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ANALYTICAL APPROXIMATION SOLUTION OF A MATHEMATICAL MODELING OF REACTION-DIFFUSION BRUSSELATOR SYSTEM BY REDUCED DIFFERENTIAL TRANSFORM METHOD

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Abstract. In this paper an approximate analytical solution of a mathematical modeling of reaction-diffusion Brusselator system with fractional time derivative will be obtained with the help of the reduced differential transform method. Fractional reaction-diffusion Brusselator system is used for modeling of certain chemical reaction-diffusion processes. The fractional derivatives are described in the Caputo sense. It is indicated that the solutions obtained by the reduced differential transform method are reliable and present an effective method for strongly nonlinear partial equations.

Key Words: Fractional Brusselator system, Fractional calculus, Reduced differential transform method.

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1. Introduction

We consider 2-dimensional time fractional reaction-diffusion Brusselator system

$$\begin{aligned} D_t^{\alpha} u &= B + u^2 v - (A+1)u + \lambda\Big(u_{xx} + u_{yy}\Big) \\ D_t^{\beta} v &= Au - u^2 v + \lambda\Big(v_{xx} + v_{yy}\Big) \end{aligned} \tag{1.1}$$

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subject to the initial conditions

$$\begin{aligned} u(x,y,0) &= e^{-x-y} \\ v(x,y,0) &= e^{x+y}, \end{aligned} \tag{1.2}$$

where $u = u(x,y,t)$, $v = v(x,y,t)$ are chemical concentrations of reaction products, and A, B are constant concentrations of input reagents, and λ is a constant based on a diffusion coefficient. Also D_t^α, D_t^β is used to represent the Caputo-type fractional derivative of order α, β [1, 7].
Fractional differential operators have played a very important role in various fields such as electrical circuits, biology, biomechanics, viscoelasticity, etc. [14, 9, 13, 6, 8]. Recently various methods such as the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational iteration method (VIM) and the homotopy analysis method (HAM) have been applied for fractional PDEs [10, 11, 12].
The Reduced differential transform method (RDTM) has been used by many authors to obtain analytical and approximate solutions to nonlinear problems [4]. In the present work, we are concerned with the application of the reduced differential transform method (RDTM) [5, 3], for the 2-dimensonal time fractional reaction-diffusion Brusselator system. The Brusselator system occurs in a large number of physical problems such as the formation of ozone by atomic oxygen through a triple collision and enzymatic reactions [7].

2. Fractional calculus

In this section, we present a review of the notations, definitions and preliminary of fractional calculus according to the refferences [3-6].

Definition 2.1. A real function $f(x), x > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $q(> \mu)$, such that $f(x) = x^q g(x)$, where $g(x) \in C[0,\infty]$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2. For a function $f \in C_\mu, \mu \geq -1$, the Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, is defined as

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, & \alpha > 0, x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{2.1}$$

For $f \in C_\mu, \mu \geq -1$ and $\forall \alpha, \beta \in \mathbb{R}_+$, the operator J^α has the properties:
$i) J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$.
$ii) J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$.

We all know that, the Riemann-Liouville approach leads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives which there is no known physical interpretation for such types of initial condition. A modified fractional differential operator D^α which proposed by Caputo in his work on the theory of viscoelasticity [2] is

$$D^\alpha f(x) = J^{m-\alpha} f^{(m)}(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{2.2}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$ and $f \in C_{-1}^m$.
The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.

Definition 2.3. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

(2.3)

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m u(x,s)}{\partial s^m} ds, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in \mathbb{N}. \end{cases}$$

3. REDUCED DIFFERENTIAL TRANSFORM METHOD

In this section, we apply the reduced differential transform method for three variables function $u(x,y,t)$ which has been developed in [4] and [5].
Consider a function of three variables $u(x,y,t)$ which is analytic and differentiated continuously in the domain of interest, and suppose that it can be represented as $u(x,y,t) = f(x,y)g(t)$.

Definition 3.1. If function $u(x,y,t)$ is analytic and differentiated continuosly with respect to x, y and t in the domain of interest, then let

$$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,y,t)\right]_{t=0}, \tag{3.1}$$

where the t-dimensional spectrum function $U_k(x,y)$ is the transformed function which is called T-function.

