Journal of Hyperstructures 4 (1) (2015), 11-19.
ISSN: 2322-1666 print/2251-8436 online

ON FORMAL LOCAL HOMOLOGY MODULES

MOHAMMAD HOSSEIN BIJAN-ZADEH AND SEBAR GHADERI *

Abstract. Throughout R is a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. In this paper we study formal homology modules of Artinian R-modules. We obtain an expression of the formal homology in terms of a certain local homology module. Also we consider their behavior with respect to the $\mathfrak{a}$-torsion functor, and exact sequences for various situations and related ideas.

Key Words: Local homology, Formal local homology.
2010 Mathematics Subject Classification: Primary: 35R35; Secondary: 49M15.

1. Introduction

Throughout this paper, R is a commutative and Noetherian ring with a nonzero identity element, $\mathfrak{a}$ is an ideal of R. N. T. Coung and T. T. Nam [4] defined the local homology module $H_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ by

$$H_i^{\mathfrak{a}}(M) = \varprojlim_n \operatorname{Tor}_i^R(R/\mathfrak{a}^n, M).$$

This definition is, in some sense, dual to the definition of usual local cohomology modules appeared in the literature in recent decades. For each $i \geq 0$, the local cohomology module $H_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

Suppose that the ideal $\mathfrak{a}$ is generated by r elements $x_1, \ldots, x_r$ of R, set $\underline{x}(t) = (x_1^t, \ldots, x_r^t)$ and $K.(\underline{x}(t))$ the Koszul complex of R with respect to $\underline{x}(t)$. For an R-module M, $K.(\underline{x}(t), M) = K.(\underline{x}(t)) \otimes M$ is defined as

---

Received: 28 April 2015, Accepted: 21 July 2015. Communicated by Mirela Stefanescu;
*Address correspondence to S. Ghaderi; E-mail: Email:ghaderi.sebar@gmail.com

the Koszul complex of M with respect to $\underline{x}(t)$ where $H.(\underline{x}(t), M)$ denotes its homology modules. Using the results of N. T. Coung and T. T. Nam [4], one can prove

$$H_i^{\mathfrak{a}}(M) \cong \varprojlim_t H_i(\underline{x}(t), M).$$

We study the behavior of the family of local homology modules $\{H_i^{\mathfrak{b}}(0 :_M \mathfrak{a}^n)\}_{n\in\mathbb{N}}$ for $i \in \mathbb{Z}$, where $\underline{x}$ denotes a system of elements of R such that $Rad\underline{x} = \mathfrak{b}$.

When R is a local ring with the maximal ideal $\mathfrak{m}$ and $\mathfrak{b} = \mathfrak{m}$, we define the i-th formal local homology module of M with respect to $\mathfrak{a}$ by

$$\mathfrak{F}_i^{\mathfrak{a}}(M) := \varinjlim_n H_i^{\mathfrak{m}}((0 :_M \mathfrak{a}^n)).$$

This notion can be viewed as a dual of formal local homology module due to Schenzel [13]. For more details on the notation of formal local cohomology, we refer the reader to [1, 5, 7]. In this paper we consider right derived functor of $\Gamma_{\mathfrak{a}}(.)$ on the formal local homology modules. As an immediate consequence of this result, it can be observed that, under some additional conditions, the formal homology modules are $\mathfrak{a}$-torsion modules. We also obtain some results for the vanishing of the formal homology. Furthermore, we study several results about the behavior of the formal homology modules with respect to the exact sequences of R-modules. Finally, we relate the $\mathfrak{a}$-formal homology to the $(\mathfrak{a}, x)$-formal homology for any $x \in \mathfrak{b}$.

2. Main results

Let $\underline{x} = x_1, \ldots, x_r$ be a system of elements of R, and $\mathfrak{b} = Rad(\underline{x})$ and let $C^{\cdot}_{\underline{x}}$ denote the $\widetilde{C}$ech complex of R with respect to $\underline{x}$, (see[3]). For an R-module M and an ideal $\mathfrak{a}$, the direct system of R-modules $\{(0 :_M \mathfrak{a}^n)\}_{n\in N}$ induces a direct system of R-complexes $\{\mathrm{Hom}(C^{\cdot}_{\underline{x}}, (0 :_M \mathfrak{a}^n))\}_{n\in N}$ which its direct limit plays an important role in our investigation in this section.

