0)\} \\
&= min\{f_a^+(x), f_a^+(y)\} \\
f_a^+(xy) &= f_a(xy) + 1 - f_a(0) \\
&\geq max\{f_a(x), f_a(y)\} + 1 - f_a(0) \\
&= max\{f_a(x) + 1 - f_a(0), f_a(y) + 1 - f_a(0)\} \\
&= max\{f_a^+(x), f_a^+(y)\}
\end{aligned}$$

If $x = 0$, then $f_a^+(0) = 1$ and $f_a \subset f_a^+$. Hence (f^+, A) is a normal fuzzy soft semiring over semiring S and (f, A) is a subset of (f^+, A). □

Theorem 2.17. *Let (f, A) and (g, B) be fuzzy soft ideals over semiring S. Then $(f, A) \cap (g, B)$ is a fuzzy soft ideal over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft ideals over semiring S. By Definition 1.6, we have $(f, A) \cap (g, B) = (h, C)$ where $C = A \cap B$.

If $c \in A \cap B$, and $x, y \in S$ then $h_c = f_c \cap g_c$

$$\begin{aligned}
h_c(x+y) &= min\{f_c(x+y), g_c(x+y)\} \\
&\geq min\{min\{f_c(x), f_c(y)\}, min\{g_c(x), g_c(y)\}\} \\
&= min\{min\{f_c(x), g_c(x)\}, min\{f_c(y), g_c(y)\} \\
&= min\{f_c \cap g_c(x), f_c \cap g_c(y)\} \\
&= min\{h_c(x), h_c(y)\} \\
\text{and } h_c(xy) &= min\{f_c(xy), g_c(xy)\} \\
&\geq min\{max\{f_c(x), f_c(y)\}, max\{g_c(x), g_c(y)\}\} \\
&= max\{min\{f_c(x), g_c(x)\}, min\{f_c(y), g_c(y)\}\} \\
&= max\{f_c \cap g_c(x), f_c \cap g_c(y)\} \\
&= max\{h_c(x), h_c(y)\}
\end{aligned}$$

Hence h_c is a fuzzy ideal of S. Thus $(f, A) \cap (g, B)$ is a fuzzy soft ideal over semiring S. □

Theorem 2.18. *Let (f, A) and (g, B) be two fuzzy soft ideals over semiring S. Then $(f, A) \cup (g, B)$ is a fuzzy soft ideal over semiring S.*

Proof. Let (f, A) and (g, B) be two fuzzy soft ideals over semiring S. By Definition 1.7, we have $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$.

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B, \\ g_c, & \text{if } c \in B \setminus A; \\ f_c \cup g_c, & \text{if } c \in A \cap B. \end{cases}$$

Case(i): If $c \in A \setminus B$ then $h_c = f_c, h_c$ is a fuzzy ideal of S since (f, A) is a fuzzy soft ideal over semiring S.
Case(ii): If $c \in B \setminus A$ then $h_c = g_c, h_c$ is a fuzzy ideal of S since (g, B) is a fuzzy soft ideal over S.

Case(iii): If $c \in A \cap B$ then for all $x, y \in S$,

$$\begin{aligned} h_c(x+y) =& f_c \cup g_c(x+y) \\ =& \ max\{f_c(x+y), g_c(x+y)\} \\ \geq& \ max\{min\{f_c(x), f_c(y)\}, min\{g_c(x), g_c(y)\}\} \\ =& \ min\{max\{f_c(x), g_c(x)\}, max\{f_c(y), g_c(y)\} \\ =& \ max\{(f \cup g)_c(x), (f \cup g)_c(y)\} \\ h_c(xy) =& \ (f_c \cup g_c)(xy) \\ =& \ max\{f_c(xy), g_c(xy)\} \\ \geq& \ max\{max\{f_c(x), f_c(y)\}, max\{g_c(x), g_c(y)\} \\ =& \ max\{max\{f_c(x), g_c(x)\}, max\{f_c(y), g_c(y)\}\} \end{aligned}$$

Hence h_c is a fuzzy ideal of S. Therefore (h, C) is a fuzzy soft ideal over semiring S. □

Theorem 2.19. *Let (f, A) and (g, B) be two fuzzy soft ideals over semiring S. Then $(f, A) \wedge (g, B)$ is a fuzzy soft ideal over semiring S.*

