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COMMON FIXED POINT THEOREMS FOR COMMUTATIVE AND WEAKLY COMPATIBLE MAPPINGS IN DIGITAL METRIC SPACE

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Abstract. In this paper, a common fixed point theorem for four commutative mappings in the setting of digital metric space is proved with a supportive example. Also, we established some common fixed point theorems for weakly compatible mappings that satisfy certain contractive conditions in digital metric space.

Key Words: Fixed point, Digital Metric Space, Digital Image, Commutative mapping, Weakly compatible mapping.
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1. Introduction

In the fixed points theory, there exist many generalities of metric space and one of them is the digital metric space introduced by Ozgur Ege and Ismet Karaca [7]. The concept of digital metric space is related to digital topology in which we study the topological and geometrical digital properties of an image. An Image is used as an object in computer graphic design and other computer-related business works. In this type of work, a digital image is taken as a set of arranged points called pixels or voxels. In digital topology, we study these points and the adjacency relation between them. Rosenfeld [9] was the first to use digital topology as an apparatus and studied the properties of almost fixed points of a digital image. Later, Boxer [5, 6] gives the topological concept in

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the digital form. Based on this concept Ozgur Ege and Ismet Karaca [7] established digital metric space in 2015 and proved the "Banach Contraction Principle" and several other fixed-point results in this space. In the whole article, *DMS* illustrates digital metric space.

The study of common fixed points for different types of maps has always been a very interesting area in the theory of fixed points. Jungck [2] was the first who introduced commutative mappings to complete metric space in 1976 and by using the properties of these mapping he proved some common fixed point results. After that, many authors generalize and extend many results for commutative mapping with different contractive conditions in several ways. In 1982, Sessa [12] define weakly commutative mappings. These mappings are more general than commutative mappings that every commutative mapping is weakly commutative, but the converse may not be true. Again, G. Jungck [3, 4] defines compatible and weakly compatible mappings that are more general in nature and give fixed point results using their variants. Recently, Asha Rani et al. [1] introduced weakly commutative and commutative mappings to digital metric space, and Sunjay Kumar et al. [11], Sumitra Dalal [10], and Rashmi Rani [8] present some results for weakly compatible, compatible and commutative maps in *DMS*. With the motivation in this paper, a common fixed point theorem for four commutative mappings in *DMS* is presented. This result generalizes and extends the result of Rashmi Rani [8]. Also, we established some results for weakly compatible mappings that satisfy certain contractive conditions on *DMS*. Before we prove our main results, the following definitions are needed.

2. Preliminaries

Let $F \subseteq \mathbb{Z}^n$, n $\in \mathbb{N}$ where $\mathbb{Z}^n$ is a lattice point set in the Euclidean n - dimensional space and $(F,\ \Upsilon)$ represent a digital image, with Υ -adjacency relation between the members of F and $(F, \Phi,\ \Upsilon)$ represent a *DMS*, where $(F,\ \Phi)$ is a metric space.

Definition 2.1. [6] "Let $l,\ n$ be two positive integers, where $1 \leq l \leq n$ and g, h are two distinct points,

$$g = (g_1,\ g_2, ...g_n)\,, h = (h_1,\ h_2, ...h_n) \in \mathbb{Z}^n.$$

Then the points g and h are said to be Υ_1- adjacent if there are at most l indices i such that $|g_i - h_i| = 1$ and for all other indices j, $|g_j - h_j| \neq 1,\ g_j = \mathrm{h}_j$."

Definition 2.2. [6] Let $\kappa \in \mathbb{Z}^n$, then the set –

$$N_{\Upsilon}(\kappa) = \{ \ \sigma \mid \sigma \text{ is } \Upsilon - \text{adjacent to } \kappa \ \}$$

represent the Υ – neighbourhood of κ for $n \in \{1, 2, 3\}$. Where $\Upsilon \in \{2, 4, 6, 8, 18, 26\}$.

Definition 2.3. [6] Let δ, $\sigma \in \mathbb{Z}$ where $\delta < \sigma$, then the digital interval is -

$$[\delta, \ \sigma]_{\alpha} = \{\alpha \in \mathbb{Z} \mid \delta \leq \alpha \leq \sigma \ \}.$$

Definition 2.4. [7] "The digital image $(F, \ \Upsilon) \ \subseteq \mathbb{Z}^n$ is called Υ – connected if and only if for every pair of different points g, $h \in F$, there is a set $\{g_0, \ g_1, ...g_s\}$ of points of digital image $(F, \ \Upsilon)$, such that $g = g_0$, $h = g_s$, and g_e and g_{e+1} are Υ -neighbours where $e = 0,1,2,\ldots\ldots.$ s-1."

Definition 2.5. [7] Let K: $F \rightarrow$ K is a function and $(F, \ \Upsilon_0) \subset \mathbb{Z}^{n}{}_0$, (K, Υ_1) $\subset \mathbb{Z}^{n}{}_1$ are two digital images. Then –

(i) K is (Υ_0, Υ_1) - continuous if there exists Υ_0 - connected subset σ of F, for every $K(\sigma)$, Υ_1- connected subset of K.
(ii) K is (Υ_0, Υ_1) -continuous if for every Υ_0 - adjacent point $\{\sigma_0, \ \sigma_1\}$ of F, either $K(\sigma_0) = K(\sigma_1)$ or $K(\sigma_0)$ and $K(\sigma_1)$ are Υ_1 -adjacent in K.
(iii) K is said to be (Υ_0, Υ_1) - isomorphism, if K is (Υ_0, Υ_1) -continuous bijective and K^{-1} is (Υ_0, Υ_1)- continuous, also it is denoted by $F \cong \mathrm{K}_{(\Upsilon_0, \Upsilon_1)}$.

