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RIEMANN SOLITONS IN PARA SASAKIAN MANIFOLDS ADMITTING DIFFERENT SEMI-SYMMETRIC STRUCTURES

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ABSTRACT. The object of the present paper is to study the Riemannian solitons on para Sasakian manifolds admitting $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$, $E \cdot \mathcal{W}_i^* = 0$, $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$, $R \cdot \mathcal{W}_i^* = 0$, $R \cdot K = 0$, $R \cdot C = 0$, $E \cdot C = 0$ and $E \cdot K = 0$, for $i = 1, 2, ..., 9$.

Key Words: Riemannian soliton, para Sasakian manifold, semi-symetry.
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1. INTRODUCTION

The idea of Ricci flow was first introduced by R. S. Hamilton [18] in 1982. This concept generalized to the idea of Riemann flow ([2], [3]). Keeping the tune with Ricci soliton, Hirica and Udriste [14] introduced and discussed Riemann soliton. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows

$$\frac{\partial}{\partial t} G\left(t\right) = -2R\left(g\left(t\right)\right), \qquad t \in \left[0, I\right],$$

where $G = \frac{1}{2} g \circledast g, \circledast$ is the Kulkarni-Nomizu product and R is the Riemann curvature tensor associated to the metric g. For $(0, 2)$-tensors

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α and β, the Kulkarni-Nomizu product $(\alpha \circledast \beta)$ is given by

$$\begin{aligned} & (\alpha \circledast \beta)(Y,U,V,Z) \\ = \;& \alpha(Y,V)\beta(U,Z) + \alpha(U,Z)\beta(Y,V) \\ & -\alpha(Y,Z)\beta(U,V) - \alpha(U,V)\beta(Y,Z). \end{aligned} \tag{1.1}$$

The authors Stepanov and Tsyganok [20] characterize the Riemann soliton in terms of infinitesimal harmonic transformation. The Riemann soliton is a smooth manifold M together with Riemannian metric g that satisfies

$$2R + \lambda(g \circledast g) + (g \circledast \pounds_W g) = 0, \tag{1.2}$$

where W is a potential vector field, $\pounds_V$ denotes the Lie-derivative along the vector field W and λ is a constant. The Riemann soliton also corresponds to the Riemann flow as a fixed point, and on the space of Riemannian metric modulo diffeomorphism they can be seen as a dynamic system. A Riemann soliton is called expanding, steady and shrinking when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Definition 1.1. Let T and D be two tensors of type (0, 4). A Riemannian (or semi-Riemannian) manifold is said to be D-semisymmetric type if $T(X,Y) \cdot D = 0$ for all X, $Y \in \chi(M)$, the set all vector fields of the manifold M where $T(X,Y)$ acts on D as derivation of tensor algebra. The above condition is often written as $T \cdot D = 0$. Especially, if we consider $T = D = R$, then the manifold is called semisymmetric [22]. Details about the semisymmetry and other conditions of semisymmetry type are available in : [6], [15], [4], [23] and also references therein.

In 2013, Kundu and Shaikh ([19], Table 2) investigated the equivalency of the various geometric structures. They have established the following conditions

i) $E \cdot R = 0$, $E \cdot P = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ...9$) are equivalent and named such a class by C_1;

ii) $R \cdot R = 0$, $R \cdot P = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$ and $R \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ...9$) are equivalent and named such a class by C_2;

iii) $R \cdot K = 0$ and $R \cdot C = 0$ are equivalent and named such a class by C_3;

iv) $E \cdot C = 0$ and $E \cdot K = 0$ are equivalent and named such a class by C_4; where the symbols C, E, P, K, $\mathcal{M}$ and $\mathcal{W}_i$ stand for conformal curvature tensor [17], concircular curvature tensor [16], projective curvature tensor [16], conharmonic curvature tensor [21], M-projective curvature tensor [11], $\mathcal{W}_i$-curvature tensor, ([11], [12], [13]) and $\mathcal{W}_i^*$-curvature tensor [11] respectively.