Proof. Let (f, A) and (g, B) be two fuzzy soft ideals over semiring S. By Definition 1.8, $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$. Let $c = (a, b) \in C = A \times B$ and $x, y \in S$. Then

$$\begin{aligned} h_c(x+y) =& f_a \wedge g_b(x+y) \\ =& min\{f_a(x+y), g_b(x+y)\} \\ \geq& min\{min\{f_a(x), f_a(y)\}, min\{g_b(x), g_b(y)\}\} \\ =& min\{min\{f_a(x), g_b(x)\}, min\{f_a(y), g_b(y)\}\} \\ =& min\{f_a \wedge g_b(x), f_a \wedge g_b(y)\} \\ =& min\{h_c(x), h_c(y)\} \\ h_c(xy) =& f_a \wedge g_b(xy) \\ =& min\{f_a(xy), g_b(xy)\} \\ \geq& min\{max\{f_a(x), f_a(y)\}, max\{g_b(x), g_b(y)\}\} \\ =& max\{min\{f_a(x), g_b(x)\}, min\{f_a(y), g_b(y)\}\} \\ =& max\{h_c(x), h_c(y)\} \end{aligned}$$

Hence h_c is a fuzzy soft ideal of S. Therefore $(h, A \times B)$ is a fuzzy soft ideal over S. □

The following proof of theorem is similar to that of Theorem (2.19) and using Definition 1.9.

Theorem 2.20. *Let (f, A) and (g, B) be fuzzy soft ideals over semiring S. Then $(f, A) \vee (g, B)$ is a fuzzy soft ideal over semiring S.*

Theorem 2.21. *If (f, A) and (g, B) are fuzzy soft ideals over semiring S with an identity element then $(fog, A \cup B)$ is a fuzzy soft ideal over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft ideals over semiring S. Then for any $c \in A \cup B$ and $x, y \in S$.
Case(i) : If $c \in A \setminus B$ then, by Definition 2.7, $(fog)_c = f_c$, which is a fuzzy ideal of S :
Case(ii) : If $c \in B \setminus A$ then, by Definition 2.7, $(fog)_c = g_c$, which is a fuzzy ideal of S.
Case(iii) : If $c \in A \cap B$ and $x, y \in S$ then, by Definition 2.7,

$$\begin{aligned}(fog)_c(y) &= \sup_{y=ab} min\{f_c(a), g_c(b)\}\\ &\leq \sup_{xy=xab} min\{f_c(xa), g_c(b)\}\\ &= \sup_{xy=lm} min\{f_c(l), g_c(m)\}\\ &=(fog)_c(xy).\end{aligned}$$

Therefore $(fog)_c(xy) \geq (fog)_c(y)$, similarly we can prove that $(fog)_c(xy) \geq (fog)_c(x)$ and $(fog)_c(xy) \geq min\{(fog)_c(x), (fog)_c(y)\}$.

Let e be an identity element of S. Then

$$\begin{aligned}(fog)_c(x+y) &= \sup_{(x+y)e} min\{f_c(x+y), g_c(e)\}\\ &\geq \sup_{(x+y)e} min\{min\{f_c(x), f_c(y)\}, g_c(e)\}\\ &= \sup_{(x+y)e} min\{min\{f_c(x), g_c(e)\}, min\{f_c(y), g_c(e)\}\}\\ &=min\{\sup_{xe}\{min\{f_c(x), g_c(e)\}\}, \sup_{ye} min\{f_c(y), g_c(e)\}\}\\ &=min\{(fog)_c(x), (fog)_c(y)\}.\end{aligned}$$

Therefore $(fog)_c$ is a fuzzy ideal of S.
Hence (fog, C) ,where $C = A \cup B$ is a fuzzy soft ideal over semiring S. □

3. Fuzzy soft $k-$ideal and $k-$fuzzy soft ideal

In this section the concept of fuzzy soft k-ideal, $k-$fuzzy soft ideal and fuzzy soft ideal of a fuzzy soft semiring is introduced and study the properties related to them.

Definition 3.1. Let (f, A) be a fuzzy -soft ideal over semiring S. If f_a is a fuzzy $k-$ideal of semiring S, for all $a \in A$ then (f, A) is said to be fuzzy soft $k-$ideal over semiring semi ring S.

