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NUMERICAL SOLUTION OF SOME CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS BY USING LEGENDRE-BERNSTEIN BASIS

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ABSTRACT. In this article, a numerical method is developed to solve the linear integro-differential equations. To this end, it will be divided in two forms, Fredholm integro-differential equations (FIDE) and Volterra integro-differential equations (VIDE). So that, the kernel and other known functions have been approximated using the least-squares approximation schemes based on Legender-Bernstein basis. The Legender polynomials are orthogonal and this property improve the accuracy of the approximations. Also the unknown function and its derivatives have been approximated by using the Bernstein basis. The useful properties of Bernstein polynomials help us to transform integro-differential equations to solve a system of linear algebraic equations. Of course, the solution way of (FIDE) case is different from (VIDE).

Key Words: Linear integro-differential equations, Fredholm integral equations, Volterra integral equations, Bernstein basis, Legendre basis, Orthogonal polynomials.
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1. Introduction

As mentioned, in this paper linear integro-differential equations are considered in two forms, Fredholm integro-differential equations (FIDE)

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and Volterra integro-differential equations (VIDE), respectively by the general forms

$$
\begin{aligned}
(1.1)\ \sum_{i=0}^{L} \varphi_i(s) g^{(i)}(s) &= f(s) + \lambda \int_0^1 k(s,t) g(t) dt, \\
(1.2)\ \sum_{i=0}^{L} \varphi_i(s) g^{(i)}(s) &= f(s) + \lambda \int_0^s k(s,t) g(t) dt \quad ; \quad 0 \le s \le 1,
\end{aligned}
$$

under the mixed conditions

$$g^{(i)}(0) = b_i \quad ; \quad i = 0, 1, \cdots (L-1),$$

where the parameter λ and functions $f(s), k(s,t)$ and $\varphi_i(s)$, $\{i = 0, 1, \cdots, L\}$, are known and $g(s)$ and so its derivatives are unknown functions. Also has assumed that all of these functions are L_2-Functions on $[0,1]$, and $g(s) \in C^{L+1}[0,1]$. The Bernstein form of a polynomial offers valuable insight into its geometrical behavior, and has thus won widespread acceptance as the basis for Bézier curves and surfaces. For least-squares approximation problems, on the other hand, the use of orthogonal bases, such as the Legendre polynomials [2, 3], permits simple and efficient constructions for convergent sequences of approximants.

In the following we'll introduce the Legendre and Bernstein polynomials and some properties of them that have been used in this article.

1.1. **Legendre polynomials.** To emphasize symmetry properties of Legendre polynomials, they are traditionally defined on the interval $[-1,+1]$, but for our purposes it is preferable to map this to $[0,1]$. The Legendre polynomials $L_k(u)$ on $u \in [0,1]$, can be generated through the recurrence relation

$$(1.3)\quad (k+1)L_{k+1}(u) = (2k+1)(2u-1)L_k(u)kL_{k1}(u) \quad ; \quad k = 1, 2, \cdots,$$

commencing with $L_0(u) = 1$ and $L_1(u) = 2u - 1$.
This gives, in the first few instances

$$\begin{array}{rcl} L_0(u) &=& 1, \\ L_1(u) &=& 2u-1, \\ L_2(u) &=& 6u^2-6u+1, \\ L_3(u) &=& 20u^3-30u^2+12u-1, \\ &\vdots& . \end{array}$$

The orthogonality of these polynomials is expressed by the relation

$$\int_0^1 L_j(u)L_k(u)du = \begin{cases} \frac{1}{2k+1} & j=k \\ 0 & j \neq k \end{cases} .$$

Now for arbitrary function $f(u)$ on $[0,1]$, we can express it in the Legendre form,

$$f(u) \simeq P_N(u) = \sum_{j=0}^{N} l_j L_j(u), \tag{1.4}$$

where the coefficients l_j, for Legendre polynomials are obtained from following relation

$$l_k = (2k+1)\int_0^1 L_k(u)f(u)du \quad ; \quad k = 0, 1, \cdots, N. \tag{1.5}$$

1.2. **Bernstein polynomials.** $(N+1)$-Bernstein basic function on $[0,1]$, are defined by using the following relation

$$B_{i,N}(u) = \binom{N}{i} u^i (1-u)^{N-i} \quad ; \quad i = 0, 1, \cdots, N. \tag{1.6}$$

