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GEOMETRY OF WARPED PRODUCT PSEUDO SLANT SUBMANIFOLDS IN A NEARLY LORENTZIAN PARA SASAKIAN MANIFOLD

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Abstract. The object of the present paper is to study Lorentzian para-Sasakian manifold on a pseudo slant submanifold and using some properties like warped product on manifolds, totally geodesic foliation, integrability on the properties of nearly Lorentzian para-Sasakian manifold we find some results.

Key Words: Pseudo slant submanifolds; nearly Lorentzian para-Sasakian manifold; totally geodesic foliation; warped product; distribution vector fields.
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1. Introduction

The geometry on warped product submanifolds studied by B.V.Chen [8] who observed the concept of CR-warped product submanifolds in a Kahlerian manifold. Further in the different geometric aspect, he tried to find the warping function in the form of some partial differential equations. After then many research articles have been appeared on the distinct forms of warped product submanifolds of different class of structures ([1], [2], [3], [4], [9], [13], [15], [16], [17]). Recently, in [14], the author established general sharp inequality for the second fundamental form in terms of the warping function f and Sasakian and cosymplectic in warped product submanifolds of nearly Lorentzian para-Sasakian manifold. S. Uddin et al. in [18] also studied some existence results for

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warped product pseudo-slant submanifolds in terms of endomorphism in a nearly Cosymplectic manifold. In this paper we investigate the properties of nontrivial warped product pseudo slant submanifold of the form $M_\perp \times_y M_\theta$ which are the natural extension of CR - warped product submanifolds. It is clear that every CR -warped product submanifold of the form $M_\perp \times_y M_\theta$ and $M_\theta \times_y M_\perp$ with slant angle $\theta = 0$. The warped product pseudo slant submanifold never induces the CR- warped product submanifolds. Here we consider M $=M_\perp \times_y M_\theta$ such that M_θ and $M_\perp$ proper slant and anti invariant submanifolds. Finally we estublish some necessary and sufficient conditions involving some geometric conditions and properties of nearly Lorentzian para-Sasakian manifold with the warped product submanifolds in a pseudo slant structure.

2. Preliminaries

If $\tilde{M}$ is n-dimensional almost contact metric manifold of equipped with an almost contact metric structure (f, ξ, η, g) consisting of a (1, 1) tensor field f, a vector field ξ, a one form η and a Riemannian metric g which satisfy

$$f^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta(f) = 0, \tag{2.1}$$

$$f(\xi) = 0, \quad \eta(U) = g(U, \xi),$$

$$g(fU, fV) = g(U, V) + \eta(U)\eta(V), \tag{2.2}$$

$$g(fU, V) = g(U, fV) = \psi(U, V)$$

for all $U, V \in \chi(M)$, then the structure(f, ξ, η, g) is termed as Lorentzian para Sasakian structure. Also in a Lorentzian para Sasakian manifold structure rank $f = n-1$, a Lorentzian para contact manifold $\tilde{M}$ is called Lorentzian para Sasakian manifold if the following conditions hold:

$$(\tilde{\nabla}_U \phi)V = g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi \tag{2.3}$$

$$\tilde{\nabla}_U \xi = fU \tag{2.4}$$

holds for all vector fields U, V tangent to $\tilde{M}$ and $\tilde{\nabla}$ is the Levi- Civita connection associated to $\tilde{M}$. An almost contact metric manifold $\tilde{M}$ on (f, ξ, η, g) is called nearly Lorentzian para Sasakian manifolds if

$$(\tilde{\nabla}_U f)V + (\tilde{\nabla}_V f)U = 2g(U, V)\xi + \eta(V)U + \eta(U)V + 4\eta(U)\eta(V)\xi \tag{2.5}$$

