Journal of Hyperstructures 11 *(2) (2022), 329-337.*
ISSN: 2322-1666 print/2251-8436 online

ON THE SOLUTION OF THE EXPONENTIAL DIOPHANTINE EQUATION $2^x + m^{2y} = z^2$, FOR ANY POSITIVE INTEGER m

MRIDUL DUTTA, PADMA BHUSHAN BORAH

Abstract. It is well known that the exponential Diophantine equation $2^x + 1 = z^2$ has the unique solution $x = 3$ and $z = 3$ in non-negative integers, which is closely related to the Catlan's conjecture. In this paper, we show that for $m \in \mathbb{N}, m > 1$, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = 2^\alpha M_n, \alpha \geq 0$ for some Mersenne number M_n. When $m = 2^\alpha M_n, \alpha \geq 0$, the unique solution is $(x, y, z) = (2 + n + 2\alpha, 1, 2^\alpha(2^n + 1))$. Finally, we conclude with certain examples and non-examples alike! The novelty of the paper is that we mainly use elementary methods to solve a particular class of exponential Diophantine equations.

Key Words: Mersenne numbers, Catalan's Conjecture, Exponential Diophantine equations.
2010 Mathematics Subject Classification: Primary: 11D61, 11D72; Secondary: 11D45.

1. Introduction

The literature contains a large number of articles about a single non-linear equation, involving various prime numbers and powers. Researchers are now very interested in determining the solutions of various Diophantine equations because these equations have many applications in the fields of algebraic topology, coordinate geometry, trigonometry, applied algebra, coding theory, cryptography, etc [4, 12].

Received: 02 February 2022, Accepted: 27 October 2023. Communicated by Mohammad Zarebnia;
∗Address correspondence to Mridul Dutta; E-mail: mridulduttamc@gmail.com.

The general theory of this type of equation is not available; special cases such as Catalan's conjecture has been resolved [17].

Many researchers have studied the exponential Diophantine equation. In 1844, the great Mathematician, Eugene Charles Catalan formulated a conjecture that the diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$[5]. Several authors (J. H. E. Cohn, N. Terai, J. W. S. Cassels, S. A. Arif, F. S. Abu Muriefah etc.) have done their research works on the Diophantine equations like $x^2 + c = y^n, x^4 - Dy^2 = 1, a^x + b^y = c^z, x^2 + 2^k = y^n$ etc. in the period of 1993-1997 [8, 16]. Famous Mathematicians F. Luca, Z. Cao, F. Beukers et. al. have done considrable works (approx $1995 - 2001$) on various aspects of the Diophantine equations $x^2 + 3^m = y^n, a^x + b^y = c^z, Ax^p + By^q = Cz^r$ etc.[6]. The Catalan conjecture was eventually proved by Preda Mihailescu [15] in 2002. S. A. Arif and F. S. A. Muriefah [16] have done works on the Diophantine equation $x^2 + q^{2k+1} = y^n$ in 2002 .

In 2007, Acu[3] proved that the Diophantine equation $2^x + 5^y = z^2, x, y, z \in \mathbb{Z}^+$ has only two solutions i.e. $(3, 0, 3)$ and $(2, 1, 3)$. A number of researchers have studied the exponential Diophantine equations in the period $2010 - 2016$. Suvarnamani, Singta, Chotchaisthit, B. Sroysang, Saritz, H. Kishan, R. Megha, J. J. Bravo, F. Luca etc.[1, 2, 9] have done their extensive works on the different types of Diophantine equations $4^x + 7^y = z^2, 4^x + 11^y = z^2, 4^x + 13^y = z^2, 4^x + 17^y = z^2, A^x + B^y = C^z, 3^x + 5^y = z^2, 8^x + 19^y = z^2, 31^x + 32^y = z^2, 7^x + 8^y = z^2, F_n + F_m = 2^a$ etc. J. F. T. Rabago [7] studied on the Diophantine equations $3^x + 19^y = z^2, 3^x + 91^y = z^2$. Moreover, recently, authors N. Burshtein [12], S. Kumar, S. Gupta and H. Kishan [17] have done works relating to Diophantine equation. Researcher M. Somanath, K. Raja, J. Kannan, K. Kaleeswari, A. Akila, M. Mahalakshmi etc. have done their works on the Diophantine equations $x^2 = 29y^2 - 7^t, t \in \mathbb{N}, x^2 = 9y^2 + 11z^2, \alpha^2 - 90\beta^2 - 10\alpha - 1260\beta = 4401$ in 2020 [11]. W. S. Gayo and J. B. Bacan [18] studied and solved the exponential Diophantine equation of the form $M_p^x + (M_q + 1)^y = z^2$ for Mersenne primes M_p and M_q and non-negative integers x, y, and z. In 2022, authors P. B. Borah and M. Dutta did related works [10, 13, 14].

