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# A STUDY ON EXISTENCE AND GLOBAL ASYMPTOTICAL MITTAG-LEFFLER STABILITY OF FRACTIONAL BLACK-SCHOLES EUROPEAN OPTION PRICING EQUATION

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ABSTRACT. In this paper, the application of asymptotic expansion method on fractional perturbated equations are studied. Furthermore, the proposed scheme is employed to obtain an analytical solution of fractional Black-Scholes equation for a European option pricing problem. Finally, the asymptotical Mittag-Leffler stability of this problem will be discussed.

## 1. INTRODUCTION

In 1973, Fischer Black and Myron Scholes [5] derived the famous theoretical valuation formula for options. The main conceptual idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such an approach strengthens the use of the no-arbitrage principle as well. Thus, the Black-Scholes formula is used as a model for valuing European (the option can be exercised only on a specified future date) or american (the option can be exercised at any time up to the date, the option expires) call and put options on a non-dividend paying stock by Manale and Mahomed [23]. Derivation of a closed-form solution to the Black-Scholes equation depends on the fundamentals solution of the heat equation. Hence, it is important, at this point, to transform the Black-Scholes equation to the heat equation by change of variables. Having found the closed form solution to the heat equation, it is possible to transform it back to find the corresponding solution of the Black-Scholes PDE. Financial models were generally formulated in terms of stochastic differential equations. However, it was

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soon found that under certain restrictions these models could written as linear evolutionary PDEs with variable coefficients by Gazizov and Ibragimov [12]. Thus, the Black-Scholes model for the value of an option is described by the equation

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} + r(t)x\frac{\partial v}{\partial x} - r(t)v = 0, \quad (x,t)\in R^+\times(0,T), \tag{1.1}$$

where $v(x,t)$ is the European call option price at asset price $x$ and at time $t$, $T$ is the maturity, $r(t)$ is the risk free interest rate, and $\sigma(x,t)$ represents the volatility function of underlying asset. Let us denote by $v_c(x,t)$ and $v_p(x,t)$ the value of the European call and put options, respectively. Then, the payoff functions are

$$v_c(x,t) = max(x-E,0), \quad v_p(x,t) = max(E-x,0), \tag{1.2}$$

where $E$ denotes the expiration price for the option and the function $max(x,0)$ gives the larger value between $x$ and $0$. During the past few decades, many researchers studied the existence of solutions of the Black Scholes model using many methods in [1–3, 6–9, 11, 13].

The seeds of fractional calculus (that is, the theory of integrals and derivatives of any arbitrary real or complex order) were planted over 300 years ago. Since then, many researchers have contributed to this field. Recently, it has turned out that differential equations involving derivatives of non-integer order can be adequate models for various physical phenomena Podlubny [28]. The book by Oldham and Spanier [27] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in Miller and Ross [24], Kilbas and Srivastava [20]. For more details see [15–18].

## 2. Some preliminaries in fractional calculus

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper. For more details, see [20, 28].

**Definition 2.1.** The Mittag-Leffler function $E_{\alpha,\beta}(z)$ with $\alpha > 0$, $\beta > 0$ is defined by the following series representation, valid in the whole complex plane [28]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \quad z \in \mathbb{C}. \tag{2.1}$$

For $\beta = 1$ we obtain the Mittag-Leffler function in one parameter:

$$E_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} \equiv E_{\alpha}(z). \tag{2.2}$$

**Definition 2.2.** The Riemann-Liouville integral operator of order $\alpha$ on the usual Lebesgue space $L_1[a,b]$ is defined as [20, 28]:

$$I_t^{\alpha}\xi(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & \alpha > 0,\ t > 0, \\ \xi(t), & \alpha = 0. \end{cases} \tag{2.3}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

**Definition 2.3.** The left sided Riemann-Liouville fractional derivative of order $\alpha$ is defined as [20, 28]:

$$D_t^{\alpha}\xi(t) = \frac{d^m}{dt^m} I_t^{m-\alpha}\xi(t), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.4}$$