In the follow, some properties of Bernstein polynomials have been expressed that in this article have been used of them ,

- The product of a power basic function and a Bernstein basic function,

$$u^m B_{i,N}(u) = \frac{\binom{N}{i}}{\binom{N+m}{i+m}} B_{i+m,N+m}(u). \tag{1.7}$$

- The product of two Bernstein basic functions,

$$B_{i,j}(u)B_{k,m}(u) = \frac{\binom{j}{i}\binom{m}{k}}{\binom{j+m}{i+k}} B_{i+k,j+m}(u). \tag{1.8}$$

- The expression of power basic functions in the Bernstein form and vice versa,

$$B_{k,N}(u)=\sum_{i=k}^{N}(-1)^{i-k}\binom{N}{i}\binom{i}{k}u^{i}. \tag{1.9}$$

Let $B_s^t=[B_{0,N}(s),B_{1,N}(s),\cdots,B_{N,N}(s)]$ and $S^t=[1,s,s^2,\cdots,s^N]$ then

$$B_s=MS \quad \text{and} \quad S=M^{-1}B_s, \tag{1.10}$$

where

$$M=\begin{bmatrix} (-1)^0\binom{N}{0}\binom{0}{0} & (-1)^1\binom{N}{1}\binom{1}{0} & \cdots & (-1)^N\binom{N}{N}\binom{N}{0} \\ 0 & (-1)^0\binom{N}{1}\binom{1}{1} & \cdots & (-1)^{N-1}\binom{N}{N}\binom{N}{1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^0\binom{N}{N}\binom{N}{N} \end{bmatrix}. \tag{1.11}$$

- All the basis functions have the same definite integral over $[0,1]$, namely

$$\int_0^1 B_{i,N}(u)du=\frac{1}{N+1} \quad ; \quad i=0,1,\cdots,N. \tag{1.12}$$

Therefore by (1.8),(1.12) produced matrix from the integration over the product of two bases in form $T=\int_0^1 B_sB_s^tds$, can be obtained. That T is a $(N+1)\times(N+1)$ matrix by elements in the following forms,

$$T_{i+1,j+1}=\frac{\binom{N}{i}\binom{N}{j}}{(2N+1)\binom{2N}{i+j}} \quad ; \quad i,j=0,1,\cdots,N. \tag{1.13}$$

Also, if $A^t=[a_0,a_1,\cdots,a_N]$, is a known vector of order $(N+1)$, then $B_sB_s^tA$, can be written again in the Bernstein form. To this end, by using the (1.8) and (1.10), we have

$$\begin{aligned} B_sB_s^tA &= M\tau\left(\sum_{k=0}^{N}a_kB_{k,N}(s)\right) \\ &= M\begin{bmatrix} \sum_{k=0}^{N}a_kB_{k,N}(s) \\ \sum_{k=0}^{N}a_ksB_{k,N}(s) \\ \vdots \\ \sum_{k=0}^{N}a_ks^NB_{k,N}(s) \end{bmatrix}. \end{aligned} \tag{1.14}$$

Now, we approximate all functions $s^j B_{k,N}(s)$ in terms of B_s. Namely

$$s^j B_{k,N}(s) \simeq B_s^t e_{j,k} \quad ; \quad j,k = 0,1,\cdots,N, \tag{1.15}$$

where $e_{j,k}$, is a approximation coefficients vector as follows

$$e_{j,k} = \begin{bmatrix} e_0^{j,k} \\ e_1^{j,k} \\ \vdots \\ e_N^{j,k} \end{bmatrix}. \tag{1.16}$$