The differential inverse transform of $U_k(x,y)$ is defined as

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^{k\alpha}, \tag{3.2}$$

combining Eqs. (3.1) and (3.2) gives that

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,y,t)\right]_{t=0} t^{k\alpha}. \tag{3.3}$$

In real applications, by consideration of $U_0(x,y) = h(x,y)$ as transformaiton of initial condition

$$u(x,y,0) = h(x,y), \tag{3.4}$$

the function $u(x,y,t)$ can be approximated by a finite series of Eq. (3.2) as

$$\widetilde{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^{k\alpha}. \tag{3.5}$$

A straightforward iterative calculations, gives the $U_k(x,y)$ values for $k = 1,2,\cdots,n$. Then the inverse transformation of the $\{U_k(x,y)\}_{k=0}^{n}$ gives the approximation solutoin as $\widetilde{u}_n(x,y,t)$, where n is order of approximation solution. Next, the exact solutoin is obtained by $u(x,y,t) = \lim_{n\longrightarrow\infty}\widetilde{u}_n(x,y,t)$.

Some basic properties of the reduced differential transformation obtained from Eqs. (3.1) and (3.2) are summarized in Table 1. Note, in this table

$$\Gamma(z) := \int_0^{\infty} e^{-t}t^{z-1}dt, \qquad z \in \mathbb{C}.$$

Function Form	Transformed Form
$u(x,y,t)=v(x,y,t)+w(x,y,t)$	$U_k(x,y)=V_k(x,y)+W_k(x,y)$
$u(x,y,t)=cv(x,y,t)$	$U_k(x,y)=cV_k(x,y)$ $(c\ is\ a\ constant)$
$u(x,y,t)=ct^n v(x,y,t)$	$U_k(x,y)=cv(x,y)\delta(k-\alpha n)=\begin{cases}1 & k=\alpha n\\ 0 & k\neq\alpha n\end{cases}$
$u(x,y,t)=v(x,y,t)w(x,y,t)$	$U_k(x,y)=\sum_{k_1=0}^{k} V_{k_1}(x,y)W_{k-k_1}(x,y)$
$u(x,y,t)=v(x,y,t)w(x,y,t)z(x,y,t)$	$U_k(x,y)=\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} V_{k_1}(x,y)W_{k_2-k_1}(x,y)Z_{k-k_2}(x,y)$
$u(x,y,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}v(x,y,t)$	$U_k(x,y)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}V_{k+N}(x,y)$
$u(x,y,t)=\frac{\partial^{m+n}}{\partial x^m\partial y^n}v(x,y,t)$	$U_k(x,y)=\frac{\partial^{m+n}}{\partial x^m\partial y^n}V_k(x,y)$

Table 1. Some basic reduced differential transformations.

According to the RDTM and table 1, we can construct the following iteration for the Eq. (1.1) as

$$\begin{aligned}\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y) &= B\delta(k)+\sum_{s=0}^{k}\sum_{r=0}^{s}V_r(x,y)U_{s-r}(x,y)U_{k-s}(x,y)-(A+1)U_k(x,y)\\ &\quad+\lambda\Big(\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)\Big)\\ \frac{\Gamma(k\beta+\beta+1)}{\Gamma(k\beta+1)}V_{k+1}(x,y) &= AU_k(x,y)-\sum_{s=0}^{k}\sum_{r=0}^{s}V_r(x,y)U_{s-r}(x,y)U_{k-s}(x,y)\\ &\quad+\lambda\Big(\frac{\partial^2}{\partial x^2}V_k(x,y)+\frac{\partial^2}{\partial y^2}V_k(x,y)\Big).\end{aligned}\tag{3.6}$$

Initial condition (1.2) gives

$$\begin{aligned}U_0(x,y)&=e^{-x-y}\\ V_0(x,y)&=e^{x+y}.\end{aligned}\tag{3.7}$$

By substituding of Eqs. (3.7) into (3.6) we can obtain the next terms of $U_k(x,y)$ and $V_k(x,y)$ as