Remark 2.1. (i) Let $K.(\underline{x}(t))$ denote the Koszul complex of R with respect to $\underline{x}$. The Koszul cocomplex $K^{\cdot}(\underline{x}(t))$ is obtained by applying $\mathrm{Hom}(. , R)$ to the Koszul complex. For an arbitary R-module$M$,
we define

$$K.(\underline{x}(t), M) = K.(\underline{x}(t)) \otimes M \quad \textit{and} \quad K^{\cdot}(\underline{x}(t), M) = \mathrm{Hom}(K.(\underline{x}(t)), M).$$

(see [2],&9). There are the following Koszul duality isomorphisms:

$$K_{\cdot}(\underline{x}(t), M) \cong \operatorname{Hom}(K^{\cdot}(\underline{x}(t)), M) \quad \textit{and} \quad K^{\cdot}(\underline{x}(t), M) \cong K^{\cdot}(\underline{x}(t)) \otimes M.$$

If $C^{\cdot}_{\underline{x}}$ denotes the $\widetilde{C}$ech complex, then there is an isomorphism of complexes (see [12]):

$$\varinjlim_{t} K^{\cdot}(\underline{x}(t)) \cong C^{\cdot}_{\underline{x}}.$$

So another representative of $\operatorname{Hom}(C^{\cdot}_{\underline{x}}, (0 :_M \mathfrak{a}^n))$ is

$$\varprojlim_{t} K_{\cdot}(\underline{x}(t), (0 :_M \mathfrak{a}^n)).$$

(ii) Let $\Gamma_{\mathfrak{a}}(M) = \varinjlim_t \operatorname{Hom}(R/\mathfrak{a}^t, M)$ denote the $\mathfrak{a}$-torsion submodule of M. It is clear that

$$\varinjlim_{n} \operatorname{Hom}(C^{\cdot}_{\underline{x}}, (0 :_M \mathfrak{a}^n)) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}(C^{\cdot}_{\underline{x}}, M)).$$

In the derived category, this complex is isomorphic to $\Gamma_{\mathfrak{a}}(\Lambda_{\mathfrak{b}}(P))$, where $P \tilde{\to} M$ denotes a projective resolution of M and the $\mathfrak{b}$-adic completion functor $\Lambda_{\mathfrak{b}}(.)$ is defined by $\Lambda_{\mathfrak{b}}(P) = \varprojlim P/\mathfrak{b}^t P$ (see [12]).
(iii) For the exactness of the inverse limit, we say the Mittag-leffler (ML) condition. An inverse system $\{M_i, \phi_{ji}\}$ of R-modules satisfies the Mittag-Leffler condition, if for each i, there is a $s \geq i$, such that for all $j, k \geq s$, $\phi_{ji}(M_j) \cong \phi_{ki}(M_k)$ as submodules of M_i. If all the homomorphisms ϕ_{ji} are surjective or $\{M_i\}$ is an inverse system of Artinian R-modules then $\{M_i\}$ satisfies (ML). If $\{M_i\}$ is an inverse system of finite-dimensional vector spaces over a field then $\{M_i\}$ satisfies (ML). (see [15]).

Definition 2.2. Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module and $\underline{x}$ a system of elements of R such that $\mathfrak{b} = Rad(\underline{x})$. The i-th $\mathfrak{a}$-formal homology with respect to b is defined by $H_i(\varinjlim_n \operatorname{Hom}(C^{\cdot}_{\underline{x}}, (0 :_M \mathfrak{a}^n)))$.

Theorem 2.3. *Let M be an Artinian R-module. Then there are isomorphisms*

$$H_i(\varprojlim_{t} K_{\cdot}(\underline{x}(t), M)) \cong \varprojlim_{t} H_i(\underline{x}(t), M),$$

for all $i \in \mathbb{Z}$.