**Definition 2.4.** The left sided Caputo fractional derivative of order $\alpha$ is defined as [20, 28]:

$$D_{*t}^{\alpha}\xi(t) = \begin{cases} \left[I_t^{m-\alpha}\xi^{(m)}(t)\right] & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \frac{d^m}{dt^m}\xi(t) & \alpha = m. \end{cases} \tag{2.5}$$

**Definition 2.5.** The Laplace transform of $f(t)$ is defined as follows:

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt. \tag{2.6}$$

**Definition 2.6.** The Laplace transform of Caputo fractional derivative is defined as follows [24]:

$$L[D^{\alpha} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{2.7}$$

**Proposition 2.7.** *We have the following pair of laplace transform for Mittag-Leffler function [28]:*

$$L[t^{\beta-1}E_{\alpha,\beta}(ct^{\alpha})] \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^{\alpha-\beta}}{s^{\alpha}-c}. \tag{2.8}$$

## 3. Asymptotic expansion method for singular perturbated problems

The asymptotic expansion method is a kind of iterative methods for obtaining approximate solution to problems involving a small parameter. The procedure of this method is based on expanding the dependent variable in a power series depending on the small parameter $\epsilon$. One assumes the solution of the form

$$u(t) = \sum_{n=0}^{\infty} \epsilon^n u_n(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots, \tag{3.1}$$

substitute this series into the original equation, expand all of equations and equate the terms corresponding to different power of the small parameter $\epsilon^n$. One considers a part of solution by choosing a value of $n$, say, $N$. In this case, the order of approximation is $O(\epsilon^{N+1})$. Methods for constructing asymptotic expansions are discussed in a number of references (see [4, 14, 19, 26, 29]).
For general $N$ the asymptotic solution is

$$u(t) = \sum_{n=0}^{N} \epsilon^n u_n(t) + O(\epsilon^{N+1}). \tag{3.2}$$

Expansion (3.2) implies that the error of the asymptotic expansion for initial value problems is given by

$$Error = |u_{exact}(t,\epsilon) - u_{asymptotic}(t,\epsilon)| = |u_{exact}(t,\epsilon) - \sum_{n=0}^{N} \epsilon^n u_n(t)| = O(\epsilon^{N+1}). \tag{3.3}$$

In order to elucidate the solution procedure of the fractional asymptotic expansion method, we consider the following fractional differential equation:

$$\epsilon \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) + a \frac{\partial^\beta}{\partial t^\beta} u(x,t) + bu(x,t) = 0. \tag{3.4}$$

We assume that

$$u(x,t) = u_0(x,t) + \epsilon u_1(x,t) + \cdots. \tag{3.5}$$

Substituting (3.5) in (3.4)

$$\epsilon \frac{\partial^\alpha}{\partial t^\alpha}(u_0(x,t) + \epsilon u_1(x,t) + \cdots) + a \frac{\partial^\beta}{\partial t^\beta}(u_0(x,t) + \epsilon u_1(x,t) + \cdots) + b(u_0(x,t) + \epsilon u_1(x,t) + \cdots) = 0. \tag{3.6}$$

Equating coefficients of $\epsilon^0$ and $\epsilon$ to zero yields

$$\frac{\partial^\beta}{\partial t^\beta} u_0(x,t) + bu_0(x,t) = 0, \tag{3.7}$$

and

$$\frac{\partial^\beta}{\partial t^\beta}u_1(x,t)+bu_1(x,t)=-a\frac{\partial^\alpha}{\partial t^\alpha}u_0(x,t). \tag{3.8}$$

The solution of Eq. (3.7) by Laplace transform method is:

$$u_0(x,t)=u_0(x,0)E_\beta(bt^\beta). \tag{3.9}$$

Substituting $u_0(x,t)$ in Eq. (3.8) we can obtain $u_1(x,t)$ by Laplace transform method and from (3.5) we have an asymptotic solution of Eq. (3.4).