Theorem 3.2. *Let (f, A) and (g, B) be two fuzzy soft $k-$ideals over semiring S. Then $(f, A) \cap (g, B)$ is a fuzzy soft $k-$ideal if it is non null.*

Proof. Let (f, A) and (g, B) be two fuzzy soft $k-$ideals over semiring By Theorem 2.17, $(f, A) \cap (g, B)$ is a fuzzy soft ideal over S. Let $(f, A) \cap (g, B) = (h, C)$ where $C = A \cap B$. If $c \in A \cap B$ then $h_c = f_c \cap g_c$ and

$$\begin{aligned} h_c(x) =& (f_c \cap g_c)(x) \\ =& min\{f_c(x), g_c(x)\} \\ \geq& min\{min\{f_c(x+y), f_c(y)\}, min\{g_c(x+y), g_c(y)\}\} \\ =& min\{min\{f_c(x+y), g_c(x+y)\}, min\{f_c(y), g_c(y)\}\} \\ =& min\{f_c \cap g_c(x+y), f_c \cap g_c(y)\} \text{ for all} x, y \in S. \end{aligned}$$

Hence $f_c \cap g_c$ is a fuzzy $k-$ideal of S.
Therefore $(f, A) \cap (g, B)$ is a fuzzy soft $k-$ideal over semiring S. □

Theorem 3.3. *Let (f, A) and (g, B) be two fuzzy soft $k-$ideals over semiring S. Then $(f, A) \cup (g, B)$ is a fuzzy soft $k-$ideal over semiring S.*

Proof. Let (f, A) and (g, B) be two fuzzy soft $k-$ideals over semiring S. By Theorem 2.18, $(f, A) \cup (g, B)$ is a fuzzy soft ideal over semiring S. By Definition 1.7, $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$.
If $c \in A \setminus B$ then $h_c = f_c, h_c$ is a fuzzy $k-$ideal since (f, A) is a fuzzy soft $k-$ideal.
If$c \in B \setminus A$ then $h_c = g_c, h_c$is a fuzzy $k-$ideal since (g, B) is a fuzzy soft $k-$ideal.
If$c \in A \cap B$ then $h_c = f_c \cup g_c$. Clearly h_c is a fuzzy ideal. Let $x, y \in S$.

Then

$$\begin{aligned}h_c(x) =&(f_c \cup g_c)(x)\\ =&max\{f_c(x), g_c(x)\}\\ \geq&max\{min\{f_c(x+y), f_c(x)\}, min\{g_c(x+y), g_c(x)\}\}\\ =&max\{min\{f_c(x+y), g_c(x+y)\}, min\{f_c(x), g_c(x)\}\\ =&min\{max\{f_c(x+y), g_c(x+y)\}, max\{f_c(x), g_c(x)\}\}\\ =&min\{f_c \cup g_c(x+y), f_c \cup g_c(x)\}\end{aligned}$$

Hence h_c is a fuzzy $k-$ideal of S. Therefore (h, C) is a fuzzy soft $k-$ideal over semiring S. □

Theorem 3.4. *Let (f, A) and (g, B) be fuzzy soft $k-$ideals over semiring S. Then "(f, A) OR (g, B)" is denoted by $(f, A) \vee (g, B) = (h, A \times B)$, where $h_c = f_a \cup g_b$ is a fuzzy soft $k-$ideal over semiring S,for all $c = (a, b) \in A \times B$*

Proof. Let (f, A) and (g, B) be fuzzy soft $k-$ideals over semiring S and $c = (a, b) \in A \times B, x \in S$. Then, by Theorem 2 .20, $(f, A) \vee (g, B)$ is a fuzzy soft ideal over semiring S and

$$\begin{aligned}h_c(x) =& f_a \cup g_b(x)\\ =&max\{f_a(x), g_b(x)\}\\ \geq& max\{min\{f_a(x+y), f_a(y)\}, min\{g_b(x+y), g_b(y)\}\}\\ =&max\{min\{f_a(x+y), g_b(x+y)\}, min\{f_a(y), g_b(y)\}\\ =&min\{max\{f_a(x+y), g_b(x+y)\}, max\{f_a(y), g_b(y)\}\}\\ =&min\{f_a \cup g_b(x+y), f_a \cup g_b(x)\}\end{aligned}$$

Hence h_c is a fuzzy $k-$ideal of S. Therefore $(h, A \times B)$ is a fuzzy soft $k-$ideal over semiring S. □

Theorem 3.5. *Let (f, A) and (g, B) be fuzzy soft $k-$ideals over semiring S. Then $(f, A) \wedge (g, B)$ is a fuzzy soft $k-$ideal over semiring S.*