By multiplying B_s, in both sides of (1.15), and integration of them, and by using of (1.13), we have

$$\begin{aligned} e_{j,k} &= T^{-1}\int_0^1 s^j B_{k,N}(s) B_s ds \\ &= T^{-1}\begin{bmatrix} \int_0^1 s^j B_{k,N}(s)B_{0,N}(s)ds \\ \int_0^1 s^j B_{k,N}(s)B_{1,N}(s)ds \\ \vdots \\ \int_0^1 s^j B_{k,N}(s)B_{N,N}(s)ds \end{bmatrix} = \frac{T^{-1}\binom{N}{k}}{2N+j+1}\begin{bmatrix} \frac{\binom{N}{0}}{\binom{2N+j}{k+j}} \\ \frac{\binom{N}{1}}{\binom{2N+j}{k+j+1}} \\ \vdots \\ \frac{\binom{N}{N}}{\binom{2N+j}{k+j+N}} \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=0}^N a_k s^j B_{k,N}(s) &\simeq \sum_{k=0}^N a_k B_s^t e_{j,k} = \sum_{k=0}^N a_k \left(\sum_{i=0}^N e_i^{j,k} B_{i,N}(s)\right) \\ &= \sum_{i=0}^N B_{i,N}(s)\left(\sum_{k=0}^N a_k e_i^{j,k}\right) = \begin{bmatrix} \sum_{k=0}^N a_k e_0^{j,k} \\ \sum_{k=0}^N a_k e_1^{j,k} \\ \vdots \\ \sum_{k=0}^N a_k e_N^{j,k} \end{bmatrix} B_s \\ &= A^t \begin{bmatrix} e_{j,0}^t \\ e_{j,1}^t \\ \vdots \\ e_{j,N}^t \end{bmatrix} B_s = A^t E_{j+1} B_s, \end{aligned}$$

that E_{j+1} is a $(N+1)\times(N+1)$ matrix that,it has vectors $e_{j,k}^t, \quad j=0,1,\cdots,N$, for each row.Therefore we define $\widehat{E_{j+1}} =$

$A^t E_{j+1}$ for $j = 0, 1, \cdots, N$. So

$$\sum_{k=0}^{N} a_k s^j B_{k,N}(s) \simeq \widehat{E_{j+1}} B_s \quad ; \quad j = 0, 1, \cdots, N. \tag{1.17}$$

Now by substituting (1.17), into (1.14), we have

$$B_s B_s^t A = M \begin{bmatrix} \widehat{E_1} B_s \\ \widehat{E_2} B_s \\ \vdots \\ \widehat{E_{N+1}} B_s \end{bmatrix}. \tag{1.18}$$

If we define matrix G_A as follows

$$G_A = \begin{bmatrix} \widehat{E_1} \\ \widehat{E_2} \\ \vdots \\ \widehat{E_{N+1}} \end{bmatrix},$$

that G_A is a $(N+1) \times (N+1)$ matrix that,it has vectors $\widehat{E_{j+1}}, \quad j = 0, 1, \cdots, N$, for each row.Therefore we can write

$$B_s B_s^t A = M G_A B_s, \tag{1.19}$$

- Operational matrix of integration
  Let $B_t^t = [B_{0,N}(t), B_{1,N}(t), \cdots, B_{N,N}(t)]$, and $\tau^t = [1, t, t^2, \cdots, t^N]$, then the integration of vector B_t is given by

$$\int_0^s B_t dt \simeq P B_s, \tag{1.20}$$

where P is the $(N+1)\times(N+1)$ operational matrix for integration and is given in [4]. By using of (1.11), we have

(1.21)

$$\int_0^s B_t dt = \int_0^s M \tau dt = M \int_0^s \tau dt = M \begin{bmatrix} s \\ \frac{1}{2} s^2 \\ \vdots \\ \frac{1}{N+1} s^{N+1} \end{bmatrix} = M M_p S_p,$$

where$S_p^t = [s, s^2, \cdots, s^{N+1}]$, and M_p is the following matrix

$$M_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \tag{1.22}$$

According to (1.11), we had $S = M^{-1}B_s$. Therefore for $k = 0, 1, \cdots, N$, we have

$$s^k = M^{-1}_{[k+1]}B_s, \tag{1.23}$$

where $M^{-1}_{[k+1]}$ is $(k+1)$-th row of M^{-1} for $k = 0, 1, \cdots, N$. We just need to approximate
$s^{N+1} \simeq B_s^t C_{N+1}$. By product both sides of it at B_s and integration on $[0, 1]$, we have

$$\begin{aligned} C_{N+1} &= T^{-1}\int_0^1 s^{N+1}B_s ds \\ &= T^{-1}\begin{bmatrix} \int_0^1 s^{N+1}B_{0,N}(s)ds \\ \int_0^1 s^{N+1}B_{1,N}(s)ds \\ \vdots \\ \int_0^1 s^{N+1}B_{N,N}(s)ds \end{bmatrix} = \frac{T^{-1}}{2N+2}\begin{bmatrix} \frac{\binom{N}{0}}{\binom{2N+1}{N+1}} \\ \frac{\binom{N}{1}}{\binom{2N+1}{N+2}} \\ \vdots \\ \frac{\binom{N}{N}}{\binom{2N+1}{2N+1}} \end{bmatrix}. \end{aligned} \tag{1.24}$$