## 4. On the stability and convergence of singular perturbated problems

In this section the stability of fractional differential equations of the form

$$\epsilon\frac{d^\alpha u}{dt^\alpha}=g(t,u,\epsilon), \tag{4.1}$$

where $\epsilon$ is a real parameter near zero, is studied. It is shown that if the reduced problem

$$g(t,u,0)=0, \tag{4.2}$$

is stable, and certain other conditions which ensure that the method of asymptotic expansions can be used to construct solutions are satisfied, then the full problem is asymptotically stable as $t\to\infty$.

**Definition 4.1.** The fractional differential equation

$$\frac{d^\alpha u}{dt^\alpha}=g(t,u), \tag{4.3}$$

is asymptotically Mittag-Leffler stable if there exist positive constants $K$ and $\lambda$ such that

$$|u(t)|\le KE_\alpha(-\lambda(t-s)^\alpha), \tag{4.4}$$

for $0\le s\le t<\infty$.

**Theorem 4.2.** *For each small $\epsilon>0$, the initial value problem*

$$\epsilon\frac{d^\alpha u}{dt^\alpha}=g(t,u,\epsilon),\quad u(0)=\xi(\epsilon), \tag{4.5}$$

*has a unique solution $u(t,\epsilon)$ for $0\le t<\infty$, that can be written as*

$$u=u^*(t,\epsilon)+U(t/\epsilon,\epsilon), \tag{4.6}$$

*where $U$ satisfies $U(\infty,\epsilon)=0$ and*

$$u^*(t,\epsilon)=\sum_{r=0}^{N}u_r^*(t)\epsilon^r+O(\epsilon^{N+1}). \tag{4.7}$$

*Finally, the functions $u_r^*$ are determined successively by solving the differential equations*

$$g(t,u_0^*,0)=0, \tag{4.8}$$

*and for* $r = 1, \cdots, N$,

$$\frac{du^*_{r-1}}{dt} = g(t, u^*_r, 0). \tag{4.9}$$

*The function* $u^*(t, \epsilon)$ *is called the outer solution of the problem. The function* $U(t/\epsilon, \epsilon)$ *is called the boundary layer solution of the problem, and it can be shown to approach zero exponentially as* $t/\epsilon \to \infty$ *at a rate independent of* $\epsilon$ *[21].*

**Theorem 4.3.** *Let Eq. (4.2) be asymptotically Mittag-Leffler stable and the domain of stability be denoted by* $D$*. If* $u(t, \epsilon)$ *and* $\tilde{u}(t, \epsilon)$ *are solutions of (4.1) such that* $u(0, \epsilon)$ *and* $\tilde{u}(0, \epsilon)$ *define functions* $\xi(\epsilon)$ *and* $\tilde{\xi}(\epsilon)$ *where* $\xi(0)$, $\tilde{\xi}(0) \in D$*, then for small* $\epsilon > 0$

$$\lim_{t\to\infty} |u(t, \epsilon) - \tilde{u}(t, \epsilon)| = 0. \tag{4.10}$$

*Proof.* The difference $\tilde{u} - u$ satisfies the equation

$$\epsilon \frac{d^\alpha(\tilde{u} - u)}{dt^\alpha} = g(t, \tilde{u} - u, \epsilon) + G, \tag{4.11}$$

where the function $G$ depends on $t$, $\tilde{u} - u$ and $\epsilon$. If we set

$$z = \tilde{u} - u, \tag{4.12}$$

then it follows from the form of $G$ that

$$G = O(z^2). \tag{4.13}$$

It follows from [21] that for the following differential equation

$$\frac{d^\alpha u}{dt^\alpha} = \frac{1}{\epsilon} g(t, u, \epsilon), \tag{4.14}$$

there are positive constants $C$ and $\mu$ such that that for sufficiently small $\epsilon > 0$

$$|g(t, s, \epsilon)| \leq C E_\alpha(-\mu(t - s)^\alpha) \quad for\ 0 \leq s \leq t < \infty. \tag{4.15}$$