Proof. Since M is an Artinian R-module and $K.(\underline{x}(t))$ is a complex of finitely generated free modules, it is clear that $K_p(\underline{x}(t))\otimes M$ satisfies the (ML) condition for all t, p. This result is obtained by [[4], *Lemma* 2.2]. □

Theorem 2.4. *Let M be an R-module. Suppose that the ideal $\mathfrak{a}$ is generated by r elements $x_1, \ldots, x_r$ of R and $K.(\underline{x}(t))$ be the Koszul complex of R with respect to $\underline{x}(t) = (x_1^t, \ldots, x_r^t)$. Then*

$$H_i^{\mathfrak{a}}(M) \cong \varprojlim_t H_i(\underline{x}(t), M),$$

for all $i \geq 0$.

Proof. See [[4], *Theorem* 3.6]. □

Theorem 2.5. *Let M be an Artinian R-module and $\underline{x} = x_1, \ldots, x_r$ a system of elements of R, such that $\mathfrak{b} = Rad(\underline{x})$. Then there are isomorphisms*

$$H_i(\varinjlim_n \operatorname{Hom}(C^{\cdot}_{\underline{x}}, (0 :_M \mathfrak{a}^n))) \cong \varinjlim_n H_i^{\mathfrak{b}}((0 :_M \mathfrak{a}^n)),$$

for all $i \in \mathbb{Z}$.

Proof. According to Theorem 2.4, for any R-module M and $i \in \mathbb{Z}$, there is an isomorphism $H_i^{\mathfrak{b}}(M) \cong \varprojlim_t H_i(\underline{x}(t), M)$. Since the direct limit is an exact functor, the result follows by using Remark 2.1 and Theorem 2.3. □

Notation. Let $(R, \mathfrak{m})$ be a local ring. If in the above theorem, we put $\mathfrak{b} = \mathfrak{m}$, we denote the module $\varinjlim_n H_i^{\mathfrak{m}}((0 :_M \mathfrak{a}^n))$ by $\mathfrak{F}_i^{\mathfrak{a}}(M)$ and we called it, the i-th formal local homology module of M with respect to $\mathfrak{a}$.

We recall that a Noetherian ring R is said to be semi-local if it has only finitely many maximal ideals. We now have the following theorem.

Theorem 2.6. *Let R be a semi-local ring and M be Artinian R-module. Then $\varinjlim H_i^{\mathfrak{b}}((0 :_M \mathfrak{a}^n)) \cong \varinjlim H_i^{\widehat{\mathfrak{b}}}((0 :_M \mathfrak{a}^n))$ for all $i \in \mathbb{Z}$, where $\widehat{\ }$ is the completion functor with respect to the jacobson radical of R.*

Proof. Assume that X is an Artinian R-module. In view of [[9], *Proposition* 3.14], we have $X \otimes \widehat{R} \cong X$ and X is also Artinian as an $\widehat{R}$-module. As $\widehat{R}$ is a flat R-module, by using [[6], *Theorem* 2.5.15], there are isomorphisms $H_i^{\mathfrak{b}}(X) \cong H_i^{\hat{\mathfrak{b}}}(X)$, for all $i \in \mathbb{Z}$. Now taking $X = (0 :_M \mathfrak{a}^n)$ and passing the direct limit to it, we obtain the desired result. □

Theorem 2.7. *Let $\mathfrak{a}$ be an ideal of $(R,\mathfrak{m})$ and $\mathfrak{b}=\mathfrak{m}$. Let M be an Artinian R-module. If $H_j^{\mathfrak{m}}((0:_M \mathfrak{a}^n))$ is an Artinian R-module for an integer $j\in\mathbb{Z}$, then the following isomorphisms hold*

$$H_{\mathfrak{a}}^i(\mathfrak{F}_j^{\mathfrak{a}}(M)) \cong \begin{cases} 0 & i\neq 0, \\ \mathfrak{F}_j^{\mathfrak{a}}(M) & i=0. \end{cases}$$

where $H_{\mathfrak{a}}^i$ denotes the right derived functors of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$. Moreover $\mathfrak{F}_i^{\mathfrak{a}}(M)=0$ whenever $\mathfrak{a}H_j^{\mathfrak{m}}((0:_M \mathfrak{a}^n))=H_j^{\mathfrak{m}}((0:_M \mathfrak{a}^n))$.