Now assume

$$B = \begin{bmatrix} M^{-1}_{[2]} \\ M^{-1}_{[3]} \\ \vdots \\ M^{-1}_{[N+1]} \\ C^t_{N+1} \end{bmatrix}, \tag{1.25}$$

then $S_p \simeq BB_s$. Therefore we have the operational matrix of integration $P = MM_pB$.

1.3. **The expression of the Legendre polynomials in the Bernstein form.** In this scale, we expand a favorite polynomial such as $P_N(s)$ in terms of Legendre-Bernstein basis. That is, we combine two bases Legendre and Bernstein, and then calculate expansion coefficients. The Legendre polynomials $L_k(s)$ can be expressed in the Bernstein basis B_s of degree N as

$$L_k(s) = \sum_{j=0}^{N} \Lambda_{k,j} B_{j,N}(s) \quad ; \quad k = 0, 1, \cdots, N, \tag{1.26}$$

where [1],
(1.27)

$$\Lambda_{k,j} = \frac{1}{\binom{N}{j}} \sum_{i=max(0,j+k-N)}^{min(j,k)} (-1)^{k+i} \binom{k}{i} \binom{k}{i} \binom{N-k}{j-i} \quad ; \quad j, k = 0, 1, \cdots, N.$$

Now consider the polynomial $P_N(s)$ of degree N, as expressed in (1.4), we can transform it in the Bernstein form as

$$P_N(s) = \sum_{k=0}^{N} l_k L_k(s) = \sum_{k=0}^{N} l_k \left(\sum_{j=0}^{N} \Lambda_{k,j} B_{j,N}(s) \right) = \sum_{j=0}^{N} b_j B_{j,N}(s),$$

that by (1.5) and (1.26), we have

$$\begin{aligned} l_k &= \frac{\langle f(s), L_k(s) \rangle}{\langle L_k(s), L_k(s) \rangle} \\ &= (2k+1) \int_0^1 f(s) L_k(s) ds = (2k+1) \int_0^1 f(s) \left(\sum_{j=0}^{N} \Lambda_{k,j} B_{j,N}(s) \right) ds \\ &= (2k+1) \sum_{j=0}^{N} \Lambda_{k,j} \int_0^1 f(s) B_{j,N}(s) ds \quad ; \quad k = 0, 1, \cdots, N, \end{aligned}$$

where

$$b_j = \sum_{k=0}^{N} l_k \Lambda_{k,j} \quad ; \quad j, k = 0, 1, \cdots, N \qquad or \qquad b = l^t \Lambda.$$

That b_j are expansion coefficients of $P_N(s)$, in terms of Legendre-Bernstein basis. Similarly, we can calculate expansion coefficients of least squares approximation of kernel $k(s,t)$, based on Legendre-Bernstein basis. Let

$L_s^t = [L_0(s), L_1(s), \cdots, L_N(s)]$, then for $k(s,t)$ we have

$$\begin{aligned}
k(s,t) &= L_s^t K L_t \\
&= \sum_{m=0}^{N}\sum_{n=0}^{N} L_m(s) k_{m,n} L_n(t) \\
&= \sum_{m=0}^{N}\sum_{n=0}^{N} \left(\sum_{i=0}^{N} \Lambda_{m,i} B_{i,N}(s)\right) k_{m,n} \left(\sum_{j=0}^{N} \Lambda_{n,j} B_{j,N}(t)\right) \\
&= \sum_{i=0}^{N}\sum_{j=0}^{N} B_{i,N}(s) \left(\sum_{m=0}^{N}\sum_{n=0}^{N} \Lambda_{m,i} k_{m,n} \Lambda_{n,j}\right) B_{j,N}(t),
\end{aligned}$$