Proof. Let (f, A) and (g, B) be fuzzy soft $k-$ideals over semiring S. By Definition 1.8, $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$. Let $c = (a, b) \in C = A \times B$ and $x, y \in S$. Then, by Theorem 2.19, $(f, A) \wedge (g, B)$ is a fuzzy soft ideal over semiring S and

$$\begin{aligned} h_c(x) =& f_a \wedge g_b(x) \\ =& min\{f_a(x), g_b(x)\} \\ \geq& min\{min\{f_a(x+y), f_a(y)\}, min\{g_b(x+y), g_b(y)\}\} \\ =& min\{min\{f_a(x+y), g_b(x+y)\}, min\{f_a(y), g_b(y)\}\} \\ =& min\{f_a \wedge g_b(x+y), f_a \wedge g_b(y)\} \end{aligned}$$

Hence h_c is a fuzzy $k-$ideal of S. Therefore $(f, A) \wedge (g, B)$ is a fuzzy soft $k-$ideal over S. □

Theorem 3.6. *Let S be a semiring, E be a parameter set and $A \subseteq E$. Then (f, A) is a fuzzy soft $k-$ideal over S if and only if for each $a \in A, (f_a)_t (t \in Im(f_a))$ is a $k-$ideal of S where f_a is the fuzzy subset of S.*

Proof. Let S be a semiring , E be a parameter set and $A \subseteq E$. Suppose (f, A) is a fuzzy soft $k-$ideal over semiring$S$, $a \in A$ and $t \in Im(f_a)$ and $x, y \in (f_a)_t, r \in S$. Then
$f_a(x+y) \geq min\{f_a(x), f_a(y)\} \geq min\{t, t\} = t$ and
$f_a(xy) \geq max\{f_a(x), f_a(y)\} = max\{t, t\} = t$ and hence $x+y, xy \in (f_a)_t$. Let $x \in (f_a)_t, r \in S$. Then $f_a(xr) \geq max\{f_a(x), f_a(r)\} = f_a(x) \geq t$ and $f_a(rx) \geq max\{f_a(r), f_a(x)\} = f_a(x) \geq t$. Hence $rx, xr \in (f_a)_t$. Therefore, for each $t \in Im(f_a), (f_a)_t$ is an ideal of S. Let $x+y \in (f_a)_t, y \in (f_a)_t$. Then $f_a(x+y) \geq t, f_a(y) \geq t$ and hence $f_a(x) \geq t$ since f_a is a fuzzy $k-$ideal. Therefore $x \in (f_a)_t$. Hence $(f_a)_t$ is a $k-$ideal of S. Conversely, suppose that $(f_a)_t$ is a $k-$ideal of S for each $t \in Im(f_a)$ and corresponding to each $a \in A$ and $x, y \in S$. Suppose $f_a(x+y) < min\{f_a(x), f_a(y)\} = t_1$(say). Then $x, y \in (f_a)_{t_1}$, but $x+y \neq (f_a)_{t_1}$ which is a contradiction. So $f_a(x+y) \geq min\{f_a(x), f_a(y)\}$. Suppose $f_a(xy) < max\{f_a(x), f_a(y)\} = t_2$(say). Then $x, y \in (f_a)_{t_2}$, but $xy \neq (f_a)_{t_2}$ which is a contradiction. Therefore $f_a(xy) \geq max\{f_a(x), f_a(y)\}$. Let $x, y \in S$ and $a \in A, min\{f_a(x+y), f_a(y)\} = t_1$ (say). Then $y, x+y \in (f_a)_{t_1}$ implies that $x \in (f_a)_{t_1}$ so that $f_a(x) \geq t_1 = min\{f_a(x+y), f_a(y)\}$. Hence f_a is a fuzzy $k-$ideal of S. Therefore (f, A) is a fuzzy soft $k-$ideal over semiring S. □

Definition 3.7. A fuzzy soft ideal (f, A)over semiring S is said to be $k-$fuzzy soft ideal over semiring Sif f_a is a $k-$fuzzy ideal of semiring S, for all $a \in A$.

Theorem 3.8. *Every fuzzy soft $k-$ideal over semiring is a $k-$fuzzy soft ideal over semiring.*

Proof. Let (f, A) be a fuzzy soft $k - ideal$ over semiring S and $a \in A$. Then f_a is a fuzzy $k-$ideal since (f, A) is fuzzy soft $k-$ideal. Let $x, y \in S$ such that $f_a(x + y) = f_a(0)$ and $f_a(y) = f_a(0)$. Then

$$\begin{aligned} f_a(x) \geq & min\{f_a(x+y), f_a(y)\} \\ \geq & min\{f_a(0), f_a(0)\} \\ = & f_a(0) \end{aligned}$$

Therefore $f_a(x) \geq f_a(0)$. Also, we have $f_a(0) \geq f_a(x)$. Hence $f_a(x) = f_a(0)$.
By definition of $k-$ fuzzy ideal, f_a is a $k-$fuzzy ideal of S.
Hence (f, A) is a $k-$fuzzy soft ideal over semiring S. □

Definition 3.9. Let (f, A) and (g, B) be fuzzy soft semirings over semiring S. Then (f, A) is a fuzzy soft subsemiring of (g, B), if it satisfies the following conditions
(i) $A \subseteq B$
(ii) f_a is a fuzzy subsemiring of g_a for all $a \in A$.

