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FIXED POINTS AND COMMON FIXED POINTS FOR FUNDAMENTALLY NONEXPANSIVE MAPPINGS ON BANACH SPACES

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Abstract. In this paper, we present some fixed point theorems for fundamentally nonexpansive mappings in Banach spaces and give one common fixed point theorem for a commutative family of demiclosed fundamentally nonexpansive mappings on a nonempty weakly compact convex subset of a strictly convex Banach space with the Opial condition and a uniformly convex in every direction Banach space, respectively; moreover, we show that the common fixed points set of such a family of mappings is closed and convex.

Key Words: Common fixed point, Fixed point, Fundamentally nonexpansive mappings, nonexpansive mappings, Opial's condition, Uniformly convex in every direction Banach spaces.

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1. Introduction and preliminaries

The existence of the fixed points have been studied for nonexpansive mappings by many authors (see, for example [3, 5, 7, 10]). Recently, Suzuki [9] introduced a condition on mappings, called condition (C) which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Author gave some interesting fixed point theorems and convergence theorems for such mappings. In [6], some basic properties have been given for fundamentally nonexpansive mappings as well as some fixed point theorems have been presented for such mappings.

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It is easy to verify that nonexpansiveness implies fundamental nonexpansiveness, but the converse is not true in general (see Examples 1.1). Also, there is a fundamentally nonexpansive mapping which dose not satisfy condition (C) (see Examples 1.1).

The aim of this paper is to prove the existence of a fixed point and a common fixed point for fundamentally nonexpansive mappings on Banach spaces and then to study the structure of the common fixed points set. In section 2, some fixed point theorems have been presented for fundamentally nonexpansive mappings. Finally, in section 3, some common fixed point theorems have been given for a commutative family of fundamentally nonexpansive mappings on a nonempty weakly compact convex subset of a Banach space having certain conditions, and the structure of the common fixed points set is studied.

We now review the needed definitions and results. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers, respectively. In what follows, $(X, \|.\|)$ is a Banach space, K is a nonempty subset of X, and T is a self-mapping of K. We denote by $F(T)$ the fixed points set of T, i.e., $\{x \in K : Tx = x\}$ and by $x_n \rightharpoonup x$ the weak convergence of the sequence $\{x_n\}$ in X to $x \in X$, respectively. The mapping T is called (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$; (ii) quasi-nonexpansive if its fixed points set is nonempty and $\|Tx - u\| \leq \|x - u\|$ for all $x \in K$ and $u \in F(T)$; (iii) to satisfy condition (C) [9] if $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, then $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$; (iv) fundamentally nonexpansive [4] if $\left\|T^2x - Ty\right\| \leq \|Tx - y\|$ for all $x, y \in K$; (v) to preserve convexity if $T(C)$ is convex for all convex subset C of K; (vi) demiclosed if a sequence $\{x_n\}$ in K converges weakly to $u \in K$ and the sequence $\{Tx_n\}$ converges strongly to $y \in K$, then $Tu = y$. Let S be a family of self-mappings of K, the common fixed points set of S is denoted by $F(S) = \bigcap_{T \in S} F(T)$.

Example 1.1. Define a mapping T on $[0, 4]$ as follows:

$$Tx = \begin{cases} 1 & \text{if } x \neq 4 \\ 2.5 & \text{if } x = 4 \end{cases}$$

for all $x \in [0, 4]$. Then T is fundamentally nonexpansive, but it dose not satisfy condition (C). Therefore, T is not nonexpansive.

The Banach space X is said to (i) satisfy the Opial condition [7] if whenever a sequence $\{x_n\}$ in X converges weakly to $x \in X$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in X$ with $x \neq y$; (ii) be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ (see [1]); (iii) be uniformly convex in every direction (UCED, for short) [9] if for any $\epsilon \in (0,2]$ and $z \in X$ with $\|z\| = 1$, there exists $\delta = \delta(\epsilon, z) > 0$, whenever $x, y \in X$, $\|x\| \leq 1, \|y\| \leq 1$ and $x - y \in \{tz : t \in [-2, -\epsilon] \cup [\epsilon, 2]\}$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$; (iv) be uniformly convex [1] if for any $\epsilon \in (0,2]$ there exists some $\delta = \delta(\varepsilon) > 0$, whenever $x, y \in X$, $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. It is clear that uniform convexity implies UCED, and UCED implies strict convexity.

