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NOTES ON REDUCED, ARTINIAN AND MULTIPLICATION MODULES

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Abstract. Let M be a unitary module over a commutative ring R with identity. In this paper we consider the concepts of Artinian, semi-Artinian, reduced and multiplication modules . Also we call an R-module M radical, if it has no maximal submodule. By $P(M)$ we denote the sum of the radical submodules of M and we show that $P(M/(P(M)) = 0$.

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1. Introduction

In this note all rings are commutative rings with identity and all modules are unital. Let R be a ring and M an R-module, then M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$.
Like in [4], we call an R-module M radical, if it has no maximal submodules. By $P(M)$ we denote the sum of the radical submodules of M, $P(M)$ is the largest radical submodule of M, If $P(M) = 0$, M is called reduced.
An R-module M is called semi-Artinian if every proper submodule of M contains a minimal submodule. We denote by $L(M)$ the sum of

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all Artinian submodules of M. $L(M)$ is the largest semi-Artinian R-module and always has a decomposition $L(M) = \oplus_{m\in Max(R)} L_m(M)$, where $L_m(M) = \Sigma_{n=1}^{\infty}(0 :_M m^n)$ and Max(R) is the set of all maximal ideal of R.
For each R-module L, we denote by $\mathrm{Ass}_R L$ the set of all associated prime ideals of L. Also we denote by $J(R)$ the radical jacobson of R which is the intersection of all maximal ideals of R. For any unexplained notation and terminology we refer the reader to [1] and [3].

2. Reduced modules

Theorem 2.1. *Let M be an R-module. Then $L(M)$ is reduced and artinian R-module if and only if $L(M)$ is Noetherian R-module.*

Proof. Suppose that $L(M)$ is reduced and artinian. Let $\mathrm{Supp}\, L(M) = \{m_1, ..., m_n\}$ and set $I = m_1...m_n$. Consider the following descending chain of submodule of $L(M)$ such that

$$L(M) \supseteq IL(M) \supseteq I^2L(M) \supseteq ...$$

Since $L(M)$ is artinian, it follows that there exists $t \in \mathbb{N}$, such that

$$I^tL(M) = I^{t+1}L(M) = ...$$

Set $N = I^tL(M)$, therefore $N = IN$. We show that N is a radical submodule of $L(M)$. Let K be a maximal submodul of N. Then there exists maximal ideal $\mathfrak{m}$ of R such that $\frac{N}{K} \approx \frac{R}{\mathfrak{m}}$. This isomorphism shows that $\mathfrak{m} \in \mathrm{Supp}\, L(M)$ and so there is a $1 \leqslant i \leqslant n$, such that $\mathfrak{m} = \mathfrak{m}_i$. Now $\mathfrak{m} \in \mathrm{Supp}\, L(M)$ and $N = IN \subseteq \mathfrak{m}N \subseteq K \subseteq N$ and so $N = K$ which is a contradiction.
Therefore N has no maximal submodule and so N is a radical submodule of $L(M)$. Since $L(M)$ has no radical submodule then $N = 0$ and so we have the following

$$N = 0 \Longrightarrow IN = 0 \Longrightarrow 0 = IN = I.I^tL(M) = I^{t+1}L(M)$$

Then $L(M)$ is Noetherian.
converse follows from definition. □

Lemma 2.2. *Let R be a ring and M be an R-module. Then $P(M/P(M))$ $= 0$.*

Proof. Let $T/P(M)$ be a radical submodule of $M/P(M)$. We show that $T/P(M) = 0$. By definition $T/P(M)$ has no maximal submodule. Therefore $T/P(M)\otimes_R R/\mathfrak{m} = 0$. (Otherwise $T/(\mathfrak{m}T + P(M))$ is a vector space over the field $R/\mathfrak{m}$ and so has a maximal subspace, consequently $T/P(M)$ has a maximal submodule which is a contradiction). To show that $T/P(M) = 0$ it is enough to prove that T is a radical submodule of M. Let T be not a radical submodule of M, so by definition T has a maximal submodule . Let L be a maximal submodule of T. Hence $R/\mathfrak{m} \simeq T/L$ for some maximal ideal of R and we have $\mathfrak{m}T \subseteq L \neq T$. Therefore $T/\mathfrak{m}T \neq 0$. Consider the exact sequence

$$0 \to P(M) \to T \to T/P(M) \to 0$$

Which implies the following exact sequence:

$$0 \to P(M) \otimes_R R/\mathfrak{m} \to T \otimes_R R/\mathfrak{m} \to T/P(M) \otimes_R R/\mathfrak{m} = 0 \to 0.$$