Theorem 2.1. *On a nearly Lorentzian para Sasakian manifold $\tilde{M}$ the following holds,*

$$g(\tilde{\nabla}_V \xi, U) + g(\tilde{\nabla}_U \xi, V) - 2g(U, fV) = 0$$

for any vector field U, V tangent to $\tilde{M}$. We denote the tangential and normal parts of $(\tilde{\nabla}_U f)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ such that

$$(\tilde{\nabla}_U f)V = \mathcal{P}_U V + \mathcal{Q}_U V$$

Then in a nearly Lorentzian para Sasakian manifold, we have

$$\mathcal{P}_U V + \mathcal{P}_V U = 2g(U, V)\xi + \eta(V)U + \eta(U)V + 4\eta(U)\eta(V)\xi$$

$$\mathcal{Q}_U V + \mathcal{Q}_V U = 0$$

for any U, V are tangent to $\tilde{M}$. It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$ [19],

$$\begin{aligned}
(i) \qquad & \mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W \\
(ii) \qquad & \mathcal{Q}_{U+V} W = \mathcal{Q}_U W + \mathcal{Q}_V W \\
(iii) \qquad & \mathcal{P}_U (W + W) = \mathcal{P}_U W + \mathcal{P}_U W \\
(iv) \qquad & \mathcal{Q}_U (W + W) = \mathcal{Q}_U W + \mathcal{Q}_U W \\
(v) \qquad & g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W) \\
(vi) \qquad & g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N) \\
(vii) \qquad & \mathcal{P}_U fV + \mathcal{Q}_U fV = -f(\mathcal{P}_U V + \mathcal{Q}_U V)
\end{aligned}$$

A Riemannian manifold M is isometrically immersed into almost contact metric manifold $\tilde{M}$ and let g denote the Riemannian metric induced on M. Suppose that $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ be the Lie algebra of the vector fields tangent to M and normal to M, respectively and $\nabla^{\perp}$ the induced connection on $(T^{\perp}M)$. It is represented by $\mathfrak{f}(M)$ the algebra of smooth functions on M and $\Gamma(TM)$, the $\mathfrak{f}(M)$ - module of smooth sections of TM over M which are also denoted by ∇ the Levi - Civita connections of M then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.6}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{2.7}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$ where h and A_N are the second fundamental form and the shape operator(corresponding to the normal vector field N), respectively V for the immersion of M into $\tilde{M}$, they are defined by

$$g(h(U,V),N) = g(A_N U,V) \tag{2.8}$$

now for any $U \in \Gamma(TM)$, we have

$$fU = rU + sU \tag{2.9}$$

where rU, sU are tangential and normal component of fU, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$fN = iN + kN, \tag{2.10}$$

where iN (res. kN) are tangential (res. normal) components of fN. A submanifold M is said to be totally geodesic and totally umbilical, if $h(U,V) = 0$ and $h(U,V) = g(U,V)H$, respectively. Now we defined a class of submanifolds which are the slant submanifold.

Definition 2.2. For each non zero vector U tangent to M at P, such that U is not proportional to ξ_p, we denote by $0 \leq \theta(U) \leq \frac{\pi}{2}$, the angle between fU and T_pM is called the Wirtinger angle. If the angle $\theta(U)$ is constant for all $U \in T_pM- <\xi>$ and $p \in M$ then M is said to be a slant submanifold [11] and the angle θ is called slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \frac{\pi}{2}$, M is anti - invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti - invariant

Proposition 2.3. *If M is a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then M is slant if and only if there exists a constant $\lambda \in$ [0, 1] such that [7]*

$$r^2 = \lambda(I + \eta \otimes \xi). \tag{2.11}$$

Furthermore, in such a case, if θ is slant angle, then it satiesfies that $\lambda = \cos^2\theta$. For a slant submanifold M of an almost contact metric manifold the followings are cosequences of above

$$g(rU,rV) = \cos^2\theta(g(U,V) + \eta(U)\eta(V)) \tag{2.12}$$

$$g(sU,sV) = \sin^2\theta(g(U,V) + \eta(U)\eta(V)) \tag{2.13}$$

for any $U,V \in \Gamma(TM)$. Also we proceed to give an another characterization which is directly related the consequence of above propostition.