Example 1.2. (i) All Hilbert spaces and $l^p (1 \leq p < \infty)$ have the Opial condition (see [2]);
(ii) it is easy to verify that $\mathbb{R}^2$ with the norm $\|(x,y)\|_2 = \sqrt{x^2 + y^2}$ for all $x, y \in \mathbb{R}$, is uniformly convex, but it is not strictly convex with the norm $\|(x,y)\|_1 =\mid x \mid + \mid y \mid$ for all $x, y \in \mathbb{R}$.

A function $f : X \to \mathbb{R}$ is called strictly quasiconvex if

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$$

for all $\lambda \in (0,1)$ and $x, y \in X$ with $x \neq y$.

The following results are useful to prove our results.

Lemma 1.3. *(see Lemma 2 of* [9]*) Let X be a Banach space, the following statements are equivalent:*
(i) X is uniformly convex in every direction.
(ii) For any bounded sequence $\{x_n\}$ in X, the function f on X defined by $f(x) = \limsup_{n\to\infty} \|x_n - x\|$ is strictly quasiconvex.

Lemma 1.4. *(see Lemma 2.1 of* [6]*) Let K be a nonempty subset of a normed space X, and let $T : K \to X$ be a fundamentally nonexpansive mapping. Then*

$$\|x - Ty\| \leq 3\,\|x - Tx\| + \|x - y\|$$

holds for each $x, y \in K$.

Lemma 1.5. *(see Lemma 2.3 of* [6]*) Let T be a fundamentally nonexpansive self-mapping on a nonempty subset K of a Banach space X, and* *let $T(K)$ be bounded and convex. Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and*

$$Tx_{n+1} = \lambda T^2 x_n + (1-\lambda) Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $\lambda \in (0,1)$. Then $\lim_{n\to\infty} \left\|Tx_n - T^2 x_n\right\| = 0$.

Proposition 1.6. *(see Proposition 2.4 of [6]) Let $T : K \to K$ be a fundamentally nonexpansive mapping, where K is a nonempty subset of a Banach space X. Then $F(T)$ is closed. Moreover, if X is strictly convex, and $T(K)$ is convex, then $F(T)$ is also convex.*

2. Fixed point theorems

In this section, the fixed points existence theorems are established for fundamentally nonexpansive mappings.

To prove the following result, refer to Theorem 2.6 of [6].

Theorem 2.1. *Let K be a nonempty weakly compact subset of a Banach space X with the Opial condition. Suppose $T : K \to K$ is a fundamentally nonexpansive mapping, and $T(K)$ is convex. Then T has a fixed point.*

Proposition 2.2. *Let T be a fundamentally nonexpansive self-mapping of a nonempty subset K of a Banach space X with the Opial condition. If a sequence $\{x_n\}$ in K converges weakly to $u \in K$, and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then u is a fixed point of T.*

Proof. By Lemma 1.4, we have

$$\|x_n - Tu\| \leq 3\,\|x_n - Tx_n\| + \|x_n - u\| \text{ for all } n \in \mathbb{N}.$$

This implies $\liminf_{n\to\infty} \|x_n - Tu\| \leq \liminf_{n\to\infty} \|x_n - u\|$. Since X has the Opial condition, we obtain $Tu = u$. Therefore, u is a fixed point of T. □

Theorem 2.3. *Let T be a fundamentally nonexpansive self-mapping of a nonempty subset K of a uniformly convex in every direction Banach space X such that $T(K)$ is a weakly compact convex subset of K. Then the fixed points set of T is nonempty, closed and convex.*

Proof. Let x_1 be an arbitrary element of K. Define a sequence $\{Tx_n\}$ in $T(K)$ as follows:

$$Tx_{n+1} = \tfrac{1}{2} T^2 x_n + \tfrac{1}{2} Tx_n \quad \text{for all } n \in \mathbb{N}.$$