Motivated by all these results, we study the exponential Diophantine equation $2^x + m^{2y} = z^2$ and we show that for any positive integer $m, m > 1$, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = 2^\alpha M_n, \alpha \geq 0$ for some Mersenne number M_n. When $m = 2^\alpha M_n, \alpha \geq 0$, the unique solution is $(x, y, z) = (2 + n + 2\alpha, 1, 2^\alpha(2^n + 1))$.

2. Preliminaries

2.1. **Mersenne numbers.** The Mersenne numbers have played an important role in number theory since 1644. In particular, researchers are interested in Mersenne prime, which is closely related to finding perfect numbers and large prime numbers. Numbers of the form $M_n := 2^n - 1$ are called Mersenne numbers, where $n \in \mathbb{N}$.

If the Mersenne number is prime, it is called the Mersenne prime. For M_n to be prime, n must also be a prime. The first four Mersenne prime numbers are $3, 7, 31, 127$ (corresponding to $n = 2, 3, 5, 7$) respectively. There are only 51 known Mersenne primes [4].

2.2. **The Catalans Conjecture.** The unique solution for the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ is $(3, 2, 2, 3)$ [15].

2.3. **Lemma 1.** The exponential Diophantine equation $1 + m^{2y} = z^2$ has no solution in non-negative integers.

Proof : $1+(m^y)^2 = z^2$ implies $1 = (z-m^y)(z+m^y)$ implies $z-m^y = 1$ and $z + m^y = 1$
which is a contradiction as $z - m^y < z + m^y$.

2.4. **Lemma 2.** The exponential Diophantine equation $2^x + 1 = z^2$ has the unique solution $x = 3$ and $z = 3$ in non-negative integers.

Proof: Given equation implies $z^2 - 2^x = 1$ implies $z^2 = 1 + 2^x$ implies $z^2 \geq 2$ implies $z \geq 2 > 1$. If $x = 0, z^2 = 2$ implies no solution exists. If $x = 1, z^2 = 3$ implies no solution exists. Otherwise, $x > 1$ implies $x = 3$ and $z = 3$ is the unique solution by Catalan conjecture.

We now come to our main result. In the following, we will study all the possible solutions and we will use Catalan's conjecture and the preceding lemmas in solving the exponential Diophantine equation $2^x + m^{2y} = z^2, m \in \mathbb{N}$.

3. Main Results

3.1. **Theorem 1.** For $m \in \mathbb{N}, m \neq 1$ odd, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = M_n$ for some $n \in \mathbb{N}, n \neq 1$. When $m = M_n, n \in \mathbb{N}, n \neq 1$, the unique solution is $(x, y, z) = (n + 2, 1, 2^n + 1)$.

Proof : The equation is-

$$2^x + m^{2y} = z^2, m \in \mathbb{N} \tag{3.1}$$

Now, m is odd and x is positive integer implies z is odd too. Equation (3.1) gives $2^x = z^2 - (m^y)^2 = (z - m^y)(z + m^y)$ implies $(z - m^y) = 2^u, z + m^y = 2^{x-u}, x > 2u, u > 1$ implies $x > 2$ implies $2.m^y = 2^u(2^{x-2u} - 1)$. Since m is odd implies $u = 1$ and $2^{x-2u} - 1 = m^y, u = 1$ implies $2^{x-2} - m^y = 1$. Catalan conjecture implies that $x = 2, 3$ or $y = 1$ as $y > 0. x = 2$ implies $m^y = 0$ implies $m = 0$, which is a contradiction. $x = 3$ implies $1 = m^y$ implies $y = 0$, which is again a contradiction. Moreover, $y = 1$ implies $2^{x-2} = 1 + m^1$ implies $m = 2^{x-2} - 1$ implies $m = M_{x-2}$, is a Mersenne number. $\therefore$ Equation (3.1) implies $z^2 = 2^x + (2^{x-2} - 1)^2 = (2^{x-2} - 1)^2 + 4.2^{x-2}.1 = (2^{x-2} + 1)^2$ implies $z = 2^{x-2} + 1$ implies $(x, y, z) = (n + 2, 1, 2^n + 1), 1 < n \in \mathbb{N}$. Thus, solution of (3.1) in positive integers exist iff m is a Mersenne number. Also, when $m = 2^n - 1$, the unique solution is $(n + 2, 1, 2^n + 1), n \in \mathbb{N}, n \neq 1$.