Proposition 2.4. *If M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$, then*

$$(2.14) \qquad (i)\ isU = sin^2\theta(U + \eta(U)\xi) \qquad and \qquad (ii)\ ksU = -srU$$

3. Pseudo slant submanifolds of nearly Lorentzian para Sasakian manifold

In this section we define pseudo slant submanifolds of an almost contact manifold by using slant distribution in [6]. We find the geometry of leaves of distributions containing in the definition of pseudo slant submanifolds. We also try to evaluate some conditions for such subimmersions to be totally geodesic foliations for which later usage in characterization theorem.

Definition 3.1. In a submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be pseudo slant submanifold, if there exist two orthogonal distributions $D^{\perp}$ and D^{θ} such that

(i) $TM = D^{\perp} \oplus D^{\theta} \oplus < \xi >$, where $< \xi >$ is 1 - dimensional distribution spanned by ξ.

(ii) $D^{\perp}$ is anti-invariant distribution under f i.e., $fD^{\perp} \subseteq T^{\perp}M$.

(iii)D^{θ} is slant distribution with slant angle $\theta \neq 0, \frac{\pi}{2}$.

If m_1 and m_2 are dimensions of distributions $D^{\perp}$ and D^{θ}, respectively and $m_2 = 0$, then M is anti invariant submanifold. If $m_1 = 0$ and $\theta = 0$, then M is invariant submanifold. If $m_1 = 0$ and $\theta \neq 0, \frac{\pi}{2}$ then M is proper slant submanifold or if $\theta = \frac{\pi}{2}$ then M is anti invariant submanifold and if $\theta = 0$, then M is semi slant submanifold. If μ is an invariant subspace of normal bundle $T^{\perp}M$ can be decomposed as follows: $T^{\perp}M = fD^{\perp} \oplus nD^{\theta} \oplus \mu$, where μ is the even dimensional invariant sub bundle of $T^{\perp}M$. Now we estublish the following:

Theorem 3.2. *If M is a pseudo slant sub manifold of a nearly Lorentzian para Sasakian manifold $\tilde{M}$, then the distribution $D^{\perp} \oplus \xi$ defines as totally geodesic foliation M if and only if*

$$(3.1) \qquad g(h(W,V), srU) = -\frac{1}{2}[(g(A_{fW}V, rU) + g(A_{fV}W, rU) + \eta(W)g(V, rU) + \eta(V)g(W, rU)]$$

Proof. From

$$g(f\tilde{\nabla}_W V, fU) = g(\tilde{\nabla}_W V, U) - \eta(\tilde{\nabla}_W V)\eta(U)$$

and using (2.5), (2.9) we have

$$g(\tilde{\nabla}_W V, U) = g(\tilde{\nabla}_W fV, rU) - g((\tilde{\nabla}_W f)V, rU) - g(\tilde{\nabla}_W V, fsU),$$

using (2.5), (2.7) we get

$$g(\tilde{\nabla}_W V, U) = -g(A_{fV}W, rU) + g(\tilde{\nabla}^{\perp}_W fV, rU) + g((\tilde{\nabla}_V f)W, rU)$$

$$-[2g(W,V)\eta(rU) + 4\eta(W)\eta(V)\eta(rU) + \eta(W)g(V, rU)$$

$$+\eta(W)g(Z, rU)] - g(\tilde{\nabla}_W V, rsU) - g(\tilde{\nabla}_W V, fsU)$$

from which applying covariant derivative of endomorphism of f we get using prop. 2, it is easily seen that

$$g(\tilde{\nabla}_W V, U) = -g(A_{fV}W, rU) - g(\tilde{\nabla}_V fW, rU) - g(\tilde{\nabla}_V W, srU)$$