Proof. Since $H_j^{\mathfrak{m}}((0:_M \mathfrak{a}^n))$ is an Artinian R-module, each of its elements is annihilated by some power of $\mathfrak{m}$. Then it is an $\mathfrak{a}$-torsion module. By [[3], *Proposition* 3.4.4], the $\mathfrak{a}$-torsion functor commutes direct limit, it follows that $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion R-module. Clearly $H_{\mathfrak{a}}^i(\mathfrak{F}_i^{\mathfrak{a}}(M))=0$ for all $i\neq 0$ and $H_{\mathfrak{a}}^0(\mathfrak{F}_i^{\mathfrak{a}}(M))\cong\mathfrak{F}_i^{\mathfrak{a}}(M)$ (see [[3], *Corollary* 2.1.7]).

Now for simplicity put $X=H_j^{\mathfrak{m}}(0:_M \mathfrak{a}^n)$. By the assumption we have $X=\mathfrak{a}^nX$. It is easy to see that $\operatorname{Hom}(R/\mathfrak{a}^n,X)\cong(0:_X \mathfrak{a}^n)=0$. Therefore $X=\Gamma_{\mathfrak{a}}(X)=0$. Now taking $X=H_j^{\mathfrak{m}}((0:_M \mathfrak{a}^n))$ and passing the direct limit to it. This proves the claim. □

We recall the notion of Krull dimension of an Artinian R-module M, denoted by $\mathrm{Kdim}M$. Let M be an Artinian R-module. When $M=0$ we put $\mathrm{Kdim}M=-1$. Then by induction, for any ordinal α, we put $\mathrm{Kdim}M=\alpha$ when (i) $\mathrm{Kdim}<\alpha$ is false, and (ii) for every ascending chain, $M_0\subseteq M_1\subseteq\ldots$. of submodules of M, there exists a positive integer $\mathfrak{m}_0$ such that $\mathrm{Kdim}(M_{\mathfrak{m}+1}/M_{\mathfrak{m}})<\alpha$ for all $\mathfrak{m}\geq\mathfrak{m}_0$. Thus M is non-zero and noetherian if and only if $\mathrm{Kdim}M=0$(see [10]).

Corollary 2.8. *Let $\mathfrak{a}$ denote an ideal of the local ring $(R,\mathfrak{m})$, let $\mathfrak{m}=Rad(\underline{x})$ and let M be an Artinian R-module. If $\mathrm{Kdim}(0:_M \mathfrak{a}^n)=0$, then*

$$\mathfrak{F}_i^{\mathfrak{a}}(M)\cong\begin{cases} 0 & i\neq 0, \\ \Gamma_{\mathfrak{a}}(M) & i=0. \end{cases}$$

Proof. Since M is Artinian R-module and $\mathrm{Kdim}(0:_M \mathfrak{a}^n)=0$, it follows from [[4], *Proposition* 4.8] that $H_i^{\mathfrak{m}}((0:_M \mathfrak{a}^n))=0$ for all $i\neq 0$ and moreover

$$H_0^{\mathfrak{m}}((0:_M \mathfrak{a}^n))\cong\Lambda_{\mathfrak{m}}(0:_M \mathfrak{a}^n)\cong(0:_M \mathfrak{a}^n),$$

By passing to the direct limit, the proof is finished. □

Theorem 2.9. *Let $0 \to A \to B \to C \to 0$ denote a split short exact sequence of Artinian R-modules. For an ideal $\mathfrak{a}$ of R and $\mathfrak{b} = Rad(\underline{x})$, there is a long exact sequence*

$$\cdots \to \varinjlim_n H^{\mathfrak{b}}_{i+1}((0 :_C \mathfrak{a}^n)) \to \varinjlim_n H^{\mathfrak{b}}_i((0 :_A \mathfrak{a}^n)) \to \varinjlim_n H^{\mathfrak{b}}_i((0 :_B \mathfrak{a}^n)) \to \cdots$$

Proof. The split short exact sequence $0 \to A \to B \to C \to 0$ induces a short exact sequence of R-modules:

$$0 \to \mathrm{Hom}(R/\mathfrak{a}^n, A) \to \mathrm{Hom}(R/\mathfrak{a}^n, B) \to \mathrm{Hom}(R/\mathfrak{a}^n, C) \to 0$$