Now, consider the continuous convex function f on $T(K)$ defined by $f(Tx) = \limsup_{n\to\infty} \|Tx - Tx_n\|$ for all $x \in K$. Since X is UCED, Lemma 1.3 implies that f is strictly quasiconvex. By Proposition 2.5.3 in [1], f is lower semicontinuous in the weak topology. Hence, Theorem 2.5.5 in [1], implies that there exists $u \in K$ such that

$$f(Tu) = \inf\{f(Tx) : x \in K\}. \tag{2.1}$$

By Lemma 1.4, we have

$$\left\|Tx_n - T^2u\right\| \leq 3\left\|Tx_n - T^2x_n\right\| + \|Tx_n - Tu\|$$

for all $n \in \mathbb{N}$. The above inequality, Lemma 1.5 and (2.1) imply $f(Tu) = f(T^2u)$. We next show that $Tu = T^2u$. Suppose, for contradiction, that $Tu \neq T^2u$. Since f is strictly quasiconvex, we have

$$f(Tu) \leq f(\frac{1}{2}Tu + \frac{1}{2}T^2u) < \max\{f(Tu), f(T^2u)\} = f(Tu),$$

a contradiction. Therefore, $T(Tu) = Tu$, that is, Tu is a fixed point of T. By Proposition 1.6, $F(T)$ is closed and convex. This completes the proof. □

Example 2.4. Define a mapping T on $[0, 2]$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \in [0,1] \cup \{2\} \\ x - 1 & \text{if } x \in (1,2) \end{cases}$$

for all $x \in [0, 2]$. Then T is fundamentally nonexpansive and satisfies condition (C), but it is not nonexpansive. $T(K)$ is convex but not closed. We have $F(T) = \{0\}$ which is closed and convex.

3. Common fixed point theorems

In this section, the common fixed points existence theorems are proved for a commutative family of fundamentally nonexpansive mappings.

Theorem 3.1. *Let S be a family of commuting fundamentally nonexpansive self-mappings of a nonempty weakly compact convex subset K of a strictly convex Banach space with the Opial condition. If S preserves convexity, then $F(S)$ is nonempty, closed and convex.*

Proof. Put $\mathcal{S} = \{F(T) : T \in S\}$. Let $n \in \mathbb{N}$ be arbitrary and $\{T_1, ..., T_n\} \subseteq S$. We show that $C_n = \bigcap_{i=1}^{n} F(T_i)$ is nonempty. Set $K_i = F(T_i)$ for every $i \in \{1, 2, ..., n\}$. Theorem 2.1 implies that K_i is nonempty, and Proposition 1.6 implies that K_i is closed and convex. In particular, C_1 is nonempty. Suppose $m < n$, and C_m is nonempty. Let $x \in C_m$ and $1 \leq i \leq m$. Since $T_i \circ T_{m+1} = T_{m+1} \circ T_i$, we have

$$T_i(T_{m+1}x) = T_{m+1}(T_i x) = T_{m+1}x.$$

So, $T_{m+1}x \in C_m$. Hence, C_m is T_{m+1}-invariant. Since K_i is convex and closed (in the norm topology), by Theorem 3.12 in [8], it is weakly closed. As K is weakly compact, K_i is weakly compact. So, C_m is weakly compact and convex. By assumptions of the theorem, $T_{m+1}(C_m)$ is convex. So, using Theorem 2.1, we conclude that T_{m+1} has a fixed point in C_m. Assume that $u \in C_m \cap F(T_{m+1})$. Therefore, $u \in C_{m+1}$. By induction, $\bigcap_{i=1}^{n} F(T_i)$ is nonempty. So, $\mathcal{S}$ has the finite intersection property. Since K is weakly compact, and $F(T)$ is nonempty, weakly compact and convex for all $T \in S$; hence, $F(S) = \bigcap_{T \in S} F(T)$ is nonempty, closed and convex. □

Theorem 3.2. *Let S be a commutative family of demiclosed fundamentally nonexpansive self-mappings of a nonempty weakly compact convex subset K of a UCED Banach space X. If S preserves convexity, then $F(S)$ is nonempty, closed and convex.*

Proof. Let $T \in S$. Since K is convex, and S preserves convexity, $T(K)$ is convex. On the other hand, $T(K)$ is closed in the norm topology. Because K is weakly compact, and T is demiclosed. Hence, by Theorem 2.3 $F(T)$ is nonempty. As X is UCED, it is strictly convex, Proposition 1.6 implies that $F(T)$ is closed and convex. Now, similar to the proof of the previous theorem, one can prove that $F(S)$ is nonempty, closed and convex. □

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