$$C(V,Y) = R(V,Y) - \frac{1}{n-2}[(V \wedge_g QY) + (QV \wedge_g Y) + \frac{r}{\alpha}(V \wedge_g Y)], \tag{1.3}$$

$$E(V,Y) = R(V,Y) - \frac{r}{n(n-1)}(V \wedge_g Y), \tag{1.4}$$

$$P(V,Y) = R(V,Y) - \frac{1}{n-1}(V \wedge_S Y), \tag{1.5}$$

$$K(V,Y) = R(V,Y) - \frac{1}{n-2}[(V \wedge_g QY) + (QV \wedge_g Y)], \tag{1.6}$$

$$\mathcal{M}(V,Y) = R(V,Y) - \frac{1}{2\alpha}[(V \wedge_g QY) + (QV \wedge_g Y)], \tag{1.7}$$

$$\mathcal{W}_0(V,Y) = R(V,Y) - \frac{1}{\alpha}(V \wedge_g QY), \tag{1.8}$$

$$\mathcal{W}_0^*(V,Y) = R(V,Y) + \frac{1}{\alpha}(V \wedge_g QY), \tag{1.9}$$

$$\mathcal{W}_1(V,Y) = R(V,Y) - \frac{1}{\alpha}(V \wedge_S Y), \tag{1.10}$$

$$\mathcal{W}_1^*(V,Y) = R(V,Y) + \frac{1}{\alpha}(V \wedge_S Y), \tag{1.11}$$

$$\mathcal{W}_2(V,Y) = R(V,Y) - \frac{1}{\alpha}[(QV \wedge_g Y) + (V \wedge_g QY) - (V \wedge_S Y)], \tag{1.12}$$

$$\mathcal{W}_2^*(V,Y) = R(V,Y) + \frac{1}{\alpha}[(QV \wedge_g Y) + (V \wedge_g QY) - (V \wedge_S Y)], \tag{1.13}$$

$$\mathcal{W}_3(V,Y) = R(V,Y) - \frac{1}{\alpha}(Y \wedge_g QV), \tag{1.14}$$

$$\mathcal{W}_3^*(V,Y) = R(V,Y) + \frac{1}{\alpha}(Y \wedge_g QV), \tag{1.15}$$

$$\mathcal{W}_5(V,Y) = R(V,Y) - \frac{1}{\alpha}[(V \wedge_g QY) - (V \wedge_S Y)], \tag{1.16}$$

$$(1.17)\quad \mathcal{W}_5^*(V,Y) = R(V,Y) + \frac{1}{\alpha}\left[(V\wedge_g QY) - (V\wedge_S Y)\right],$$

$$(1.18)\quad \mathcal{W}_7(V,Y) = R(V,Y) + \frac{1}{\alpha}\left[(QV\wedge_g Y) - (V\wedge_S Y)\right],$$

$$(1.19)\quad \mathcal{W}_7^*(V,Y) = R(V,Y) - \frac{1}{\alpha}\left[(QV\wedge_g Y) - (V\wedge_S Y)\right],$$

$$(1.20)\quad \mathcal{W}_4(V,Y)Z = R(V,Y)Z - \frac{1}{\alpha}[g(V,Z)QY - g(V,Y)QZ],$$

$$(1.21)\quad \mathcal{W}_4^*(V,Y)Z = R(V,Y)Z + \frac{1}{\alpha}[g(V,Z)QY - g(V,Y)QZ],$$

$$(1.22)\quad \mathcal{W}_6(V,Y)Z = R(V,Y)Z - \frac{1}{\alpha}[S(Y,Z)V - g(V,Y)QZ],$$

$$(1.23)\quad \mathcal{W}_6^*(V,Y)Z = R(V,Y)Z + \frac{1}{\alpha}[S(Y,Z)V - g(V,Y)QZ],$$

$$(1.24)\quad \mathcal{W}_8(V,Y)Z = R(V,Y)Z - \frac{1}{\alpha}[S(Y,Z)V - S(V,Y)Z],$$