The second exact sequence shows that $P(M) \otimes_R R/\mathfrak{m} \neq 0$.On the other hand $P(M) = \Sigma K$where K is a radical submodule of M. Now we have the following relation:

$$\mathfrak{m}P(M) = \mathfrak{m}\Sigma K = \Sigma \mathfrak{m}K = \Sigma K = P(M).$$

This shows that $\mathfrak{m}P(M) = P(M)$ and so $P(M) \otimes_R R/\mathfrak{m} = 0$ which is a contradiction. □

Theorem 2.3. *Let R be a ring,and M be an R-module. Let I , J be two maximal ideal of R. Then the R-module M/IJM is a reduced $R-$ module.*

Proof. First we show that $M/IJM \simeq M/IM \oplus M/JM$. To do this consider the exact sequence

$$0 \to R/IJ \to R/I \oplus R/J \to R/I + J = 0 \to 0,$$

which implies that $R/IJ = R/(I \cap J) \simeq R/I \oplus R/J$. Hence $R/IJ \otimes M \simeq R/I \otimes M \oplus R/J \otimes M = M/IM \oplus M/JM$. It is enough to show that $M/IM \oplus M/JM$ is a reduced $R-$module. Since $M/IM$and M/JM are vector space over the fields R/I and R/J respectively, it follows that $M = M/IM \oplus M/JM$ is a direct sum of simple $R-$modules. So let $M = M/IM \oplus M/JM = \oplus_{i \in X} S_i$, where S_i is a simple $R-$module. Now we assume that K be a radical submodule of $M = M/IM \oplus M/JM$. We show that $K = 0$. Suppose on the contrary $K \neq 0$. Hence $K = \oplus_{i \in Y \subseteq X} S_i$. But K has a maximal submodule which is a contradiction. □

Theorem 2.4. *Let R be a ring, and M be an R-module. If N be a submodule of M and $P(M/N) = 0$. Then $P(M) \subseteq N$.*

Proof. Suppose on the contrary that $P(M) \nsubseteq N$. So there is a radical submodule L of M such that $L \nsubseteq N$. Since $\frac{L}{N \cap L} \approx \frac{N+L}{N} \neq 0$ and L has no maximal submodule, it follows that the R-module $\frac{N+L}{N}$ is also has no maximal submodule. Therefore $\frac{N+L}{N}$ is a radical submodule of $\frac{M}{N}$ and by hypothesis is equal to zero submodule. In this case $L \subseteq N$, which is a contradiction. □

Theorem 2.5. *Let $\{M_i\}_{i=1}^{\infty}$ be a family of submodules of M over local ring $(R, \mathfrak{m})$ such that each M_i is finitely generated and M is semi-artinian R-module. Then $\oplus_{i=1}^{\infty} M_i = K$ is a reduced R-module.*

Proof. Let N be a radical submodule of K. We show that $N = 0$. Let $N \neq 0$ and $0 \neq x \in N$, then $x \in K$ and $x = x_1 + ... + x_t$ such that $x_i \in M_i$. Since M is semi-artinian module, it follows that each M_i is artinian and so for large $s \in \mathbb{N}$, we have $\mathfrak{m}^s M_i = 0$ for $i = 1, ..., t$. Since N is a radical submodule of K, it follows that $N = \mathfrak{m}N$.
(otherwise $\frac{N}{\mathfrak{m}N}$ is a non-zero vector space over field $\frac{R}{\mathfrak{m}}$ and so has a maximal subspace).
Now $N = \mathfrak{m}N$ and so for large s, we have $N = \mathfrak{m}^s N$ consequently $x \in \mathfrak{m}^s N$. Then there is an element $b \in \mathfrak{m}^s$ and an element $y \in N$ such that x=by. Also $y \in K$ and $y = y_1 + ... + y_n$ where $y_i \in M_i$. Therefore $by_i = 0$ for $i = 1, ..., n$ and consequently $by = 0$ which is a contradiction. □