Conversely, it is immediate to see that $(n + 2, 1, 2^n + 1), n \in \mathbb{N}, n \neq 1$ is a positive integral solution of equation (3.1) for odd $m \neq 1$.

3.2. **Theorem 2.** For $m \in \mathbb{N}$ even, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = 2^\alpha M_n, \alpha \in \mathbb{N}$ for some Mersenne number M_n. When $m = 2^\alpha M_n, \alpha \in \mathbb{N}$, the unique solution is $(x, y, z) = (2 + n + 2\alpha, 1, 2^\alpha(2^n + 1))$.

Proof : Since m is even, Let $m = 2^\alpha c, \alpha \geq 1, c$ is odd. Equation (3.1) implies $2^x + (2^\alpha c)^{2y} = z^2$ implies z is even too. Let $z = 2^\beta b, \beta \geq 1, b$ is odd.

$$\Rightarrow 2^x + 2^{2\alpha y} c^{2y} = 2^{2\beta} b^2 \tag{3.2}$$

If $x = 2\alpha y = 2\beta$, equation (3.2) reduces to $1 + c^{2y} = b^2$ and from Lemma 1, we get that equation (3.2) has no solution in this case. Otherwise, let $\gamma = min\{x, 2\alpha y, 2\beta\}$. If $x \neq 2\alpha y \neq 2\beta$, equation (3.2) boils down to one of the three.
(a) $\gamma = x : 1 + 2^{2\alpha y}c^{2y} = 2^{2\beta-\gamma}b^2$, which is a contradiction by parity consideration.
or, *(b)* $\gamma = 2\alpha y : 2^{x-2\alpha y} + c^{2y} = 2^{2\beta-2\alpha y}b^2$, which is again a contradiction by parity.
or, (c) $\gamma = 2\beta : 2^{x-\gamma} + 2^{2\alpha y-\gamma}c^{2y} = b^2$, which is also a contradiction, again by parity.
$\therefore$ We must have two of $\{x, 2\alpha y, 2\beta\}$ equal for equation (3.2) to have a solution.

Case 1: $x = 2\alpha y$, In this case, we must have $\gamma = x = 2\alpha y$ as otherwise $2^{x-\gamma}+2^{2\alpha y-\gamma}c^{2y} = b^2$, which is a contradiction, by parity. Equation (3.2) implies $1 + c^{2y} = 2^{2\beta-\gamma}b^2 = 2^{2\beta-2\alpha y}b^2$ implies $1 + c^{2y} = (2^{\beta-\alpha y}b)^2$ has no solution by Lemma 1.

Case 2: $x = 2\beta$, If $\gamma = x = 2\beta$, from equation (3.2), $1 + 2^{2\alpha y-2\beta}c^{2y} = b^2$ implies $1 + (2^{\alpha y-\beta}c^y)^2 = b^2$ has no solution by Lemma 1. Otherwise, $\gamma = 2\alpha y$, Equation (3.2) implies $2^{x-\gamma} + c^{2y} = 2^{x-\gamma}b^2$ has no solution again by parity.

Case 3: $2\alpha y = 2\beta$ i.e. $\alpha y = \beta$, If $\gamma = x$, then equation (3.2) $\Rightarrow$ $1 + 2^{2\alpha y-\gamma}.c^{2y} = z^{2\beta-\gamma}b^2$ has no solution due to parity. Otherwise, $\gamma = 2\alpha y = 2\beta$, then from equation (3.2),

$$2^{x-\alpha} + c^{2y} = b^2. \tag{3.3}$$

Now, since 'b' and 'c' are both odd, this is same as 'case 2' of the equation (3.1). Since $y \neq 0$, equation (3.3) has a solution 'iff' c is a Mersenee number i.e. $c = 2^n - 1, n \in \mathbb{N}, b = 2^n + 1, y = 1 \Rightarrow \alpha = \beta$ and $x = \gamma + n + 2 = n + 2 + 2\alpha, \therefore m = 2^\alpha c = 2^\alpha(2^n - 1), z = 2^\beta b = 2^\alpha(2^n + 1), \therefore (x, y, z, m) = (n + 2 + 2\alpha, 1, 2^\alpha(2^n + 1), 2^\alpha M_n)$.

Conversely, it is easy to see that $(x, y, z) = (2 + n + 2\alpha, 1, 2^\alpha(2^n + 1)), n \in \mathbb{N}$ is a solution of equation (3.1) for $m = 2^\alpha M_n$.