$$(1.25)\quad \mathcal{W}_8^*(V,Y)Z = R(V,Y)Z + \frac{1}{\alpha}[S(Y,Z)V - S(V,Y)Z],$$

$$(1.26)\quad \mathcal{W}_9(V,Y)Z = R(V,Y)Z - \frac{1}{\alpha}[S(V,Y)Z - g(Y,Z)QV],$$

$$(1.27)\quad \mathcal{W}_9^*(V,Y)Z = R(V,Y)Z + \frac{1}{\alpha}[S(V,Y)Z - g(Y,Z)QV],$$

where

$$(1.28)\qquad \alpha = (n-1), \quad (V\wedge_B Y)Z = B(Y,Z)V - B(V,Z)Y.$$

The present paper is structured as follows. After introduction, in Section 2, we briefly recall a short description of para-Sasakian manifold. In section 3, we discussed Riemann solitons in para-Sasakian manifolds and in this case we obtained the Riemann soliton on M is always shrinking. Here also we studied the Riemann solitons in para-Sasakian manifold admitting $E\cdot R=0$, $E\cdot P=0$, $E\cdot E=0$, $E\cdot P^*=0$, $E\cdot\mathcal{M}=0$, $E\cdot\mathcal{W}_i=0$, $E\cdot\mathcal{W}_i^*=0$, $R\cdot R=0$, $R\cdot P=0$, $R\cdot E=0$, $R\cdot P^*=0$, $R\cdot\mathcal{M}=0$, $R\cdot\mathcal{W}_i=0$, $R\cdot\mathcal{W}_i^*=0$, $R\cdot K=0$, $R\cdot C=0$, $E\cdot C=0$ and $E\cdot K=0$. For all the cases we obtained the Riemann soliton on M is always shrinking.

2. Properties of para-Sasakian manifold

Let M be an n-dimensional differentiable manifold with almost para-contact Riemannian structure (ϕ,ξ,η,g) consisting of $(1,1)$ tensor field

ϕ, a 1-form η, a contravarient vector field ξ and an associated Riemannian metric g. Then it satisfies

$$\phi^2 V = -\eta(V)\xi + V, \quad rank\ \phi = n-1 \tag{2.1}$$

$$\eta(\xi) = 1, \eta(\phi V) = 0, \ \phi\xi = 0. \tag{2.2}$$

$$g(V,\phi Y) = g(\phi V, Y), g(V,\xi) = \eta(V), \forall V, Y \in TM \tag{2.3}$$

$$g(\phi V,\phi Y) = g(V,Y) - \eta(V)\eta(Y), g(QV,Y) = S(V,Y) \tag{2.4}$$

$$\nabla_V \xi = \phi V, \ d\eta = 0, \tag{2.5}$$

$$(\nabla_V \eta)Y = (\nabla_Y \eta)V = g(V,\phi Y). \tag{2.6}$$

An almost paracontact Riemannian manifold is called a P-Sasakian manifold if it satisfies

$$(\nabla_V \phi)Y = -g(V,Y)\xi - \eta(Y)V + 2\eta(V)\eta(Y)\xi. \tag{2.7}$$

In an n-dimensional para Sasakian manifold ([1], [7], [8], [9]) M, the curvature tensor R, the Ricci tensor S, and the Ricci operator Q satisfy

$$R(V,Y)\xi = \eta(V)\,Y - \eta(Y)V, \tag{2.8}$$

$$S(V,\xi) = -(n-1)\eta(V), \tag{2.9}$$

$$R(V,\xi)Y = g(V,Y)\xi - \eta(Y)V, \tag{2.10}$$

$$R(\xi,V)Y = \eta(Y)V - g(V,Y)\xi, \tag{2.11}$$

$$\eta(R(V,Y)U) = g(V,U)\eta(Y) - g(Y,U)\eta(V), \tag{2.12}$$

$$Q\xi = -(n-1)\xi, \tag{2.13}$$

$$S(\phi V,\phi Y) = S(V,Y) + (n-1)\eta(V)\eta(Y). \tag{2.14}$$

3. Riemann solitons admitting semi-symmetric strucres in para Sasakian manifolds

In this section we consider a para Sasakian manifold (M^n, g, ϕ, ξ, η) admits an Riemann soliton. Then taking account (1.1), into (1.2), we obtain