$$-g(\tilde{\nabla}_V W, r^2U) - \eta(W)g(V, rU) - \eta(V)g(W, rU)$$

$$+\sin^2\theta g(\tilde{\nabla}_W V, U) + g(\tilde{\nabla}_W V, srU).$$

Using(2.6), (2.8), (2.11), (2.14) we finally arrive at

$$\cos^2\theta g(\tilde{\nabla}_W V, U) = -g(A_{fW}V, rU) - g(A_{fV}W, rU)$$

$$-2g(h(W,V), srU) - \cos^2\theta g(\tilde{\nabla}_V W, U) - g(\tilde{\nabla}_V W, srU)$$

$$-g(\tilde{\nabla}_W V, srU) - \eta(W)g(V, rU) - \eta(V)g(W, rU).$$

Now if this is totally geodesic foliation then

$$0 = -g(A_{fW}V, rU) - g(A_{fV}W, rU) - 2g(h(W,V), srU)$$

$$-\eta(W)g(V, rU) - \eta(V)g(W, rU).$$

the result follows immediately after applying it in the last expression. □

Theorem 3.3. *A pseudo slant submanifold M if a nearly Lorentzian para Sasakian manifold $\tilde{M}$ the distribution D^θ is intregrable if and only if*

$$2g(\nabla_U V, W) = \ sec^2\theta[g(h(U, rV) + h(V, rU), fW) + g(h(U,W), srV)$$

$$+g(h(V,W), srU) + \eta(W)g(\tilde{\nabla}_U\xi, V)].$$

for every $W \in \Gamma(D^\perp \otimes \xi)$ and $U, V \in \Gamma(D^\theta)$.

Proof. By using the properties of symmetric torsion and Riemannian metric g and using (2.2), (2.5), (2.9) using properties of nearly Lorentzian para Sasakian manifold and $\eta(U)=0, \eta(V)=0$ we have,

$$g([U,V],W) = g(\tilde{\nabla}_U PV, fW) + g(\tilde{\nabla}_U FV, fW) + g((\tilde{\nabla}_V f)U, fW)$$
$$-g(\tilde{\nabla}_V U, W) + \eta(W) g(\tilde{\nabla}_U \xi, V)$$

From which we get applying covariant derivative and (2.6) we get

$$\begin{aligned} g([U,V],W) &= g(h(U,rU), fW) - g(sV, (\tilde{\nabla}_U f)W) - g(fsV, \tilde{\nabla}_U W) \\ &\quad + g(\tilde{\nabla}_V fU, fW) - 2g(\tilde{\nabla}_V U, W) + \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

Which gives at once

$$\begin{aligned} g([U,V],W) &= g(h(U,rV), fW) + g(h(V,rU), fW) \\ &\quad - g(sV, \mathcal{Q}_U W) - g(rsV, \tilde{\nabla}_U W) - g(fsV, \tilde{\nabla}_U W) \\ &\quad - g(sU, \tilde{\nabla}_V fW) - 2g(\tilde{\nabla}_V U, W) + \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

Now

$$g(fV, \mathcal{Q}_U W) = g(V, f\mathcal{Q}_U W) = 0,$$

using (2.9), we have

$$g(sV, \mathcal{Q}_U W) = g(fV, \mathcal{Q}_U W)$$

Considering $(\tilde{\nabla}_U f)V = \mathcal{P}_U V + \mathcal{Q}_U V$, where $\mathcal{P}_U V, \mathcal{Q}_U$ are the tangential; part and normal part of $(\tilde{\nabla}_U f)V$, use this in 2nd term, prop. 2, (2.6), (2.10) and (2.14), we get

$$\begin{aligned} g([U,V],W) &= g(h(U,rV), fW) + g(h(rU,V), fW) - g(V, f\mathcal{Q}_U W) \\ &\quad - \sin^2\theta g(V, \tilde{\nabla}_U W) - g(srV, \tilde{\nabla}_U W) - g(isU, \tilde{\nabla}_V W) \\ &\quad - g(fsU, \tilde{\nabla}_V W) - 2g(\tilde{\nabla}_V U, W) + \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