Then we obtain from (4.11) that for $0 \leq s \leq t < \infty$, $z$ satisfies

$$z(t) = g(t, s, \epsilon) z(s) + \frac{1}{\Gamma(\alpha)} \int_s^t (t - s')^{\alpha-1} g(t, s', \epsilon) \frac{G}{\epsilon}(s') ds'. \tag{4.16}$$

Therefore, according to (4.13) and (4.15), there is a constant $C'$ such that

$$|z(t)| \leq C E_\alpha(-\mu(t-s)^\alpha)|z(s)| + \frac{CC'}{\epsilon\Gamma(\alpha)} \int_s^t (t-s')^{\alpha-1} E_\alpha(-\mu(t-s')^\alpha)|z(s')|^2 ds'. \tag{4.17}$$

If for $s \leq s' \leq t$ we know that

$$|z(s')| \leq \frac{\epsilon\mu}{CC'}, \tag{4.18}$$

then

$$|z(t)| \leq C E_\alpha(-\mu(t - s)^\alpha)|z(s)| + \frac{\mu}{\Gamma(\alpha)} \int_s^t (t - s')^{\alpha-1} E_\alpha(-\mu(t - s')^\alpha)|z(s')| ds', \tag{4.19}$$

and so, by Gronwall's inequality for fractional integral equations [20]

$$|z(t)|\leq C|z(s)|E_{\alpha}(-\mu(t-s)^{\alpha}). \tag{4.20}$$

Therefore, (4.20) guarantees that (4.18) is satisfied provided

$$|z(s)|\leq \frac{\epsilon\mu}{C'C^2}. \tag{4.21}$$

We will now show that for any small $\epsilon > 0$, it is possible to choose $s$ so that (4.21) is satisfied. This will complete the proof.
Now, we use Theorem 1 to show

$$z(t,\epsilon) = (\tilde{u}_0 - u_0)(t) + \epsilon(\tilde{u}_1 - u_1)(t) + O(\epsilon^2). \tag{4.22}$$

Let us choose $\epsilon$ so small that the term $O(\epsilon^2)$ here satisfies

$$O(\epsilon^2)\leq \frac{\epsilon\mu}{3C'C^2}. \tag{4.23}$$

By definition if $\xi\in D$, there is a unique solution of (4.1) having $\xi$ as its initial value, and this solution approaches $u_0(t)$ as $t\to\infty$. Next since $u_0(0)$, $\tilde{u}_0(0)\in D$, we have that

$$\Delta(t) = |\tilde{u}_0(t) - u_0(t)|\to 0 \quad as\ t\to\infty. \tag{4.24}$$

Therefore, we choose $s_1$ so that

$$\Delta(t)\leq \frac{\epsilon\mu}{3C'C^2}, \tag{4.25}$$

for $t\geq s_1$. Furthermore its shown on [10] that if $z\in\mathbb{R}^k$ satisfies

$$\frac{dz}{dt} = B(t)z + b(t), \tag{4.26}$$

where $B$ is a continuous, real $k\times k$-matrix such that $\lim_{t\to\infty} B(t)$ exists and the vector $b$ is a $k$-vector such that $\lim_{t\to\infty} b(t)$ exists, then

$$\lim_{t\to\infty} z(t) = -(\lim_{t\to\infty} B^{-1}(t))(\lim_{t\to\infty} b(t)). \tag{4.27}$$