$$\begin{aligned} & 0 \\ = {} & 2R(Y,U,V,Z) + 2\lambda[g(Y,V)g(U,Z) - g(Y,Z)g(U,V)] \\ & + [g(Y,V)\pounds_\xi g(U,Z) + g(U,Z)\pounds_\xi g(Y,V) \\ & - g(Y,Z)\pounds_\xi g(U,V) - g(U,V)\pounds_\xi g(Y,Z)]. \end{aligned} \tag{3.1}$$

Now by the help of (2.1), (2.3) and (2.5) we get

$$\begin{aligned} & \pounds_\xi g(V,Y) \\ = \; & \pounds_\xi g(V,Y) - g([\xi,V],Y) - g(V,[\xi,Y]) \\ = \; & 2g(\phi V,Y). \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1), we get

$$\begin{aligned} & 0 \\ = \; & 2R(Y,U,V,Z) + 2\lambda[g(Y,V)g(U,Z) - g(Y,Z)g(U,V)] \\ & +[2g(Y,V)g(\phi Z,U) + g(U,Z)g(\phi V,Y) \\ & -g(Y,Z)g(\phi V,U) - g(U,V)g(\phi Z,Y)]. \end{aligned} \tag{3.3}$$

By the suitable contraction of (3.3), we obtain

$$\begin{aligned} & S(U,V) \\ = \; & \lambda(n-1)g(U,V) + (n-2)g(\phi U,V). \end{aligned} \tag{3.4}$$

Replacing $U = V = \xi$ in (3.4) also using (2.2) and (2.9), we get

$$\lambda = -1. \tag{3.5}$$

Thus we can state

Theorem 3.1. *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on para Sasakian manifold M^n, then the Riemann soliton on M is always shrinking.*

3.1. **Riemann solitons on para Sasakian manifolds admitting the class C_1.** Here, we choose para-Sasakian manifolds satisfying the condition

$$(E(X,Z)\cdot R)(Y,V)U = 0, \tag{3.6}$$

which implies

$$\begin{aligned} & g(E(\xi,Z)R(Y,V)U,\xi) - g(R(E(\xi,Z)Y,V)U,\xi) \\ & -g(R(Y,E(\xi,Z)V)U,\xi) - g(R(Y,V)E(\xi,Z)U,\xi) = 0. \end{aligned} \tag{3.7}$$

By the help of (1.4), (2.8)-(2.13), we get the following

$$\begin{aligned} & g(E(\xi,Z)R(Y,V)U,\xi) \\ = \; & R(Y,V,U,\xi)\eta(Z) - R(Y,V,U,Z) \\ & -\frac{r}{n(n-1)}R(Y,V,U,Z) + \frac{r}{n(n-1)}R(Y,V,U,\xi)\eta(Z), \end{aligned} \tag{3.8}$$

$$\begin{aligned} & g\left(R\left(E\left(\xi, Z\right) Y, V\right) U, \xi\right) \\ = \ & \eta\left(Y\right) R\left(Z, V, U, \xi\right) - \ g\left(Z, Y\right) R\left(\xi, V, U, \xi\right) \\ & -\frac{r}{n(n-1)} g\left(Z, Y\right) R\left(\xi, V, U, \xi\right) + \frac{r}{n(n-1)} \eta\left(Y\right) R\left(Z, V, U, \xi\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} & g\left(R\left(Y, E\left(\xi, Z\right) V\right) U, \xi\right) \\ = \ & \eta\left(V\right) R\left(Y, Z, U, \xi\right) - \ g\left(Z, V\right) R\left(Y, \xi, U, \xi\right) \\ & -\frac{r}{n(n-1)} g\left(Z, V\right) R\left(Y, \xi, U, \xi\right) + \frac{r}{n(n-1)} \eta\left(V\right) R\left(Y, Z, U, \xi\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} & g\left(R\left(Y, V\right) E\left(\xi, Z\right) U, \xi\right) \\ = \ & \eta\left(U\right) R\left(Y, V, Z, \xi\right) - \ g\left(Z, U\right) R\left(Y, V, \xi, \xi\right) \\ & -\frac{r}{n(n-1)} g\left(Z, U\right) R\left(Y, V, \xi, \xi\right) + \frac{r}{n(n-1)} \eta\left(U\right) R\left(Y, V, Z, \xi\right). \end{aligned} \tag{3.11}$$