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. Then the R-module $K = \oplus_{i=1}^{\infty}(0 :_M \mathfrak{m}^i)$ is a reduced .*

Proof. Let N be a radical submodule of K. We show that $N = 0$. Suppose on the contrary that $N \neq 0$ and $0 \neq x \in N$. Since N is a radical submodule of K, it follows that $N = \mathfrak{m}N$ (otherwise $\frac{N}{\mathfrak{m}N}$ is a non-zero vector space over field $\frac{R}{\mathfrak{m}}$ and so has a maximal subspace).
Now $x \in K$ and $x = x_{i_1} + ... + x_{i_t}$ where $x_{i_j} \in (0 :_M \mathfrak{m}^{i_j})$, therefore $x_{i_j}\mathfrak{m}^{i_j} = 0$. Then for large n, we have $x_{i_j}\mathfrak{m}^n = 0 \Longrightarrow x\mathfrak{m}^n = 0$.
On the other hand $N = \mathfrak{m}N$ and so $N = \mathfrak{m}^n N \Longrightarrow x \in \mathfrak{m}^n N \Longrightarrow x = by$; $b \in \mathfrak{m}^n$ and $y \in N$. By the above argument, $y\mathfrak{m}^n = 0$. Therefore $by = 0$ which is a contradiction. □

3. Artinian and multiplication modules

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local artinian principal ideal ring and $E(R/\mathfrak{m}^k)$ be an injective hull of $R/\mathfrak{m}^k$. Then $E(R/\mathfrak{m}^k) \approx R$.*

Proof. If $k = 1$ we show that $E(R/\mathfrak{m}) \approx R$. By [5, Lemma 6.6] , R is injective R-module and so is Gorenstein ring. Therefore by [1, Theorem 3.2.6] , $E(R/\mathfrak{m}) \approx R$.
Now let $k > 1$, in this case we have

$$Soc(R/\mathfrak{m}^k) = 0 :_{R/\mathfrak{m}^k} \mathfrak{m} = \frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k} \approx R/\mathfrak{m}$$

by[5, Proposition 3.17] , $E(Soc(R/\mathfrak{m}^k)) = E(R/\mathfrak{m}^k)$. Then by above relation we have

$$E(R/\mathfrak{m}^k) = E(Soc(R/\mathfrak{m}^k) = E(R/\mathfrak{m}) = R$$

□

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local artinian ring. Then the following are equivalent:*

(i) *R is Gorenstein ring.*
(ii) *$E(R/\mathfrak{m})$ is multiplication module.*
(iii) *for all non-zero ideals I and J ; $I \cap J \neq 0$.*

Proof. $(i \Rightarrow ii)$ Since R is Gorenstein ring, so by [1, Theorem 3.2.6] , $E(R/\mathfrak{m}) \approx R$. It follows that $E(R/\mathfrak{m})$ is cyclic module and so is multiplication module.
$(ii \Rightarrow iii)$ Let $E(R/\mathfrak{m})$ be a multiplication R-module. Let I and J be two non-zero ideales of R. We show that $I \cap J \neq 0$. Suppose on the contrary that $I \cap J = 0$.
Since $0 :_E I \leq E$ and $0 :_E J \leq E$, it follows that there exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R such that $0 :_E I = \mathfrak{a}E$ and $0 :_E J = \mathfrak{b}E$.
but E is injective and so we have $0 :_E I \cap J = 0 :_E I + 0 :_E J$. therefore we have

$$\mathfrak{a}E + \mathfrak{b}E = 0 :_E I \cap J = 0 :_E 0 = E \Longrightarrow (\mathfrak{a} + \mathfrak{b})E = E = RE$$