where

$$\begin{aligned}
k_{m,n} &= \frac{\langle\langle k(s,t), L_n(t)\rangle, L_m(s)\rangle}{\langle L_n(t), L_n(t)\rangle \langle L_m(s), L_m(s)\rangle} \\
&= (2n+1)(2m+1)\int_0^1\int_0^1 L_m(s)L_n(t)k(s,t)dtds \\
&= (2n+1)(2m+1)\sum_{i=0}^{N}\sum_{j=0}^{N} \Lambda_{m,i}\Lambda_{n,j}\int_0^1\int_0^1 B_{i,N}(s)B_{j,N}(t)k(s,t)dtds \\
&\quad ; \quad i,j = 0,1,\cdots,N.
\end{aligned}$$

Let

$$C_{i,j} = \sum_{m=0}^{N}\sum_{n=0}^{N} \Lambda_{m,i} k_{m,n} \Lambda_{n,j} \quad ; \quad i,j = 0,1,\cdots,N, \tag{1.28}$$

or

$$C = \Lambda^t K \Lambda. \tag{1.29}$$

Then

$$k(s,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} B_{i,N}(s) C_{i,j} B_{j,N}(t) = B_s^t C B_t. \tag{1.30}$$

2. Approximation of Fredholm integro-differential equations (FIDE)

Consider the equation (1.1), as follows

$$\sum_{i=0}^{L} \varphi_i(s) g^{(i)}(s) = f(s) + \lambda \int_0^1 k(s,t)g(t)dt, \tag{2.1}$$

under the mixed conditions

$$g^{(i)}(0) = b_i \quad ; \quad i = 0, 1, \cdots (L-1).$$

Let the least-squares approximation for $f(s)$ and $\varphi_i(s)$ in Legendre-Bernstein basis as follows,

$$f(s) = B_s^t F \quad and \quad \varphi_i(s) = q_i^t B_s \quad ; \quad , i = 0, 1, \cdots, L, \tag{2.2}$$

also, we approximate $g^{(L)}(s)$, by Bernstein basis as $g^{(L)}(s) = B_s^t A$, where $A^t = [a_0, a_1, \cdots, a_N]$. Then, by integration of $g^{(L)}(s)$ on $[0, s]$ and considering the mixed conditions, we can write

$$\begin{aligned}
g^{(L)}(s) &= B_s^t A \\
g^{(L-1)}(s) &= \int_0^s B_s^t A ds = \int_0^s B_s^t ds A = B_s^t P^t A + b_{L-1} \\
g^{(L-2)}(s) &= B_s^t (P^t)^2 A + b_{L-1} s + b_{L-2} \\
g^{(L-3)}(s) &= B_s^t (P^t)^3 A + b_{L-1} \frac{s^2}{2!} + b_{L-2} s + b_{L-3} \\
\vdots &= \vdots \\
g^{(1)}(s) &= B_s^t (P^t)^{L-1} A + b_{L-1} \frac{s^{L-2}}{(L-2)!} + \cdots + b_3 \frac{s^2}{2!} + b_2 s + b_1 \\
g(s) &= B_s^t (P^t)^L A + b_{L-1} \frac{s^{L-1}}{(L-1)!} + \cdots + b_2 \frac{s^2}{2!} + b_1 s + b_0.
\end{aligned}$$

No, by (1.10)and (1.11), we can write

$$\begin{aligned}
g^{(L)}(s) &= B_s^t A \\
g^{(L-1)}(s) &= B_s^t \left(P^t A + b_{L-1} d_0\right) \\
g^{(L-2)}(s) &= B_s^t \left((P^t)^2 A + b_{L-1} d_1 + b_{L-2} d_0\right) \\
g^{(L-3)}(s) &= B_s^t \left((P^t)^3 A + \frac{b_{L-1}}{2!} d_2 + b_{L-2} d_1 + b_{L-3} d_0\right) \\
\vdots &= \vdots \\
g^{(1)}(s) &= B_s^t \left((P^t)^{L-1} A + \frac{b_{L-1}}{(L-2)!} d_{L-2} + \cdots + \frac{b_3}{2!} d_2 + b_2 d_1 + b_1 d_0\right) \\
g(s) &= B_s^t \left((P^t)^L A + \frac{b_{L-1}}{(L-1)!} d_{L-1} + \cdots + \frac{b_2}{2!} d_2 + b_1 d_1 + b_0 d_0\right),
\end{aligned} \tag{2.3}$$