Thus, when m is even, the exponential Diophantine equation $2^x + m^{2y} = z^2, m \in \mathbb{N}$ admits a solution if and only if $m = 2^\alpha M_n, \alpha \in \mathbb{N}, M_n$

being the n^{th} 'Mersenne number'. In this case, the unique solution of equation (3.1) is $(x, y, z) = (2 + n + 2\alpha, 1, 2^{\alpha}(2^n + 1))$.
This completes the theorem.

3.3. **Corollary 1:** For $m > 1$, the exponential Diophantine equation $2^x + m^2 = z^2$ has a unique solution $(x, z, m) = (n + 2 + 2\alpha, 2^{\alpha}(2^n + 1), 2^{\alpha}(2^n - 1)), \alpha > 0, n \in \mathbb{N}$.

3.4. **Corollary 2:** The exponential Diophantine equation $2^x + m^{2y} = z^2$ has no solution in positive integers for $m, y > 1$ by Theorem 1 and Theorem 2. In particular, $2^x + m^4 = z^2$, $2^x + m^6 = z^2$ etc. have no solution in positive integers for $m > 1$.

3.5. **Corollary 3:** For $m > 1$, the exponential Diophantine equation $2^x + m^{2y} = w^4$ has a solution (x, z, w, m) in positive integers if and only if m is of the form 7.4^k where k is a non-negative integer. In this case, the unique solution is given by $(x, y, w) = (5 + 4k, 1, 3.2^k), k \in \mathbb{Z}, k \geq 0$.

Proof: Given equation is

$$2^x + m^{2y} = w^4, m > 1 \tag{3.4}$$

If (x, y, w, m) is a solution of equation (3.4), then $(x, y, z = w^2, m)$ is a solution of equation (3.1). By Theorem 1 and Theorem 2, $(x, y, w^2, m) = (x + 2(1 + \alpha), 1, 2^{\alpha}(2^n + 1), 2^{\alpha}(2^n - 1)), \alpha \geq 0, n \in \mathbb{N}$
implies $w^2 = 2^{\alpha}(2^n + 1)$ implies $w = 2\frac{\alpha}{2}.\sqrt{2^n + 1}$ implies $\alpha = 2k, k \geq 0$ and $2^n + 1 = v^2$ implies $n = v = 3$, by Lemma 2 implies $w = 2^k\sqrt{9} = 3.2^k$ implies $m = 2^{\alpha}(2^3 - 1) = 7.2^{2k} = 7.4^k$, Also, $x = 3 + 2(1 + 2k) = 5 + 4k$ and $y = 1$.
This completes the proof.

3.6. **Corollary 4:** The exponential Diophantine equation $2^x + m^{2y} = w^{2l}, l \geq 3$ has no solution in non negative integers.

Proof: Suppose (x, y, w, m) is a solution. Then, $(x, y, z = w^2, m)$ is a solution of equation (3.1). By Lemma 2, Theorem 1 and Theorem 2, we must have $2 = w^l = 3$ or $2^{\alpha}(2^n + 1)$. Now, $l \geq 3$ implies $w^l \neq 3$ implies

$w^l = 2^\alpha(2^n+1), w \in \mathbb{Z}$ implies $w = 2^{\frac{\alpha}{l}}(2^n+1)^{\frac{1}{l}}$ implies $2^n + 1 = v^l$. But, for $l \geq 3, n \in \mathbb{N}$, this has no solution in integers by Catalan conjecture. $\therefore$ The given equation has no solution.

This completes the Journey !

We now provide some examples and non-examples below.

Example 1 The equation $2^x + 25^y = z^2$ has no solution in positive integer as 5 is not a Mersenne number.

Example 2 $M_3 = 7 \therefore$ The equation $2^x + 49^y = z^2$ has a unique solution in positive integer $(x, y, z) = (5, 1, 9)$.

Example 3 The equation $2^x + 81^y = z^2$ has no solution in positive integers as 81 is not a Mersenne number.

Example 4 For $m = 4 = 2^2 M_1$, So $\alpha = 2, n = 1$ then the equation becomes $2^x + 16^y = z^2$ which has a unique solution $(x, y, z) = (2 + 1 + 2.2, 1, 2^2(2^1 + 1)) = (7, 1, 12)$.

Example 5 For $m = 6 = 2.3 = 2^1 M_2$, So $\alpha = 1, n = 2$ then the equation becomes $2^x + 36^y = z^2$ which has a unique solution $(x, y, z) = (2 + 2 + 2.1, 1, 2^1(2^2 + 1)) = (6, 1, 10)$.