By [[4], *Proposition* 4.1]. If M is an Artinian R-module then $L^{\mathfrak{b}}_i(M)$, the i-th left derived module of $\Lambda_{\mathfrak{b}}(M)$, is isomorphism to $H^{\mathfrak{b}}_i(M)$. By using [11] this shows that the sequence of functors $H^{\mathfrak{b}}_i(.)$ is positive strongly connected on the category of Artinian R-modules and this completes the proof. □

Corollary 2.10. *Let $\mathfrak{a}$ be an ideal of $(R, \mathfrak{m})$. Let M be an Artinian R-module and $N < M$ be a finitely generated summand of M. Then there is an exact sequence*

$$0 \to \mathfrak{F}^{\mathfrak{a}}_1(M) \to \mathfrak{F}^{\mathfrak{a}}_1(M/N) \to \mathfrak{F}^{\mathfrak{a}}_0(N) \to \mathfrak{F}^{\mathfrak{a}}_0(M) \to \mathfrak{F}^{\mathfrak{a}}_0(M/N) \to 0$$

and there are isomorphisms $\mathfrak{F}^{\mathfrak{a}}_i(M) \cong \mathfrak{F}^{\mathfrak{a}}_i(M/N)$ for all $i > 1$.

Proof. Since N is a summand of M, the short exact sequence $0 \to N \to M \to M/N \to 0$ is split. Hence applying the functor $\mathfrak{F}^{\mathfrak{a}}_i(.)$ to it and using Theorem 2.9, we have the following long exact sequence.

$$\cdots \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M/N) \to \mathfrak{F}^{\mathfrak{a}}_i(N) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to \cdots$$

Since the functor $\Lambda_{\mathfrak{b}}(.)$ is exact on the subcategory of finitely generated R-modules, it follows that $L^{\mathfrak{b}}_i((0 :_N \mathfrak{a}^n)) = 0$ for all $i > 0$. Therefore $H^{\mathfrak{b}}_i((0 :_N \mathfrak{a}^n)) = 0$ for all $i > 0$. Now the assertion follows easily. □

Theorem 2.11. *Let M be an Artinian R-modules. If $I\mathfrak{b}^n = I \cap \mathfrak{b}^n$, for every ideal I of R of and $n \epsilon N$, then*

$$H^{\mathfrak{b}}_i((0 :_{H^{\mathfrak{b}}_j((0:_M \mathfrak{a}^n))} \mathfrak{a}^n)) \cong \begin{cases} (0 :_{H^{\mathfrak{b}}_j((0:_M \mathfrak{a}^n))} \mathfrak{a}^n) & i = 0, j \geq 0, \\ 0 & i > 0, j \geq 0. \end{cases}$$

Proof. By [[4], *Lemma* 4.3], the Tor_i preserves inverse limit in the second variable. Since $I\mathfrak{b}^n = I \cap \mathfrak{b}^n$, it follows $\mathrm{Tor}^R_1(R/\mathfrak{b}^n, R/I) = 0$. By using [[8], *Theorem* 7.8], the module $R/\mathfrak{b}^n$ is flat and so in view of [[11], *Lemma* 4.86] we have the following isomorphisms.

$$H^{\mathfrak{b}}_i((0 :_{H^{\mathfrak{b}}_j((0:_M \mathfrak{a}^n))} \mathfrak{a}^n)) = \varprojlim_t \mathrm{Tor}^R_i(R/\mathfrak{b}^t, \mathrm{Hom}(R/\mathfrak{a}^n, H^{\mathfrak{b}}_j((0 :_M \mathfrak{a}^n))))$$

$$\cong \varprojlim_t \varprojlim_s \operatorname{Tor}_i^R(R/\mathfrak{b}^t, \operatorname{Hom}(R/\mathfrak{a}^n, \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n)))$$
$$\cong \varprojlim_s \varprojlim_t \operatorname{Hom}(R/\mathfrak{a}^n, \operatorname{Tor}_i^R(R/\mathfrak{b}^t, \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n))).$$

Let $\underline{x} = x_1, \ldots, x_r$ denote a system of generators of $\mathfrak{b}$ and $\underline{x}(t) = (x_1^t, \ldots, x_r^t)$. So by Theorem 2.4

$$H_i^{\mathfrak{b}}(0:_{H_j^{\mathfrak{b}}((0:_M \mathfrak{a}^n))} \mathfrak{a}^n) \cong \varprojlim_s \operatorname{Hom}(R/\mathfrak{a}^n, \varprojlim_t H_i(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n))).$$

But $\underline{x}(t)\operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n) = 0$ for all $t \geq s$. Thus the following results can be obtained:

$$\varprojlim_t H_0(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n))) \cong \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n)),$$

and $\varprojlim_t H_i(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{b}^s, (0:_M \mathfrak{a}^n))) = 0$ for all $i > 0$.