where d_i^t, is i-th row of M^{-1}. By defining $R_L = O_{(N+1)\times 1}$ and by setting $R_{L-k} = \sum_{j=1}^{k} \frac{b_{L-j}}{(k-j)!} d_{k-j} \quad ; \quad i = 1, 2, \cdots L$, and by (1.30), (2.2) and (2.3) the equation (2.1), can be written as

$$\begin{aligned}\sum_{i=0}^{L} q_i^t B_s B_s^t((P^t)^{L-i}A + R_i) &= B_s^t F + \lambda B_s^t C \int_0^1 B_t B_t^t((P^t)^L A + R_0)dt \\ &= B_s^t F + \lambda B_s^t CT((P^t)^L A + R_0).\end{aligned}$$

But by using (1.19), we can write

$$\sum_{i=0}^{L} B_s^t G_{q_i}^t M^t((P^t)^{L-i}A + R_i) = B_s^t F + \lambda B_s^t CT((P^t)^L A + R_0),$$

then

$$\sum_{i=0}^{L} G_{q_i}^t M^t((P^t)^{L-i}A + R_i) = F + \lambda CT((P^t)^L A + R_0),$$

or

$$\left(\sum_{i=0}^{L} G_{q_i}^t M^t (P^t)^{L-i} - \lambda CT(P^t)^L\right) A = F + \lambda CTR_0 - \sum_{i=0}^{L} G_{q_i}^t M^t R_i.$$

After determining A, as

$$A = \left(\sum_{i=0}^{L} G_{q_i}^t M^t (P^t)^{L-i} - \lambda CT(P^t)^L\right)^{-1} \left(F + \lambda CTR_0 - \sum_{i=0}^{L} G_{q_i}^t M^t R_i\right),$$

the unknown function $g(s)$, can be determined as

$$g(s) = B_s^t((P^t)^L A + R_0).$$

3. Approximation of Volterra integro-differential equations (VIDE)

Consider the equation (1.2), as follows

$$\sum_{i=0}^{L} \varphi_i(s) g^{(i)}(s) = f(s) + \lambda \int_0^s k(s,t) g(t) dt, \tag{3.1}$$

under the mixed conditions

$$g^{(i)}(0) = b_i \quad ; \quad i = 0, 1, \cdots (L-1).$$

By defining $\{Q_i = (P^t)^{L-i}A + R_i \quad ; \quad i = 0, 1, \cdots L\}$ such as (FIDE) kind, we have

$$\sum_{i=0}^{L} B_s^t G_{q_i}^t M^t Q_i = f(s) + \lambda B_s^t C \int_0^s B_t B_t^t Q_0 dt,$$

by using of (1.19) and (1.20), we can write

$$\begin{aligned} \sum_{i=0}^{L} B_s^t G_{q_i}^t M^t Q_i &= f(s) + \lambda B_s^t C M G_{Q_0} \int_0^s B_t dt \\ &= f(s) + \lambda B_s^t C M G_{Q_0} P B_s. \end{aligned} \tag{3.2}$$

So by putting nodes $\{s_i = \frac{i}{N} \mid i = 0, 1, \cdots, N\}$ in (3.2), we get a system of linear algebraic equations of $(N+1) \times (N+1)$ degree, with unknown coefficients $\{a_i \mid i = 0, 1, \cdots N\}$. After solving this linear system, we can approximate the solution of equation (3.1), as follows

$$g(s) = B_s^t Q_0. \tag{3.3}$$

3.1. **Error bound for approximation.** The Bernstein polynomials can be expressed in terms of some orthogonal polynomials, such as Chebychev polynomials $\chi_N(x)$ of second kind [5, 6]. It can be shown that

$$B_{i,N}(x) = \frac{1}{2^N}\binom{N}{i}\sum_{j=0}^{N} d_j^{i,N}\frac{1}{2^j}\sum_{m=0}^{[\frac{j}{2}]}\left(\binom{j}{m} - \binom{j}{m+1}\right)\chi_{j-2m}(x),$$