Definition 3.10. Let (f, A) be a fuzzy soft semiring over S. A non null fuzzy soft set (g, B) over S is called a fuzzy soft ideal of (f, A) if it satisfies the following conditions
(i) (g, B) is a fuzzy soft subsemiring of (f, A)
(ii) (g, B) is a fuzzy soft ideal over semiring S.

Theorem 3.11. *Let (f, A) and (g, B) be two fuzzy soft ideals of fuzzy soft semiring (h, C) over semiring S. Then $(f, A) \cap (g, B)$ is a fuzzy soft ideal of (h, C) if it is non null.*

Proof. Let (f, A) and (g, B) be two fuzzy soft ideals of fuzzy soft semiring (h, C) over semiring S. By Theorem2.17, $(f, A) \cap (g, B)$ is a fuzzy soft ideal over semiring S. Hence by Definition 3.10, $(f, A) \cap (g, B)$ is a fuzzy soft ideal of (h, C). □

The following theorem can be proved easily.

Theorem 3.12. *Let (f, A) and (g, B) be two fuzzy soft ideals of a fuzzy soft semiring (h, C) over semiring S. Then $(f, A) \cup (g, B)$ is a fuzzy soft ideal of (h, C) if it is non null.*

Theorem 3.13. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring S, (f_1, C) and (g_1, D) be fuzzy soft ideals of (f, A) and (g, B) respectively. Then $(f_1, C) \cap (g_1, D)$ is a fuzzy soft ideal of $(f, A) \cap (g, B)$ if it is non null.*

Proof. Let (f_1, C) and (g_1, D) be fuzzy soft ideals of (f, A) and (g, B) respectively. Since (f_1, C) and (g_1, D) are fuzzy soft ideals of (f, A) and (g, B) respectively, we have by Definition3.10(f_1, C) and (g_1, D) are fuzzy soft ideals over semiring S. By Theorem 2.17, $(f_1, C) \cap (g_1, D)$ is a fuzzy soft ideal ove semiringr S and by Theorem 2.9, $(f, A) \cap (g, B)$ is a fuzzy soft semiring over semiring S. Hence by Definition 3.10, $(f_1, C) \cap (g_1, D)$ is a fuzzy soft ideal of $(f, A) \cap (g, B)$. □

4. Fuzzy soft semiring homomorphism

In this section, the concept of fuzzy soft semiring homomorphism is introduced and study their properties.

Definition 4.1. Let (f, A) and (g, B) be fuzzy soft sets over semirings R and S respectively. Let $\phi : R \to S$ and $\psi : A \to B$ be two functions where A and B are parameter sets for the crisp sets R and S respectively. Then the pair (ϕ, ψ) is called a fuzzy soft function from R to S.

Definition 4.2. Let (ϕ, ψ) be a fuzzy soft function from R to S. The pre image of a fuzzy soft set (g, B) under the fuzzy soft function (ϕ, ψ) is denoted by $(\phi, \psi)^{-1}(g, B)$ and it is the fuzzy soft set defined by

$$(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B)).$$

Definition 4.3. Let (f, A) and (g, B) be fuzzy soft semirings over semirings R and S respectively and (ϕ, ψ) be fuzzy soft function from R to S. Then (ϕ, ψ) is said to be fuzzy soft semiring homomorphism if the following conditions hold.
(i) ϕ is a homomorphism from R onto S
(ii) ψ is a mapping from A onto B
(iii) $\phi(f_a) = g_{\psi(a)}$ for all $a \in A$

Definition 4.4. If there exists a fuzzy soft semiring homomorphism between (f, A) and (g, B) fuzzy soft semirings, we say that (f, A) is a fuzzy soft homomorphic to (g, B).