Using (3.8), (3.9), (3.10) and (3.11) in (3.7), then conntracting over Y and Z we get

$$-\left[1 + \frac{r}{n(n-1)}\right] S\left(V, U\right) = \left[1 + \frac{r}{n(n-1)}\right] \left(n-1\right) g\left(V, U\right). \tag{3.12}$$

In view of (3.4) and (3.12), we obtain

$$0 = \left[1 + \frac{r}{n(n-1)}\right] \left[\left(\lambda + 1\right)\left(n-1\right) g\left(V, U\right) + \left(n-2\right) g\left(\phi V, U\right)\right]. \tag{3.13}$$

Replacing $U = V = \xi$ in (3.13) also using (2.2) and (2.9) it follows that

$$\lambda = -1, \quad \text{provided} \ \ n(n-1) + r \neq 0. \tag{3.14}$$

Thus we can state the following theorem.

Theorem 3.2. *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on para Sasakian manifold M^n admitting the class C_1, then the Riemann soliton on M is always shrinking.*

3.2. **Riemann solitons on para Sasakian manifolds admitting the class C_2.** Here, we opt para-Sasakian manifolds satisfying the condition

$$\left(R\left(X, Z\right) \cdot R\right)\left(Y, V\right) U = 0, \tag{3.15}$$

which implies

$$\begin{aligned} & g\left(R\left(\xi, Z\right) R\left(Y, V\right) U, \xi\right) - g\left(R\left(R\left(\xi, Z\right) Y, V\right) U, \xi\right) \\ & -g\left(R\left(Y, R\left(\xi, Z\right) V\right) U, \xi\right) - g\left(R\left(Y, V\right) R\left(\xi, Z\right) U, \xi\right) = 0. \end{aligned} \tag{3.16}$$

Making use of (2.11), equation (3.16) reduces to

$$\begin{aligned} & R(Y,V,U,\xi)\,\eta(Z) - R(Y,V,U,Z) \\ = \; & \eta(Y)\,R(Z,V,U,\xi) - g(Z,Y)\,R(\xi,V,U,\xi) \\ & +\eta(V)\,R(Y,Z,U,\xi) - g(Z,V)\,R(Y,\xi,U,\xi) \\ & +\eta(U)\,R(Y,V,Z,\xi) - g(Z,U)\,R(Y,V,\xi,\xi)\,. \end{aligned} \tag{3.17}$$

Contracting over Y and Z in (3.17) and using (2.11), we obtain

$$S(V,U) = -(n-1)\,g(V,U)\,. \tag{3.18}$$

In consequence of (3.4) and (3.18), we get

$$\begin{aligned} & 0 \\ = \; & (\lambda+1)(n-1)\,g(V,U) + (n-2)\,g(\phi V,U)\,. \end{aligned} \tag{3.19}$$

Setting $U = V = \xi$ in (**3.19**), we get

$$\lambda = -1. \tag{3.20}$$

Thus we conclude that

Theorem 3.3. *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on para Sasakian manifold M^n admitting the class C_2, then the Riemann soliton on M is always shrinking.*

3.3. **Riemann solitons on para Sasakian manifolds admitting the class C_3.** Here, we take para-Sasakian manifolds satisfying the condition

$$(R(X,Z)\cdot C)(Y,V)U = 0, \tag{3.21}$$

which implies

$$\begin{aligned} & g(R(\xi,Z)\,C(Y,V)\,U,\xi) - g(C(R(\xi,Z)\,Y,V)\,U,\xi) \\ & -g(C(Y,R(\xi,Z)\,V)\,U,\xi) - g(C(Y,V)\,R(\xi,Z)\,U,\xi) = 0. \end{aligned} \tag{3.22}$$