Utilise U, V are orthogonal to W above settle to

$$\begin{aligned} g([U,V],W) &= g(h(U,rV), fW) + g(h(rU,V), fW) - g(srU, h(V,W)) \\ &\quad + \sin^2\theta g(W, \tilde{\nabla}_U V) + \sin^2\theta g(W, \tilde{\nabla}_V U) \\ &\quad + g(srV, h(U,W)) - 2g(\tilde{\nabla}_V U, W) + \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

Now calculating for a while we reach ,

$$\begin{aligned} \sin^2\theta g([U,V],W) &= g(h(U,rV) + h(rU,V), fW) + g([U,V],W) \\ &\quad - 2\cos^2\theta g(W, \tilde{\nabla}_U V) + g(h(U,W), srV) \\ &\quad + g(h(V,W), srU) + \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

If D^θ is integrable then we finally have from above

$$\begin{aligned} 2\cos^2\theta g(W, \tilde{\nabla}_U V) &= g(h(U, rV) + h(rU, V), fW) \\ &+ g(h(U, W), srV) + g(h(V, W), srU) \\ &+ \eta(W) g(\tilde{\nabla}_U \xi, V). \end{aligned}$$

□

Which leads the proof .

4. Warped product submanifold of the form $M_\perp \times_y M_\theta$

One of the important part of Riemannian product is warped product with warping function y, and was introduced by Bishop and Neil [5]. They express this matter as follows, if y is a positive differentiable function which always be defined on leaves and (D, g_D) and (E, g_E) are two Riemannian manifolds $D \times_y E = (D \times_y E, g)$, where $g = g_D + y^2 g_E$. For a warped product, we have

$$\nabla_U W = \nabla_W U = U \ln y W \tag{4.1}$$

for any vector field U, W and tangents to D and E, respectively, where ∇ denotes the Levi - Civita connection on M. On the other hand , $\nabla \ln y$ is the gradient of $\ln y$ which is defined as $g(\nabla \ln y, U) = U \ln y$. If the warping function y is constant then a warping product manifold $M = D \times_y E$ is called simply Riemannian product or trivial warped product manifold. For a warped product $M = D \times_y E$, D is said to be totally geodesic and D is totally umbilical submanifolds of M, respectively. Now, we obtain some results in the next section.

Proposition 4.1. *If $M = M_\perp \times_y M_\theta$ is a warped product pseudo slant submanfold of a nearly Lorentzian para Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to $M_\perp$, then for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$*

$$\begin{aligned} 2g(h(U, W), srU) &= -(W \ln y)\cos^2\theta \parallel U \parallel^2 \\ &+ g(h(U, rU), yW) - g(h(W, PU), sU). \end{aligned}$$

Proof. Suppose $M = M_\perp \times_y M_\theta$ be a warped product pseudo slant submanifold of a nearly Lorentzian para Sasakian manifold $\tilde{M}$. Using (2.6) we get

$$g(\tilde{\nabla}_W U, srU) = g(h(U, W), srU),$$

ξ is tangent to $M_\perp$ then use (2.9), prop. 1 and (4.1) we have and using prop. of nearly Lorentzian para Sasakian manifold we have after some straight forward calculations, we have

$$g(h(U,W),srU) = g(f\tilde{\nabla}_W U, rU) - \cos^2\theta(W\ln y) \parallel U \parallel^2,$$

and then

$$\begin{aligned} g(h(U,W),srU) &= g(rU,\tilde{\nabla}_W PU) - g(FU,\tilde{\nabla}_W PU) \\ &\quad + g(sU,(\tilde{\nabla}_U y)W) - \cos^2\theta(W\ln y) \parallel U \parallel^2 . \end{aligned}$$

Now using (2.1), (2.5), (2.6), (2.7), (2.9), (2.14) and (4.1) after a few step calculation we arrive at

$$\begin{aligned} g(h(U,W),srU) &= -g(A_{yW}U,sU) - g(h(U,W),srU) \\ &\quad - g(h(W,rU),sU) - \cos^2\theta(W\ln y) \parallel U \parallel^2 . \end{aligned}$$