Theorem 4.5. *Let (f, A) be a fuzzy soft semiring over semiring S and $\theta : R \to S$ be an onto homomorphism for each $a \in A$. If $(\theta f)_a(x) = f_a(\theta(x))$, for all $x \in R$ then $(\theta f, A)$ is a fuzzy soft semiring over semiring S.*

Proof. Let $x, y \in R, a \in A$. Then

$$\begin{aligned}(\theta f)_a(x+y) =& f_a(\theta(x+y)) \\ =& f_a[\theta(x)+\theta(y)] \\ \geq& min\{f_a(\theta(x)), f_a(\theta(y))\} \\ =& min\{(\theta f)_a(x), (\theta f)_a(y)\} \\ (\theta f)_a(xy) =& f_a(\theta(xy)) \\ =& f_a[\theta(x)\theta(y)] \\ \geq& min\{f_a(\theta(x)), f_a(\theta(y))\} \\ =& min\{(\theta f)_a(x), (\theta f)_a(y)\}\end{aligned}$$

Hence $(\theta f)_a$ is a fuzzy subsemiring of S. Therefore $(\theta f, A)$ is a fuzzy soft semiring over S. □

Theorem 4.6. *Let (α, A) be a fuzzy soft semiring over S , θ be an endomorphism of S and define $(\alpha\theta)_a = \alpha_a\theta$ for each $a \in A$ .Then $(\alpha\theta, A)$ is a fuzzy soft semiring over semiring S.*

Proof. Let $x, y \in S, a \in A$. Then

$$\begin{aligned}(\alpha\theta)_a(x+y) =& \alpha_a(\theta(x+y)) \\ =& \alpha_a[\theta(x)+\theta(y)] \\ \geq& min\{\alpha_a(\theta(x)), \alpha_a(\theta(y))\} \\ =& min\{(\alpha\theta)_a(x), (\alpha\theta)_a(y)\} \\ (\alpha\theta)_a(xy) =& \alpha_a(\theta(xy)) \\ =& \alpha_a[\theta(x)\theta(y)] \\ \geq& min\{\alpha_a(\theta(x)), \alpha_a(\theta(y))\} \\ =& min\{(\alpha\theta)_a(x), (\alpha\theta)_a(y)\}\end{aligned}$$

Hence $(\alpha\theta)_a$ is fuzzy subsemiring of S. Therefore $(\alpha\theta, A)$ is a fuzzy soft semiring over semiring S. □

Definition 4.7. Let R and S be two semirings and f be a function from R into S. If μ is a fuzzy subset of S then the pre image of μ under f is the fuzzy subset of R and it is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in R$.

Theorem 4.8. *Let $\phi : R \to S$ be an onto homomorphism of semirings and (α, A) be a fuzzy soft left ideal over semiring S. If for each $a \in A$, $\beta_a = \phi^{-1}(\alpha_a)$ then (β, A) is a fuzzy soft left ideal over semiring R.*

Proof. Let $\phi : R \to S$ be an onto homomorphism of semirings , (α, A) be a fuzzy soft left ideal over S and $a \in A$. Then α_a is a fuzzy soft left ideal over S. Let $x, y \in R$. Then

$$\begin{aligned}
\phi^{-1}(\alpha_a)(x+y) =& \alpha_a(\phi(x+y)) \\
=& \alpha_a\{\phi(x)+\phi(y)\} \\
\geq& min\{\alpha_a(\phi(x)), \alpha_a(\phi(y))\} \\
=& min\{\phi^{-1}(\alpha_a)(x), \phi^{-1}(\alpha_a)(y)\} \\
\phi^{-1}(\alpha_a)(xy) =& \alpha_a(\phi(xy)) \\
=& \alpha_a\{\phi(x)\phi(y)\} \\
\geq& \alpha_a(\phi(y)) \\
=& \phi^{-1}(\alpha_a)(y)
\end{aligned}$$

Hence $\beta_a = \phi^{-1}(\alpha_a)$ is a fuzzy left ideal of R. Hence (β, A) is a fuzzy soft left ideal over semiring R. □

Definition 4.9. Let $\phi : R \to S$ be a homomorphism of semirings and f be a fuzzy subset of R. We define a fuzzy subset $\phi(f)$ of S by

$$\phi(f)(x) = \begin{cases} \sup\limits_{y \in \phi^{-1}(x)} f(y), & \text{if } \phi^{-1}(x) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \text{for all } x \in R.$$

Lemma 4.10. *Let R and S be semirings, $\phi : R \to S$ be a homomorphism and f be a ϕ invariant fuzzy subset of R. If $x = \phi(a)$ then $\phi(f)(x) = f(a), a \in R$.*