This completes the claim. □

Corollary 2.12. *Let M be an Artinian R-module. If for every ideal I of R and $n \epsilon N$, IM be a summand of M and $I\mathfrak{b}^n = I \cap \mathfrak{b}^n$, then*

$$\varinjlim_n H_i^{\mathfrak{b}}((0:_{\bigcap_{t>0} \mathfrak{b}^t M} \mathfrak{a}^n)) \cong \begin{cases} \varinjlim_n H_i^{\mathfrak{b}}((0:_M \mathfrak{a}^n)) & i \geq 1, \\ 0 & i = 0. \end{cases}$$

Proof. Set $X = \bigcap_{t>0} \mathfrak{b}^t M$. Since M is an Artinian R-module, there exists a positive integer s such that $\mathfrak{b}^t M = \mathfrak{b}^s M$ for all $t \geq s$. So $X = \mathfrak{b}^s M$ and $\Lambda_{\mathfrak{b}}(M) \cong M/\mathfrak{b}^s M$. There is a short exact sequence of Artinian R-modules:

$$0 \to X \to M \to \Lambda_{\mathfrak{b}}(M) \to 0$$

But by assumption $\mathfrak{b}^s M$ is a summand of M, therefore the short exact sequence is split and by using Theorem 2.9, we have the following long exact sequence of local homology modules

$$\cdots \to \varinjlim_n H_{i+1}^{\mathfrak{b}}((0:_{\Lambda_{\mathfrak{b}}(M)} \mathfrak{a}^n)) \to \varinjlim_n H_i^{\mathfrak{b}}((0:_X \mathfrak{a}^n)) \to \varinjlim_n H_i^{\mathfrak{b}}((0:_M \mathfrak{a}^n))$$
$$\to \varinjlim_n H_i^{\mathfrak{b}}((0:_{\Lambda_{\mathfrak{b}}(M)} \mathfrak{a}^n)) \to \cdots \to \varinjlim_n H_1^{\mathfrak{b}}((0:_{\Lambda_{\mathfrak{b}}(M)} \mathfrak{a}^n)) \to \varinjlim_n H_0^{\mathfrak{b}}((0:_X \mathfrak{a}^n))$$
$$\to \varinjlim_n H_0^{\mathfrak{b}}((0:_M \mathfrak{a}^n)) \to \varinjlim_n H_0^{\mathfrak{b}}((0:_{\Lambda_{\mathfrak{b}}(M)} \mathfrak{a}^n)) \to 0.$$

This completes the proof by Theorem 2.11.

□

Theorem 2.13. *Let $\mathfrak{a}$ be an ideal of R. Let M be an Artinian R-module and $x \in \mathfrak{b}$. Then there is a long exact sequence*

$$\cdots \to \varinjlim_n H_i^{\mathfrak{b}}((0 :_M (\mathfrak{a}, x)^n)) \to \varinjlim_n H_i^{\mathfrak{b}}((0 :_M \mathfrak{a}^n)) \to R_x \otimes \varinjlim_n H_i^{\mathfrak{b}}((0 :_M \mathfrak{a}^n)) \to \cdots$$

for all $i \in \mathbb{Z}$.