where

$$d_j^{i,N} = \sum_k (-1)^{j-k}\binom{i}{k}\binom{N-i}{j-k}.$$

Expand $f(x)$ in the approximated form of Bernstein polynomials

$$f(x) \simeq P_N(x) = \sum_{i=0}^{N} a_i B_{i,N}(x).$$

Thus, it is eventually expressed as

$$P_N(x) = \sum_{j=0}^{N} b_j \chi_j(x),$$

where b_j can be expressed in terms of a_i; $\quad i, j = 0, 1, \cdots, N$. If $u_j(x) = \sqrt{\frac{2}{\pi}}\chi_j(x)$, then $u_j(x), j = 0, 1, \cdots, N$, form an orthogonal polynomial basis in $[-1, 1]$ with respect to weight function $\omega(x) = (1 - x^2)^{\frac{1}{2}}$, that can be mapped to $[0, 1]$. Therefor, this procedure yields

$$P_N(x) = \sum_{j=0}^{N} \sqrt{\frac{\pi}{2}} b_j u_j(x),$$

Golberg and Chen [7], proved that when a continuously differentiable function ($f \in C^r, r > 0$) is approximated by Chebychev polynomials, then

$$\|f - P_N\|_\infty < c_0 N^{-r}, \tag{3.4}$$

where c_0 is some constant. Now we find error bound for (VIDE) and so, for (FIDE) kind is as the same. Assume $P_N(s)$ and $g(s)$ be approximate and exact solutions of the equation (3.1), respectively, so

$$\sum_{i=0}^{L} \varphi_i(s) P_N^{(i)}(s) - \lambda \int_0^s k(s,t) P_N^{(0)}(t) dt = f(s) + R_N(s), \tag{3.5}$$

where $R_N(s)$ is the perturbation function that depends only on $P_N(s)$, and $P_N^{(i)}(s)$; $\quad i = 0, 1, \cdots, L$, are i-th derivative of the $P_N(s)$. As previously mentioned $g(s) \in C^{L+1}[0, 1]$ and by (3.4), we can write

$$\|g^{(i)}(s) - P_N^{(i)}(s)\|_\infty < c_i N^{-(L+1)+i} \quad ; \quad i = 0, 1, \cdots, L. \tag{3.6}$$

Let $M \equiv \sup_{0 \leq s,t \leq 1} |k(s,t)| < \infty$ and $\phi = \sup_{0 \leq s \leq 1} |\varphi_i(s)|$. By subtracting equation (3.5), from equation (3.1), we have

$$|R_N(s)| \leq \sum_{i=0}^{L} \phi c_i N^{-(L+1)+i} + |\lambda| M c_0 N^{-(L+1)},$$

Let $c = \sup |c_i|$; $\quad i = 0, 1, \cdots, L$, then

$$\begin{aligned} |R_N(s)| &\leq \left(\phi c \sum_{i=0}^{L} N^i + |\lambda| M c \right) N^{-(L+1)} \\ &= \left(\phi c (\frac{1 - N^{L+1}}{1 - N}) + |\lambda| M c \right) N^{-(L+1)}, \end{aligned}$$

so, an error bound obtained for the perturbation function $R_N(s)$ such as

$$|R_N(s)| \leq \left(\phi c (\frac{N^{-(L+1)} - 1}{1 - N}) + |\lambda| M c N^{-(L+1)} \right). \tag{3.7}$$

4. Illustrations

Example 4.1. Consider linear integro-differential equation [8]:

$$g'(s) = e^s + se^s - s + \int_0^1 sg(t)dt \quad ; \quad 0 \leq s, t \leq 1, \tag{4.1}$$

with the initial condition $g(0) = 0$ and the exact solution $g(s) = se^s$. Table 1 shows the numerical results for Example 4.1 in comparison with method of [8].

Table 1: Numerical results of the absolute error functions $E_7(x_i)$ of $g(x)$ for Example 4.1.