$$
\begin{aligned}
U_1(x,y) &= \Big(1-4A+4\lambda+4Be^{x+y}\Big)\frac{e^{-(x+y)}}{4\Gamma(1+\alpha)},\\
U_2(x,y) &= \Bigg[\Big(-16+16A+4(1+4\lambda)e^{2(x+y)}\Big)\Gamma(1+\alpha)\\
&\quad - e^{2(x+y)}\Big(-5-16A^2-16Be^{x+y}-24\lambda-16\lambda^2+24A\\
&\quad +16Be^{x+y}+32\lambda\Big)\Gamma(1+\beta)\Bigg]\frac{e^{-3(x+y)}}{16\Gamma(1+2\alpha)\Gamma(1+\beta)},\cdots.
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
V_1(x,y) &= \Big(-4+4A+(1+4\lambda)e^{2(x+y)}\Big)\frac{e^{x+y}}{4\Gamma(1+\beta)},\\
V_2(x,y) &= \Bigg[\Big(-4+(1+4\lambda)e^{2(x+y)}\Big)\Big(-4+4A+(1+4\lambda)e^{2(x+y)}\Big)\Gamma(1+\alpha)\\
&\quad +4\big(-2+A\big)e^{2(x+y)}\Big(1-4A+4\lambda+4Be^{x+y}\Big)\Gamma(1+\beta)\Bigg]\\
&\quad \frac{e^{-3(x+y)}}{16\Gamma(1+\alpha)\Gamma(1+2\beta)},\cdots.
\end{aligned}
\tag{3.9}
$$

The time fractional reaction-diffusion Brusselator system (1.1) when $A=1$, $B=0$, $\alpha=\beta=1$ and $\lambda=0.25$ has the exact solution $u(x,y,t)=e^{-x-y-0.5t}$ and $v(x,y,t)=e^{x+y+0.5t}$ [7]. By considering of ten terms, we can get the approximation solution $\widetilde{u}_{10}(x,y,t)$ and $\widetilde{v}_{10}(x,y,t)$, which is very close to exact solution. A comparison between the exact and the approximation solution have been presented in Table 2. Also, the exact and approximation solution in this case and approximate solution for different values of α and β have been indicated in Figure 1 for $u(x,y,t)$ and Figure 2 for $v(x,y,t)$ respectively. finally in figure 3, the exact and approximate solution at $x=y=0.5$ and $0\le t\le 1$, have been compared.

FIGURE 1. Solutions using the $\widetilde{u}_{10}(x, y, t)$ for different values of α and β when $\lambda = 0.25$: (a) exact ($\alpha = \beta = 1$), (b) ($\alpha = \beta = 1$), (c) $\alpha = \beta = 0.75$ and (d) $\alpha = \beta = 0.5$.

FIGURE 2. Solutions using the $\widetilde{v}_{10}(x, y, t)$ for different values of α and β when $\lambda = 0.25$: (a) exact ($\alpha = \beta = 1$), (b) ($\alpha = \beta = 1$), (c) $\alpha = \beta = 0.75$ and (d) $\alpha = \beta = 0.5$.

FIGURE 3. Comparison between exact and approximate solution at $x = y = 0.5$ and $0 \leq t \leq 1$: (a) $u(x, y, t)$ and (b) $v(x, y, t)$

x	y	t	$\lvert u(x,t) - \widetilde{u}_{10}(x,y,t)\rvert$	$\lvert v(x,t) - \widetilde{v}_{10}(x,y,t)\rvert$
x=0.3	y=0.3	t=1	6.44401E-12	2.32552E-11
		t=2	1.26854E-8	4.97669E-8
x=0.6	y=0.6	t=1	3.53662E-12	4.2375E-11
		t=2	6.96188E-9	9.06812E-8
x=0.9	y=0.9	t=1	1.94088E-12	7.72058E-11
		t=2	3.82076E-9	1.65232E-7
x=1.2	y=1.2	t=1	1.06519E-12	1.40684E-10
		t=2	2.09688E-9	3.01072E-7
x=1.5	y=1.5	t=1	5.84588E-13	2.56335E-10
		t=2	1.15079E-9	5.48589E-7

Table 2. Comparison between the exact solution and the approximate solution for $\alpha = \beta = 1$ and $\lambda = 0.25$.

4. Conclusion

In this paper, the reduced differential transform method was successfully applied for the time fractional reaction-diffusion Brusselator system. Figurse 1, 2, 3 and Table 2 indicate that, the mentioned method is a very powerful and effcient technique for finding approximate solutions for nonlinear problems, and does not require linearization, discretization or perturbation.

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