By use it we have that

$$\Delta_1(t) = |\tilde{u}_1(t) - u_1(t)|\to 0, \tag{4.28}$$

as $t\to\infty$. We therefore choose $s_2\geq s_1$ so that

$$\Delta_1(t)\leq \frac{\epsilon\mu}{3C'C^2}, \tag{4.29}$$

for $t\geq s_2$. It follows that if $s\geq s_2$, (4.21) is satisfied and therefore

$$|z(t)|\leq C|z(s_2)|E_{\alpha}(-\mu(t-s_1)^{\alpha}). \tag{4.30}$$

This completes the proof of Theorem. □

## 5. Application on Black-Scholes equation

In this section, we discuss the implementation of asymptotic expansion method on fractional Black-Scholes equation. The simplicity and accuracy of the method is illustrated through the following problem and asymptotical Mittag-Leffler stability of the problem is studied.
Consider the fractional Black-Scholes European option pricing equation:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} + \frac{\partial^\alpha v}{\partial t^\alpha} - rv = 0, \quad (x,t)\in R^+ \times (0,T), \tag{5.1}$$

subject to conditions

$$v(x,0) = max(x - EE_\alpha(-rT^\alpha), 0), \tag{5.2}$$

$$\frac{\partial v}{\partial x} = \begin{cases} 1 & \text{if} \quad x > EE_\alpha(-r(T^\alpha - t^\alpha)), \\ 0 & \text{if} \quad x < EE_\alpha(-r(T^\alpha - t^\alpha)). \end{cases} \tag{5.3}$$

We try an expansion in powers of $\sigma$ in the form

$$v(x,t) = v_0 + \sigma v_1 + \cdots, \tag{5.4}$$

which, when substituted into Eq. (5.1) and initial condition (5.2) gives

$$\frac{1}{2}\sigma^2 x^2 (\frac{\partial^2 v_0}{\partial x^2} + \sigma\frac{\partial^2 v_0}{\partial x^2} + \cdots) + rx(\frac{\partial v_0}{\partial x} + \sigma\frac{\partial v_1}{\partial x} + \cdots) + (\frac{\partial^\alpha v_0}{\partial t^\alpha} + \sigma\frac{\partial^\alpha v_1}{\partial t^\alpha} + \cdots) - r(v_0 + \sigma v_1 + \cdots) = 0, \tag{5.5}$$

and

$$v_0(x,0) + \sigma v_1(x,0) + \cdots = max(x - EE_\alpha(-rT^\alpha), 0). \tag{5.6}$$

Equating coefficients of $\sigma^0$ and $\sigma$ and $\cdots$ to zero yields

$$\frac{\partial^\alpha v_0}{\partial t^\alpha} + rx\frac{\partial v_0}{\partial x} - rv_0 = 0, \tag{5.7}$$

$$v_0(x,0) = max(x - EE_\alpha(-rT^\alpha), 0), \tag{5.8}$$

$$\frac{\partial v_0}{\partial x} = \begin{cases} 1 & \text{if} \quad x > EE_\alpha(-r(T^\alpha - t^\alpha)), \\ 0 & \text{if} \quad x < EE_\alpha(-r(T^\alpha - t^\alpha)), \end{cases} \tag{5.9}$$

and

$$\frac{\partial^\alpha v_1}{\partial t^\alpha} + rx\frac{\partial v_1}{\partial x} - rv_1 = 0, \tag{5.10}$$

$$v_1(x,0) = 0, \tag{5.11}$$

$$\frac{\partial v_1}{\partial x} = 0, \tag{5.12}$$

$$\vdots$$

Applying Laplace transform on Eq. (5.7) and using the differentiation property of Laplace transform, for $x > EE_\alpha(-r(T^\alpha - t^\alpha))$ we have

$$s^\alpha L(v_0) - s^{\alpha-1}v_0(x,0) + s^{-1}rx - rL(v_0) = 0, \tag{5.13}$$

and from here

$$L(v_0)(s^\alpha - r) = s^{\alpha-1}v_0(x,0) - s^{-1}rx, \tag{5.14}$$

or

$$L(v_0) = \frac{s^{\alpha-1}}{s^\alpha - r}v_0(x,0) + x(\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha - r}). \tag{5.15}$$