Since E is multiplication and faithfull, it follows from [2, Theorem 3.1] that $\mathfrak{a} + \mathfrak{b} = R$. On the other hand $\mathfrak{a} \subset \mathfrak{m}$ and $\mathfrak{b} \subset \mathfrak{m}$. (otherwise if $\mathfrak{a} \nsubseteq \mathfrak{m}$, then $\mathfrak{a} = R$ and so

$$0 :_E I = \mathfrak{a}E = RE = E \Longrightarrow IE = 0 \Longrightarrow I \subseteq 0 :_R E = 0$$

Consequently $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{m} \Longrightarrow R = \mathfrak{m}$ which is a contradiction).
$(iii \Rightarrow i)$ By [1, Theorem 3.2.10], it is enough to show that $r(R) = 1$. Suppose on the contrary that $r(R) > 1$. Since $r(R) = \dim_K \operatorname{Hom}(K, R)$, it follows that there exist subspaces U and V of a vector space $Hom_R(K, R)$ such that $\operatorname{Hom}_R(K, R) = U \oplus V$. In this case $U \cap V = 0$. But

$\mathrm{Hom}_R(K, R)$ is isomorphic with a submodule of R and so R has ideals I and J such that $I \cap J = 0$, which is a contradiction. $\square$

Theorem 3.3. *Let R be an artinian ring and p and q be prime ideals of R such that $p \neq q$. Then $R_p \otimes_R R_q = 0$*

Proof. Let $R_p \otimes_R R_q \neq 0$, Then $Supp_R(R_p \otimes_R R_q) \neq \emptyset$. Let $p' \in Supp_R(R_p \otimes_R R_q)$ so $(R_p)_{p'} \otimes_{R_{p'}} (R_q)_{p'} \neq 0$. It follows that $(R_p)_{p'} \neq 0$ and $(R_q)_{p'} \neq 0$. In this case we have $p' \subseteq q$ and $p' \subseteq p$. (otherwise if $p' \nsubseteq p \Longrightarrow \exists t \in p' \backslash p$ and $R_p \xrightarrow{t} R_p$ is an isomorphism, consequently $(R_p)_{p'} \xrightarrow{t/1} (R_p)_{p'}$ is an isomorphism. Therefore $(R_p)_{p'} = t/1(R_p)_{p'}$ and so $t/1$ is invertible. On the other hand $t/1 \in p'R_{p'}$ which is a contradiction). Since $p' \subseteq q$ and $p' \subseteq p$, it follows that $p' = p = q$. $\square$

Theorem 3.4. *Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R and M be an R-module . If $Hom(R/\mathfrak{a}, M)$ is artinian , then $Hom(R/\mathfrak{a}^n, M)$ for all $n \in \mathbb{N}$ is artinian R-module.*

Proof. We use induction on n. The case $n = 1$ is true by hypothesis. Now, let $n > 1$ and suppose that the result has been proved for $n - 1$. We know that

$$\mathrm{Hom}_R(R/\mathfrak{a}^n, M) \simeq 0 :_M \mathfrak{a}^n.$$

Consider the exact sequence

$$0 \to 0 :_M \mathfrak{a} \to 0 :_M \mathfrak{a}^n \xrightarrow{f} a_1(0 :_M \mathfrak{a}^n) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^n) \to 0,$$

where $\mathfrak{a} = (a_1, \ldots, a_t)$ and f is defined by $f(x) = (a_1 x, \ldots, a_t x)$. Clearly, $a_i(0 :_M \mathfrak{a}^n)$ is a submodule of $0 :_M \mathfrak{a}^{n-1}$ for all $i = 1, 2, \ldots, t$. Therefore, by induction hypothesis, $a_i(0 :_M \mathfrak{a}^n)$ is Artinian for all $i = 1, 2, \ldots, t$. Thus $a_1(0 :_M \mathfrak{a}^n) \oplus \cdots \oplus a_t(0 :_M \mathfrak{a}^n)$ is Artinian. Hence $0 :_M \mathfrak{a}^n$ is Artinian. $\square$

Theorem 3.5. *Let M be a non-semi-artinian R-module over noetherian local ring $(R, \mathfrak{m})$. Then there exists a submodule N of M such that N is isomorphic to $\frac{R}{P}$ for some*
$\mathfrak{m} \neq P \in Spec(R)$.