Proof. The $\widetilde{C}$ech complex $\tilde{C}_x$ of the single element x is the fiber of the natural homomorphism $R \to R_x$. So there is a split exact sequence $0 \to R_x[-1] \to \tilde{C}_x \to R \to 0$. Let $\underline{\acute{x}}$ be a system of elements of R such $Rad\underline{\acute{x}} = Rad\mathfrak{a}$. By tensoring the short exact sequence with $C^{\cdot}_{\underline{\acute{x}}} \otimes Hom(C^{\cdot}_{\underline{x}}, M)$, we have an exact sequence of R complexes

$$0 \to C^{\cdot}_{\underline{\acute{x}}} \otimes \mathrm{Hom}(C^{\cdot}_{\underline{x}}, M) \otimes R_x[-1] \to C^{\cdot}_{\underline{\acute{x}},x} \otimes \mathrm{Hom}(C^{\cdot}_{\underline{x}}, M) \to$$
$$C^{\cdot}_{\underline{\acute{x}}} \otimes \mathrm{Hom}(C^{\cdot}_{\underline{x}}, M) \otimes R \to 0$$

where $\underline{x}$ denotes a system of elements of R such that $\mathfrak{b} = Rad\underline{x}$. By taking the long exact cohomology sequence, this provides

$$\cdots \to H^{-i}_{(\mathfrak{a},x)}(\mathrm{Hom}(C^{\cdot}_{\underline{x}}, M)) \to H^{-i}_{\mathfrak{a}}(\mathrm{Hom}(C^{\cdot}_{\underline{x}}, M)) \to$$
$$R_x \otimes H^{-i}_{\mathfrak{a}}(\mathrm{Hom}(C^{\cdot}_{\underline{x}}, M)) \to \cdots$$

We get the result by *Remark* 2.1 and *Theorem* 2.5. □

Corollary 2.14. *Let M be an Artinian R-module and $x \in \mathfrak{b}$. Then there is a short exact sequence*

$$\ldots \to \varinjlim_n H_i^{\mathfrak{b}}((0 :_M x^n)) \to H_i^{\mathfrak{b}}(M) \to R_x \otimes H_i^{\mathfrak{b}}(M) \to \ldots$$

for all $i \in \mathbb{Z}$.

Proof. This is a special case of *Theorem* 2.13 for $\mathfrak{a} = 0$. □

References

[1] M. Asgharzadeh K. and Divaani-Aazar, *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness,* Comm. Algebra, **39** (2011), 1082–1103.

[2] N. Bourbaki, *Algebre, Chap. X:Algebre homologique,* Masson, Paris, 1980.

[3] M. P. Brodman and R. Y. Sharp , *Local cohomology, An algebraic introduction with geometric applications,* Cambridge University Press, 1998.

[4] N. T. Coung and T. T. Nam, *The I-adic completion and local homology,* J, Algebra, **149** (1992), 438–453.

[5] M. Eghbali, *On Artinianness of Formal Local cohomology, colocalization and coassociated primes,* Math. Scand, **113** (2013), 5–19.

[6] E. E. Enochs and O. M. G. Jenda , *Relative homological Algebra,* Walter De Gruyter, expositions in Mathematics Berline, NewYork, 2000.

[7] A. Mafi, *Results of formal local cohomology modules,* Bull. Malays Math. Sci. Soc, **36** (2013), 173–177.
[8] H. Matsumura, *Commutative ring theory,* Cambridge University Press, 1986.
[9] A. Ooishi, *Matlis duality and the width of a module,* Hiroshima Math. J, **6** (1976), 573–587.
[10] R. N. Roberts, *Krull dimension for Artinian modules over quasi-local commutative rings,* Quart. J. Math Oxford, **26** (1975), 177–195.
[11] J. J. Rotman, *An introduction to homological Algebra,* London Academic Press, 1979.
[12] P. Schenzel, *Proregular sequences, local cohomology and completion*, Math Scand, **92** (2003), 161–180.
[13] P. Schenzel, *On formal local cohomolgy and connectedness*, Journal Of Algebra, **315** (2007), 894–923.
[14] Z. Tang, *Local homology theory Artinian module,* Comm Algebra, **22** (1994), 1675–1684.
[15] C. A. Weibel, *An introduction to homological Algebra,* Cambridge University Press NewYork, 1994.

M.H. Bijan-Zadeh
Department of Mathematics, Payame Noor University P.O.BOX 19395-3697, Tehran, IRAN.
Email: Profbijanzadeh@gmail.com

S. Ghaderi
Department of Mathematics, Payame Noor University P.O.BOX 19395-3697, Tehran, IRAN.
Email:ghaderi.sebar@gmail.com