By the help of (1.6), (2.8)-(2.13), we get the following

$$g(R(\xi,Z)\,C(Y,V)\,U,\xi) = C(Y,V,U,\xi)\,\eta(Z) - C(Y,V,U,Z)\,, \tag{3.23}$$

$$g(C(R(\xi,Z)\,Y,V)\,U,\xi) = \eta(Y)C(Z,V,U,\xi) - g(Z,Y)\,C(\xi,V,U,\xi)\,, \tag{3.24}$$

$$g(C(Y,R(\xi,Z)\,V)\,U,\xi) = \eta(V)C(Y,Z,U,\xi) - g(Z,V)\,C(Y,\xi,U,\xi)\,, \tag{3.25}$$

$$g(C(Y,V)\,R(\xi,Z)\,U,\xi) = \eta(U)C(Y,V,Z,\xi) - g(Z,U)\,C(Y,V,\xi,\xi)\,. \tag{3.26}$$

Using (3.23), (3.24), (3.25) and (3.26) in (3.22), then conntracting over Y and Z we obtain

$$\begin{aligned} & S(V,U) \\ = \; & \frac{(n+r-1)}{(n-1)} g(V,U) - \frac{(n^2-n+r)}{(n-1)} \eta(V)\eta(U). \end{aligned} \tag{3.27}$$

Now in consequence of (3.4) and (3.27), we get

$$\begin{aligned} & \left[\lambda(n-1) - \frac{(n+r-1)}{(n-1)}\right] g(V,U) \\ & + \frac{(n^2-n+r)}{(n-1)} \eta(V)\eta(U) + (n-2)\, g(\phi V,U) \\ = \; & 0. \end{aligned} \tag{3.28}$$

Setting $U = V = \xi$ in (**3.10**), we get

$$\lambda = -1. \tag{3.29}$$

Thus we can state

Theorem 3.4. *If* $(g,\ \phi,\ \xi,\ \eta)$ *is a Riemann soliton on para Sasakian manifold* M^n *admitting the class* C_3*, then the Riemann soliton on* M *is always shrinking.*

3.4. **Riemann solitons on para Sasakian manifolds admitting the class C_4.** Here, we consider para-Sasakian manifolds satisfying the condition

$$(E(X,Z)\cdot C)(Y,V)U = 0, \tag{3.30}$$

which implies

$$\begin{aligned} & g(E(\xi,Z)C(Y,V)U,\xi) - g(C(E(\xi,Z)Y,V)U,\xi) \\ & - g(C(Y,E(\xi,Z)V)U,\xi) - g(C(Y,V)E(\xi,Z)U,\xi) = 0. \end{aligned} \tag{3.31}$$

Now by the help of [5], we get

$$\left(\frac{r}{n(n-1)}+1\right)\left[S(V,U) - \left(\frac{r}{n-1}+1\right)g(V,U) - \left(\frac{r}{n-1}-n\right)\eta(V)\eta(U)\right] = 0. \tag{3.32}$$

$$S(V,U) = \left(1+\frac{r}{n-1}\right) g(V,U) + \left(-n+\frac{r}{1-n}\right)\eta(V)\eta(U). \tag{3.33}$$

In view of (3.4) and (3.33), we obtain

$$0 = \left[\lambda(n-1) - \left(1 + \frac{r}{n-1}\right)\right] g(V,U)$$

$$- \left(-n + \frac{r}{1-n}\right) \eta(V)\,\eta(U) + (n-2)\,g(\phi V, U)\,. \tag{3.34}$$

Replacing $U = V = \xi$ in (3.34) also using (2.2) and (2.9) it follows that

$$\lambda = -1. \tag{3.35}$$

Thus we can state

Theorem 3.5. *If $(g,\ \phi,\ \xi,\ \eta)$ is a Riemann soliton on para Sasakian manifold M^n admitting the class C_4, then the Riemann soliton on M is always shrinking.*

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