Proof. Let R and S be semirings, $\phi : R \to S$ be a homomorphism and f be a ϕ invariant fuzzy subset of R. Suppose $a \in R$ and $\phi(a) = x$. Then $\phi^{-1}(x) = a$. Let $t \in \phi^{-1}(x)$. Then $\phi(t) = x = \phi(a)$. Since f is a ϕ invariant fuzzy subset of R, $f(t) = f(a)$. Therefore $\phi(f)(x) = \sup\limits_{t \in \phi^{-1}(x)} f(t) = f(a)$ and hence $\phi(f)(x) = f(a)$. □

Theorem 4.11. *Let (α, A) be a fuzzy soft left ideal over semiring R. and ϕ be a homomorphism from semiring R onto semiring S. For each $c \in A, \alpha_c$ is a ϕ invariant fuzzy left ideal of R. If $\beta_c = \phi(\alpha_c), c \in A$ then (β, A) is a fuzzy soft left ideal ove semiringr S.*

Proof. Let (α, A) be a fuzzy soft left ideal over semiring R. , ϕ be a homomorphism from semiring R onto semiring S, $x, y \in S$ and $c \in A$. Then there exists $a, b \in R$ such that$\phi(a) = x, \phi(b) = y, x + y = \phi(a + b), xy = \phi(ab)$. Since α_c is ϕ invariant and by Lemma 4.10, we have

$$\begin{aligned}
\beta_c(x+y) =& \phi(\alpha_c)(x+y)\\
=& \alpha_c(a+b)\\
\geq& min\{\alpha_c(a), \alpha_c(b)\}\\
=& min\{\phi(\alpha_c)(x), \phi(\alpha_c)(y)\}\\
=& min\{\beta_c(x), \beta_c(y)\}\\
\beta_c(xy) =& \phi(\alpha_c)(xy)\\
=& \alpha_c(\phi(ab))\\
=& \alpha_c[\phi(a)\phi(b)]\\
\geq& \alpha_c(\phi(b))\\
=& \phi(\alpha_c)(y)\\
=& \beta_c(y).
\end{aligned}$$

Hence β_c is a left ideal of S. Therefore (β, A) is a fuzzy soft left ideal over semiring S. □

Theorem 4.12. *Let (f, A) and (g, B) be fuzzy soft semirings over semirings R and S respectively, and (ϕ, ψ) be a fuzzy soft semiring homomorphism from (f, A) onto (g, B). Then $(\phi(f), B)$ is a fuzzy soft semiring over S.*

Proof. Let (ϕ, ψ) be a fuzzy soft semiring homomorphism from (f, A) onto (g, B). By Definition4.3, ϕ is a homomorphism from R onto S and ψ is a mapping from A onto B. For each $b \in B$ there exists $a \in A$ such that $\psi(a) = b$. Define

$$[\phi(f)]_b = \phi(f_a).$$

Let $y_1, y_2 \in S$. Then there exist $x_1, x_2 \in R$ such that $\phi(x_1) = y_1, \phi(x_2) = y_2$ and $\phi(x_1 + x_2) = y_1 + y_2$ and $\phi(x_1 x_2) = y_1 y_2$. Now

$$\begin{aligned}
[\phi(f)]_{\psi(a)}(y_1 + y_2) &= \phi(f_a)(y_1 + y_2) \\
&= f_a[x_1 + x_2] \\
&\geq min\{f_a(x_1), f_a(x_2)\} \\
&= min\{\phi(f_a)(y_1), \phi(f_a)(y_2)\} \\
&= min\{\phi(f)_{\psi(a)}, \phi(f)_{\psi(a)}(y_2)\}
\end{aligned}$$

$$\begin{aligned}
[\phi(f)]_{\psi(a)}(y_1 y_2) &= \phi(f_a)(y_1 y_2) = f_a(x_1 x_2) \\
&\geq min\{f_a(x_1), f_a(x_2)\} \\
&= min\{\phi(f_a)(y_1), \phi(f_a)(y_2)\} \\
&= min\{\phi(f)_{\psi(a)}, \phi(f)_{\psi(a)}(y_2)\}
\end{aligned}$$

Therefore $\phi(f)_b$ is a fuzzy subsemiring of S. Hence $(\phi(f), B)$ is a fuzzy soft semiring over S. □

Definition 4.13. Let (f, A) and (g, B) be fuzzy soft semirings over a semiring S. Then (f, A) is a fuzzy soft subsemiring of (g, B), if it satisfies the following conditions
(i) $A \subseteq B$
(ii) f_a is a fuzzy subsemiring of g_a for all $a \in A$.

Theorem 4.14. *Let (f, A) and (g, B) be fuzzy soft semirings over S. Then the following statements are true*
(i) If $g_b \subseteq f_b$ for all $b \in B \subseteq A$ then (g, B) is a fuzzy soft subsemiring of (f, A).
(ii) $(f, A) \cap (g, B)$ is a fuzzy soft subsemiring of (f, A) and (g, B) if it is non null.