Proof. Since M be a non-semi-artinian R-module, it follows that there is a non-zero proper submodule K of M such that K not contains any minimal submodule. In this case $Ass(M) \nsubseteq \{\mathfrak{m}\}$, (otherwise there is a $0 \neq x \in M$, such that $\mathfrak{m} = 0 :_R x$ and $Rx \approx \frac{R}{0:_R x} = \frac{R}{\mathfrak{m}}$ therefore there is monomorphism $g : \frac{R}{\mathfrak{m}} \longrightarrow M$. Now we have $\emptyset \neq Ass(K) \subseteq Ass(M) =$

$\{\mathfrak{m}\} \Longrightarrow Ass(K) = \{\mathfrak{m}\}$ and so $\frac{R}{\mathfrak{m}} \approx g(\frac{R}{\mathfrak{m}})$ is a minimal submodule of K which is a contradiction).
Therefore $Ass(M) \nsubseteq \{\mathfrak{m}\}$ and then there exists $P \in Ass(M)$ such that $P \neq \mathfrak{m}$. Now there is a monomorphism $h : \frac{R}{P} \longrightarrow M$ and $\frac{R}{P} \approx h(\frac{R}{P}) := N \leqslant M$. □

Theorem 3.6. *Let M be an R-module and N a submodule of M such that $L(\frac{M}{N}) = 0$. Then every artinian submodule of M is contained in N.*

Proof. Suppose on the contrary that there is an artinian submodule L of M such that $L \nsubseteq N$. Since $\frac{L}{N \cap L} \approx \frac{N+L}{N} \neq 0$ and L is artinian, it follows that $\frac{N+L}{N}$ is artinian submodule of $\frac{M}{N}$ and so by hypothesis is equal to the zero submodule of $\frac{M}{N}$. In this case $\frac{N+L}{N} = 0$ and so $L \subseteq M$, which is a contradiction. □

Theorem 3.7. *Let M be an R-module and N be a submodule of M such that $\frac{M}{N}$ is artinian module over noetherian ring R. Then for every ideal I of R and for any positive integer n , the R-module $\frac{I^nM}{I^nN}$ is artinian R-module.*

Proof. Let $I^n =< a_1, ..., a_t >$ for some $a_i \in R$. Now we define the R-homomorphism

$$f : (\frac{M}{N})^t \longrightarrow \frac{I^nM}{I^nN}$$
$$f(x_1 + N, ..., x_t + N) = a_1x_1 + ... + a_tx_t + I^nN$$

It is clear that f is an epimorphism and so $\frac{I^nM}{I^nN}$ is artinian R-module. □

Corollary 3.8. *Let M be an R-module and I be an ideal of noetherian ring R such that $\frac{M}{IM}$ is artinian. Then for each positive integer n , the R-module $\frac{M}{I^nM}$ is artinian.*

Proof. By induction on n. If $n = 1$, by hypothesis $\frac{M}{IM}$ is artinian. Now suppose that the result has been proved for $n - 1$ and $\frac{M}{I^{n-1}M}$ be an artinian module. By theorem 3.8 and hypothesis the R-module $\frac{I^{n-1}M}{I^nM}$ is artinian. Therefore the exact sequence

$$0 \longrightarrow \frac{I^{n-1}M}{I^nM} \longrightarrow \frac{M}{I^nM} \longrightarrow \frac{M}{I^{n-1}M} \longrightarrow 0$$

Shows that the R-module $\frac{M}{I^nM}$ is also artinian. □

Corollary 3.9. *Let R be an artinian ring with radical jacobson $J = J(R)$ and M be a non- artinian R- module . Then $\frac{M}{JM}$ is not artinian R-module.*

Proof. Since R is artinian, it follows that there exists $n \in \mathbb{N}$ such that $J^n = 0$. Suppose on the contrary that $\frac{M}{JM}$ is artinian. By the argument as in Corollary 3.9 we show that the R-module M/J^nM is artinian, which is a contradiction. □

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