Operating the inverse Laplace transform on Eq. (5.15), we get

(5.16)

$$v_0 = (x - EE_\alpha(-rT^\alpha))E_\alpha(rt^\alpha) + x(1 - E_\alpha(rt^\alpha)) = x - EE_\alpha(-r(T^\alpha - t^\alpha)),$$

in the same manner for $x < EE_\alpha(-r(T^\alpha - t^\alpha))$ we have

$$v_0 = 0. \tag{5.17}$$

By above procedure we can conclude that $v_i = 0$ for $i = 1, 2, \cdots$, and when $v_0$ substituted into (5.4), gives

$$v(x,t) = max(x - EE_\alpha(-r(T^\alpha - t^\alpha)), 0). \tag{5.18}$$

Eq. (5.18) represents the closed form solution of the fractional Black Scholes equation Eq. (5.1) and is in coincide with the solution of Eq. (5.1) achieved from VIM and LHPM [22, 25]. Now for the standard case $\alpha = 1$ this series has the closed form of the solution $v(x,t) = max(x - Ee^{-r(T-t)}, 0)$ which is an exact solution of the given Black Scholes equation (5.1) for $\alpha = 1$.

From Eq. (5.15), it is not difficult to see that there exists $K > 0$ such that

$$v_0(x,t) \leq KE_\alpha(-r(T^\alpha - t^\alpha)), \tag{5.19}$$

which implies the reduced Eq. (5.7) is asymptotically Mittag-Leffler stable, and from Theorem (4.3) we conclude that the original Eq. (5.1) is too asymptotically Mittag-Leffler stable.

However the expansion (5.18) is not valid in a thin boundary layer near

$$x = EE_\alpha(-r(T^\alpha - t^\alpha)), \tag{5.20}$$

and for this reason we introduce inner variable

$$\xi = \frac{x - EE_\alpha(-r(T^\alpha - t^\alpha))}{\sigma}. \tag{5.21}$$

In terms of this scale

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{\sigma}\frac{\partial}{\partial \xi}, \tag{5.22}$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial^2}{\partial \xi^2}. \tag{5.23}$$

Then, (5.1) becomes

(5.24)
$$\frac{1}{2}\sigma^2(\sigma\xi + EE_\alpha(-r(T^\alpha - t^\alpha)))^2 \times \frac{1}{\sigma^2}\frac{\partial^2 v}{\partial \xi^2} + r(\sigma\xi + EE_\alpha(-r(T^\alpha - t^\alpha))) \times \frac{1}{\sigma}\frac{\partial v}{\partial \xi} + \frac{\partial^\alpha v}{\partial t^\alpha} - rv = 0.$$

By substituting

$$v = v_0 + \sigma v_1 + \cdots, \tag{5.25}$$

and equating coefficients of like powers of $\sigma$, we have

$$\frac{1}{2}E^2 E_\alpha(-2r(T^\alpha - t^\alpha))\frac{\partial^2 v_0}{\partial \xi^2} + r\xi\frac{\partial v_0}{\partial \xi} + \frac{\partial^\alpha v_0}{\partial t^\alpha} - rv_0 = 0, \tag{5.26}$$

and

(5.27)
$$\frac{1}{2}E^2 E_\alpha(-2r(T^\alpha - t^\alpha))\frac{\partial^2 v_1}{\partial \xi^2} + r\xi\frac{\partial v_1}{\partial \xi} + \frac{\partial^\alpha v_1}{\partial t^\alpha} - rv_1 = -\xi EE_\alpha(-r(T^\alpha - t^\alpha))\frac{\partial^2 v_0}{\partial \xi^2}.$$

Solving (5.26) and (5.27), and substituting $v_0$ and $v_1$ in (5.25) we have an inner expansion which is valid in boundary layer.

## 6. Numerical examples

In this section we present two examples with different volatility function $\sigma$ and it is shown that the value of small parameter $\sigma$ as perturbation parameter is not effective on outer expansion. The difference between Figure 1. of this paper with Figure 1. and Figure 2. of [7] is only in boundary layer of problem that can be resolves by introduce inner variable and inner expansion such as (5.26) and (5.27).