Example 6. The equation $2^x + 100^y = z^2$ has no solution in positive integers as $10 \neq 2^\alpha(2^n - 1)$ for any $\alpha \geq 0, n \in \mathbb{N}$.

4. Conclusion

In this article, we have showed that for $m \in \mathbb{N}, m \neq 1$ odd, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = M_n$ for some $n \in \mathbb{N}, n \neq 1$. When $m = M_n, n \in \mathbb{N}, n \neq 1$, the unique solution is $(x, y, z) = (n+2, 1, 2^n+1)$. We also proved that for $m \in \mathbb{N}$ even, the exponential Diophantine equation $2^x + m^{2y} = z^2$ admits a solution in positive integers (x, y, z) if and only if $m = 2^\alpha M_n, \alpha \in \mathbb{N}$ for some Mersenne number M_n. When $m = 2^\alpha M_n, \alpha \in \mathbb{N}$, the unique solution is $(x, y, z) = (2+n+2\alpha, 1, 2^\alpha(2^n+1))$.

Finally, we concluded with certain illustrations of our results.

References

[1] A. Suvarnamani, *Solution of the Diophantine equation* $p^x + q^y = z^2$, Int. J. Pure. Appl. Math., **94(4)** (2014), 457-460.

[2] B. Sroysang, *On the Diophantine equation* $7^x + 8^y = z^2$, Int. J. Pure. Appl. Math., **84 (1)** (2013), 111-114.

[3] D. Acu, *On a Diophantine equation* $2^x + 5^y = z^2$, Gen. Math., **15** (2007), 145-148.

[4] D. M. Burton, *Elementary number theory*, McGraw-Hill Education, (2006).

[5] E. Catalan, *Note extraite dúne lettre adressee à l´editeur*, J. Reine Angew. Math., **27** (1844), 192.

[6] F. Luca, *On a Diophantine equation*, Bull. Austral. Math. Soc., **61** (2000), 241-246.

[7] J. F. T. Rabago, *On the Diophantine equations* $3^x + 19^y = z^2, 3^x + 91^y = z^2$, Int. J. of Mathematics and Scientific Computing, **3(1)** (2013), 28-29.

[8] J. H. E. Cohn , *The Diophantine equation* $x^2 + c = y^n$, Acta Arith., **LXV. 4** (1993).

[9] J. J. Bravo and F. Luca, *On the Diophantine equation* $F_n + F_m = 2^a$, Quaest. Math., **39(3)** (2016).

[10] M. Dutta and P. B. Borah, *On the solution of a class of exponential Diophantine equations*, South East Asian J. Math. Math. Sci., **18(3)** (2022).

[11] M. Somanath, K. Raja, J. Kannan and A. Akila, *Integral Solutions of an infinite Elliptic Cone* $x^2 = 9y^2 + 11z^2$, Adv. Appl. Math. Sci., **19(11)** (2020), 1119-1124.

[12] N. Burshtein, *On the Diophantine equation* $2^{2x+1} + 7^y = z^2$, Ann. Pure Appl. Math., **16 (1)** (2018), 177-179.

[13] P. B. Borah and M. Dutta, *On two classes of exponential Diophantine equations*, Communications in Mathematics and Applications, **13 (1)** (2022), 137145.

[14] P. B. Borah and M. Dutta, *On the Diophantine equation* $7^x + 32^y = z^2$ *and Its generalization*, Integers, **22 (2)** (2022).

[15] P. Mihailescu, *Primary cycolotomic units and a proof of Catalans conjecture*, J. Reine Angew. Math., **27** (2004), 167-195.

[16] S. A. Arif and F. S. A. Muriefah, *On the Diophantine equation* $x^2 + q^{2k+1} = y^n$, J. Number Theory, **95** (2002), 95-100.

[17] S. Kumar, S. Gupta and H. Kishan, *On the Non-Linear Diophantine equation* $61^x + 67^y = z^2, 67^x + 73^y = z^2$, Ann. Pure Appl. Math., **18(1)** (2018), 91-94.

[18] W. S. Gayo and J. B. Bacan, *On the Diophantine equation* $M_p^x + (M_q + 1)^y = z^2$, Eur. J. Pure Appl. Math., **14 (2)** (2021), 396-403.

Mridul Dutta

Department of Mathematics, Dudhnoi College, P.O. Dudhnoi, Goalpara, Assam, India

Email: mridulduttamc@gmail.com

Padma Bhushan Borah
Department of Mathematics, Gauhati University, Guwahat, Assam, India
Email: padmabhushanborah@gmail.com