$Nodes\ s_i = \frac{i}{10}$	Method of [8] $N = 7$	Present method $N = 7$
0.0		$2.9802322388e-008$
0.1	$2.1789e-008$	$3.7999745711e-009$
0.2	$2.2665e-008$	$5.8854213447e-009$
0.3		$8.7810120286e-009$
0.4	$2.5198e-008$	$5.2150994634e-010$
0.5		$9.7645229680e-009$
0.6	$2.7325e-008$	$5.0493556003e-009$
0.7		$2.8782316974e-008$
0.8	$2.7236e-008$	$3.5657182540e-008$
0.9	$7.6359e-007$	$4.4966471435e-008$
1.0		$2.2943756228e-008$

Example 4.2. Consider linear integro-differential equation [9]:

$$g''(s) + sg'(s) - sg(s) = e^s + \frac{1}{2}s\cos(s) - \frac{1}{2}\int_0^s \cos(s)e^{-t}g(t)dt \quad ; \quad 0 \leq s, t \leq 1, \tag{4.2}$$

with the initial condition $g(0) = 1, g'(0) = 1$ and the exact solution $g(s) = e^s$. Table 2 and Figure 1 shows the numerical results for Example 4.2 in comparison with method of [9].

Table 2: Numerical results for Example 4.2.

$Nodes\ s_i = \frac{i}{10}$	Method of [9] $(N = 7, y_7(x_i))$	Present method $N = 7$	Exact solution
0.0	1.0000000000000	0.999999996780694	1.000000000000000
0.1		1.105170917959630	1.105170918075648
0.2	1.2214027614222	1.221402756883232	1.221402758160170
0.3		1.349858807645092	1.349858807576003
0.4	1.4918247044117	1.491824696834059	1.491824697641270
0.5		1.648721268500642	1.648721270700128
0.6	1.8221188108838	1.822118798699455	1.822118800390509
0.7		2.013752706070358	2.013752707470477
0.8	2.2255409520234	2.225540924921041	2.225540928492468
0.9		2.459603107273732	2.459603111156950
1.0	2.7182815307470	2.718281820338751	2.718281828459046

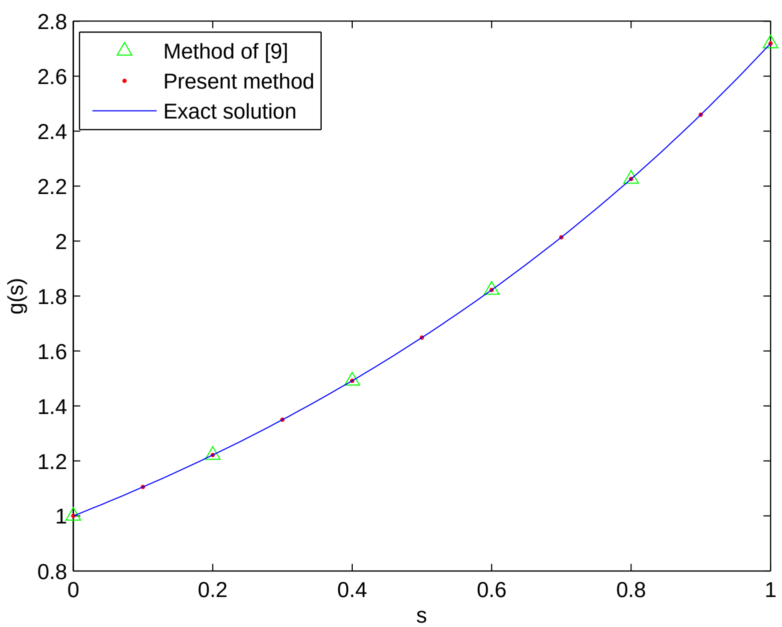

FIGURE 1. Numerical results for Example 4.2.

5. Conclusions

In this paper, solving a linear integro-differential equation became to solve a system of linear equations. To this end, the kernel of integro-differential equation and other known functions have been extended by the least squares approximation of Legendre-Bernstein basis. Also the unknown function and its derivatives have been extended in terms of Bernstein basis. The advantage of this method is that, both characteristics orthogonality of Legendre polynomials and simplification of Bernstein polynomials are used. Thus, we have accuracy and simplicity together.Where, numerical results obtained from the examples show it. So, this basis can be used as a reliable basis for approximation functions.That, its coefficients are easily calculated as, it has been shown in context.

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