Proof. (i) Since $g_b \subseteq f_b$ for all $b \in B \subseteq A$ and by Definition3.9, (g, B) is a fuzzy soft subsemiring of (f, A).
(ii) Let $(f, A) \cap (g, B) = (h, C)$ where $C = A \cap B$. By Theorem 2.9, (h, C) is a fuzzy soft semiring over S. Since $C = A \cap B \subseteq A$ and $C \subseteq B$, by Definition 3.9,(h, C) is a fuzzy soft subsemiring of (f, A) as well as (g, B). □

Theorem 4.15. *Let (f, A) and (g, B) be fuzzy soft semirings over semiring R and (f, A) be a fuzzy soft subsemiring of (g, B) and $\phi : R \to S$ be a homomorphism from R onto S then $(\phi(f), A)$ and $(\phi(g), B)$ are soft subsemirings over S and $(\phi(f), A)$ is a soft subsemiring of $(\phi(g), B)$.*

Proof. Let (f, A) and (g, B) be fuzzy soft semirings over semiring R, (f, A) be a fuzzy soft subsemiring of $(g, B) and\ \phi : R \to S$ be a homomorphism from R onto S. Since ϕ is a homomorphism from R onto S, $[\phi(f)]_a = \phi(f_a)$ is fuzzy subsemiring of S for all $a \in A$ and $[\phi(g)]_b = \phi(g_b)$ is a fuzzy subsemiring of S for all $b \in B$. Hence $(\phi(f), A)$, $(\phi(f), B)$ are fuzzy soft semirings over S. Since (f, A) is a fuzzy soft subsemiring of (g, B), f_a is a fuzzy subsemiring of g_a. Therefore $\phi(f_a)$ is a fuzzy subsemiring of $\phi(g_a)$ for all $a \in A$. Hence $(\phi(f), A)$ is a fuzzy soft subsemiring of $(\phi(g), B)$. □

Theorem 4.16. *Let (f, A) and (g, B) be fuzzy soft semirings over semirings R and S respectively. If (ϕ, ψ) is a fuzzy soft homomorphism from (f, A) onto (g, B) then the pre-image of (g, B) under fuzzy soft semiring homomorphism (ϕ, ψ) is a fuzzy soft subsemiring of (f, A) over R.*

Proof. Let (f, A) and (g, B) be fuzzy soft semirings over semirings R and S respectively. Suppose (ϕ, ψ) is a fuzzy soft homomorphism from (f, A) onto (g, B). By Definition 4.2, $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$. Define
$(\phi^{-1}(g))_a(x) = g_{\psi(a)}(\phi(x))$ for all $x \in R$ and $a \in \psi^{-1}(B)$.
Let $x_1, x_2 \in R$. Then

$$\begin{aligned}
(\phi^{-1}(g))_a(x_1 + x_2) =& g_{\psi(a)}[\phi(x_1 + x_2)] \\
=& g_{\psi(a)}[\phi(x_1) + \phi(x_2)] \\
\geq& min\{g_{\psi(a)}(\phi(x_1)), g_{\psi(a)}(\phi(x_2))\} \\
=& min\{(\phi^{-1}(g))_a(x_1), (\phi^{-1}(g))_a(x_2)\} \\
(\phi^{-1}(g))_a(x_1 x_2) =& g_{\psi(a)}[\phi(x_1 x_2)] \\
=& g_{\psi(a)}[\phi(x_1)\phi(x_2)] \\
\geq& min\{g_{\psi(a)}(\phi(x_1)), g_{\psi(a)}(\phi(x_2))\} \\
=& min\{(\phi^{-1}(g))_a(x_1), (\phi^{-1}(g))_a(x_2)\}
\end{aligned}$$

Therefore $(\phi^{-1}(g))_a$ is a fuzzy subsemiring of R for all $a \in \psi^{-1}(B)$.
Therefore $\big((\phi^{-1}(g)), (\psi^{-1}(B)\big)$ is a fuzzy soft subsemiring of (f, A) over semiring R. Hence the Theorem. □

5. Conclusion

Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. In this paper, we discussed the algebraic properties of fuzzy soft ideals, fuzzy soft $k-$ideals over semiring structures and the properties of homomorphic image of fuzzy soft soft semiring. In the next paper, we study the properties of kernel of fuzzy soft semiring homomorphism, fuzzy soft prime ideals and fuzzy soft filters over semirings.

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