*Example* 6.1. In this example, we consider the following generalized fractional Black-Scholes equation [7] as follows:

$$\frac{\partial^\alpha v}{\partial t^\alpha} + 0.08(2 + sinx)^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \quad 0 < \alpha \leq 1, \tag{6.1}$$

with initial condition $v(x, 0) = max(x - 25E_\alpha(-0.06), 0)$. The methodology consist of the applying asymptotic expansion on Eq. (6.1), we get

$$v(x, t) = max(x - 25E_\alpha(-0.06(1 - t^\alpha)), 0), \tag{6.2}$$

which is the solution of the given option pricing equation (6.1) and is in coincide with the solution of Eq. (6.1) by VIM [25]. Now the solution of the generalized Black-Scholes equation (6.1) at $\alpha = 1$ is $v(x, t) = max(x - 25e^{-0.06(1-t)}, 0)$. Furthermore from Theorem (4.3) we conclude that Eq. (6.1) is asymptotically Mittag-Leffler stable. In Figure 1. we plot European call option with parameters $x_{max} = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.4(2 + sinx)$, $E = 25$ and $v(x, 0) = x - 25e^{-0.06}$, which is in close agreement with Figure 2. in [7] associated with solution of (6.1) for $\alpha = 1$ and by using finite difference method.

*Example* 6.2. In this example, we consider the following generalized fractional Black-Scholes equation [7] as follows:

$$\frac{\partial^\alpha v}{\partial t^\alpha} + 0.02(1+te^{-x})^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \quad 0 < \alpha \leq 1, \tag{6.3}$$

with initial condition $v(x,0) = max(x - 25E_\alpha(-0.06), 0)$. The methodology consist of the applying asymptotic expansion on Eq. (6.3), we get

$$v(x,t) = max(x - 25E_\alpha(-0.06(1-t^\alpha)), 0), \tag{6.4}$$

Now the solution of the generalized Black-Scholes equation (6.3) at $\alpha = 1$ is $v(x,t) = max(x - 25e^{-0.06(1-t)}, 0)$.
In Figure 1. we plot European call option with parameters $x_{max} = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2(1+te^{-x})$, $E = 25$ and $v(x,0) = x - 25e^{-0.06}$, which is in close agreement with Figure 1. in [7] associated with solution of (6.3) for $\alpha = 1$ and by using finite difference method.

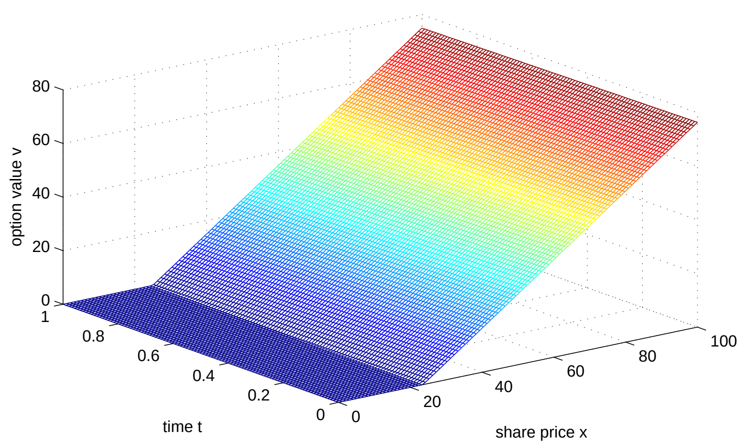


Fig. 1. ( Computed option value $v$ for Examples 1,2. )

## 7. Concluding remarks

In this paper, the asymptotic expansion method has been employed to provide a powerful tool on European option pricing problem. The asymptotical Mittag-Leffler stability of fractional Black-Scholes equation is thoroughly discussed. Numerical examples has been presented to determine the efficiency and simplicity of our scheme.

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