Which finally gives

$$\begin{aligned} 2g(h(U,W),srU) &= -g(h(rU,U),yW) - g(h(W,rU),sU) \\ &\quad -(W\ln y)\cos^2\theta \parallel U \parallel^2 . \end{aligned}$$

Which completes the proof. □

Proposition 4.2. *Assume that $M_\perp \times_y M_\theta$ is a warped product pseudo slant submanifold of a nearly Lorentzian para Sasakian manifold $\tilde{M}$ then,*

$$g(h(U,W),srU) = g(h(W,rU),sU).$$

for any $U \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. Using (2.6), (2.9), we get

$$g(h(W,rU),sU) = g(\tilde{\nabla}_W rU, yU) + g((\tilde{\nabla}_W)rU, rU).$$

Now using (2.11), (2.12) and (4.1) and properties of nearly Lorentzian para Sasakian manifold we have after performing a few steps

$$\begin{aligned} g(h(W,rU),sU) &= -g(h(U,W),srU) + g(\tilde{\nabla}_{rU}y)W,U) \\ &\quad -2(W\ln y)\cos^2\theta \parallel U \parallel^2 . \end{aligned}$$

Now using covariant derivative, nearly Lorentzian para Sasakian manifold properties (2.6), (2.7), (2.8), (2.9) and (4.1) and after some simple calculating steps we arrive at

$$\begin{aligned} 2g(h(W,rU),sU) &= -g(h(U,W),srU) - g(h(rU,U),yW) \\ &\quad -3(W\ln y)\cos^2\theta \parallel U \parallel^2 \end{aligned} \tag{4.2}$$

Now interchange U by rX and using (4.1), (2.11) we get ,

$$(4.3)\ 2g(h(W,U),srU) = -g(h(U,rU),yW) - g(h(W,rX),sX) -3(W\ln y)cos^2\theta \parallel U \parallel^2 .$$

Now adding above two equations, we get finally

$$2g(h(W,rU),sU) = 2g(h(W,U),srU) +g(h(W,U),srU) - g(h(rX,W),sU)$$

which gives

$$g(h(W,rU),sU) = g(h(rU,W),sU)$$

Which completes the proof . □

Proposition 4.3. *In a warped product pseudo slant submanifold* $M = M_\perp \times_y M_\theta$ *of a nearly Lorentzian para Sasakian manifold* $\tilde{M}$*, we have*

$$g(h(U,rU),fW) = -3g(h(U,W),srU) - (W\ln y)\cos^2\theta \parallel U \parallel^2,$$

for any $U \in \Gamma(TM_\theta)$ *and* $W \in \Gamma(TM_\perp)$.

Proof. From prop. 4.1 and prop. 4.2, we have

$$2g(h(U,W),srU) = 2g(h(rU,U),fW) - g(h(W,rU),sU) -(W\ln y)\cos^2\theta \parallel U \parallel^2,$$

and

$$g(h(U,W),srU) = g(h(W,rU),sU),$$

now adding above two equations we have after a small calculation,

$$g(h(rU,U),fW) = -(W\ln y)\cos^2\theta \parallel U \parallel^2 -3g(h(U,W),srU).$$

□

Which proves the proposition .

Theorem 4.4. *If* $\tilde{M}$ *is a nearly Lorentzian para Sasakian manifold and* M *be a proper pseudo - slant submanifold of* $\tilde{M}$*, then* $M = M_\perp \times_y M_\theta$ *is locally warped product of proper slant and anti-invariant submanifold submanifolds if and only if,*

$$3A_{srU}W + A_{fW}rU = -(W\ln y)\cos^2\theta U.$$

for any $W \in \Gamma(D^\perp \oplus \xi)$ *and* $U \in \Gamma(D^\theta)$.

Proof. From prop. 4.3, we have

$$g(h(U,rU),fW) = -3g(h(U,W),srU) - (W\ln y)\cos^2\theta \parallel U \parallel^2,$$

hence we have

$$g(A_{fW}rU,U) = -3g(A_{srU}W,U) - (W\ln y)\cos^2\theta g(U,U),$$

its ultimately give

$$3A_{srU}W + A_{fW}rU = -(W\ln y)\cos^2\theta U.$$

Which proves the necessary part. Now conversely if M be a proper pseudo slant submanifold in a nearly Lorentzian para Sasakian manifold $\tilde{M}$ with above proved equation holds, then taking the inner product with W and use the fact that U,C are orthogonal then,

$$3g(A_{srU}W,C) = -g(A_{fW}rU,C).$$

Using (2.8) we have

$$3g(h(W,C),srU) = -g(h(rU,C),fW).$$

Interchanging W by C we have ,

$$3g(h(W,C),srU) = -g(h(rU,W),fW).$$

Now adding above two equation we have ,

$$6g(h(W,C),srU) = -[g(h(rU,W),fW) + g(h(rU,W),fW)].$$

Now

$$g(h^\theta(U,V),W) = g(\tilde{\nabla}_U V,W) = g(f\tilde{\nabla}_U V,fW) - \eta(W)g(\xi,\tilde{\nabla}_U V)$$

Using covariant derivative (2.6), (2.8), (2.9), (2.11), (2.14) and U,V are orthogonal to ξ and after a few steps calculations we have

$$\begin{aligned} g(h^\theta(U,V),W) &= -g(A_{fV}rW,U) + \sin^2\theta g(\tilde{\nabla}_U W,V) \\ &\quad - \cos^2\theta\eta(W)g(V,\tilde{\nabla}_U\xi) - g(A_{srW}U,V). \end{aligned}$$

Which again gives

$$\begin{aligned} (1-\sin^2\theta)g(h^\theta(U,V),W) &= -g(A_{fV}rW,U) - g(A_{srW}U,V) \\ &\quad - \cos^2\theta\eta(W)g(V,\tilde{\nabla}_U\xi), \end{aligned}$$

hence,

$$\begin{aligned} (1-\sin^2\theta)g(h^\theta(U,V),W) &= -g(A_{srU}W,V) + g(A_{fW}rW,V) \\ &\quad - \cos^2\theta\eta(W)g(V,\tilde{\nabla}_U\xi) \end{aligned}$$

,

Now using the given condition and performing a few step calculation ,

$$\begin{aligned}(1-\sin^2\theta)g(h^\theta(U,V),W) &= g(2A_{srU}W-(W\ln y)\cos^2\theta U,V)\\ &\quad -\cos^2\theta\eta(W)g(V,\tilde{\nabla}_U\xi),\end{aligned}$$

hence,

$$g(h^\theta(U,V),W)=\frac{2}{\cos^2\theta}g(A_{srX}W,V)-(W\ln y)g(U,V)-\eta(W)g(V,\tilde{\nabla}_U\xi),$$

interchanging U and V we have from above ,

$$g(h^\theta(U,V),W)=\frac{2}{\cos^2\theta}g(A_{srV}W,U)-(W\ln y)g(U,V)-\eta(W)g(U,\tilde{\nabla}_V\xi),$$

now adding above two equation we get,

$$\begin{aligned}g(h^\theta(U,V),W) &= \frac{1}{cos^2\theta}g(A_{srU}W,V)+\frac{1}{cos^2\theta}g(A_{srV}W,U)\\ &\quad -(W\ln y)g(U,V)-\eta(W)g(U,\tilde{\nabla}_V\xi)\\ &= -(W\ln y)g(U,V)-\eta(W)g(U,\tilde{\nabla}_V\xi),\end{aligned}$$

compairing we have,

$$h^\theta(U,V)=-\nabla\lambda g(U,V)-g(U,\tilde{\nabla}_V\xi)\xi,$$

now taking inner product with fW we have ,

$$g(h^\theta(U,V),fW)=-g(U,V)g(\nabla\lambda,fW),$$

compairing we get,

$$h^\theta(U,V)=-g(U,V)\nabla\lambda,$$

this implies,

$$H^\theta=-\nabla\lambda,$$

is the mean curvature vector of M. □

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