Definition 2.6. [7] Let a $(2, \ \Upsilon)$ continuous function K: $[0, \sigma]_{\ z} \rightarrow F$ s.t. $K\ (0) = \alpha$ and $K(\sigma) = \beta$. Then in the digital image $(\ F, \ \Upsilon)$, it is called a digital Υ -path from α to β.

Definition 2.7. [9] Let K: $(F, \ \ \Upsilon\ ) \rightarrow (F, \ \ \Upsilon)$ be a $(\Upsilon, \ \ \Upsilon)$ - continuous function on a digital image $(\ F, \ \Upsilon)$, then we said that the property of fixed point satisfied by the digital image $(\ F, \ \Upsilon\ )$ if for every $(\Upsilon, \ \Upsilon)$ - continuous function K: $F \rightarrow F$ there exists $\alpha \in F$ such that $K\ (\alpha) = \alpha$.

Definition 2.8. [7] "Let $\{u_n\}$ is a sequence in digital metric space $(F, \Phi, \ \Upsilon)$, then the sequence $\{u_n\}$ is called-

(i) Cauchy sequence if and only if there exists $\varrho \in$ N such that, $\Phi(u_{n,} \ u_m) < \epsilon$, $\forall \ n, \ m > \varrho$.

(ii) Converge to a limit point $\ell \in F$ if for every $\epsilon > 0$, there exists $\varrho \in N$ such that for all $n > \varrho$, $\Phi(u_{n,}\ \Upsilon) < \epsilon$."

Theorem 2.9. [7] *"A digital metric space $(F,\ \Phi,\ \Upsilon)$ is complete."*

Definition 2.10. [7] Let K: $(F,\ \Phi,\ \Upsilon) \to (F,\ \Phi,\ \Upsilon)$ be a self-map. Then K is called a digital contraction if, for all $u,\ \sigma \in F$ there exist $\tau \in [0,\ 1)$ such that,

$$\Phi(K(u),\ K(\sigma)) \leq \tau\ \Phi(u,\ \sigma).$$

Proposition 2.11. [7] *"Every digital contraction map K: $(F,\Phi,\ \Upsilon) \to (F,\ \Phi,\ \Upsilon)$ is digitally Υ - continuous."*

Definition 2.12. [10] *Let J, K: $F \to F$* are two self- mappings on $(F,\ \Phi,\ \Upsilon)$. Then the point $\sigma \in\ F$ is said to be a coincidence point of J and K if $J(\sigma) = K(\sigma)$. Furthermore, if $J\ (\sigma) = K\ (\sigma) = \eta$ then η is said to be a point of coincidence for mappings J and K.

Definition 2.13. [1] Let J, K: $(F,\ \Phi,\ \Upsilon) \to (F,\ \Phi,\ \Upsilon)$ are two mappings defined on the digital metric space $(F,\ \Phi,\ \Upsilon)$. Then these mappings are called commutative mappings if $J(K(\sigma)) = K(J(\sigma))$, $\forall\ \sigma \in F$.

Definition 2.14. [10] Let J, K: $(F,\ \Phi,\ \Upsilon) \to (F,\ \Phi,\ \Upsilon)$ are two mappings defined on the digital metric space $(F,\ \Phi,\ \Upsilon)$. If mappings J and K commute at coincidence points, then they are called weakly compatible mappings that is if $J(\sigma) = K(\sigma)$, $\forall\ \sigma \in F$ *then* $J(K(\sigma)) = K(J(\sigma))$, $\forall\ \sigma \in F$.

Proposition 2.15. *Let J, K: $F \to F$ are two weakly compatible maps on F and if a point η is a unique point of coincidence of mappings J and K i.e., $J(\sigma) = K(\sigma) = \eta$ then η is the unique common fixed point of the mappings J and K.*

Proof. Since J, K: $F \to F$ are two weakly compatible mappings and $J(\sigma) = K(\sigma) = \eta$. Then we have $J(\sigma) = J(K(\sigma)) = K(J(\sigma)) = K(\sigma)$ i.e., $J(\sigma) = K(\sigma)$ be a point of coincidence of J and K. But we have the only point of coincidence of J and K is η. Hence $J(\eta) = K(\eta) = \eta$. Let α is another point of coincidence of J and K i.e., $J(\alpha) = K(\alpha) = \alpha$. Then by uniqueness, we get, $\eta = \alpha$. Therefore, η is the unique common fixed point of J and K. $\square$

Remark 2.16. The mappings which are commutative are evidently weakly compatible, but the converse may not be true.

Example 2.17. *Let $F = [1, \infty)$ and Φ is a usual metric on F. Let J, K: $F \to F$ are two mappings on F defined by $J(u) = 2u$-1 and $K(u) = u^2$, $\forall\ u \in F$. Then we can see that J and K are weakly compatible mappings. Since they commute at the coincidence point 1 that is, for $J(1) = K(1)$, we have, $J\ (K(1)) = K\ (J(1))$. But not commutative because $J(K(\sigma)) \neq K(J(\sigma))$, $\forall\ \sigma \in F$.*

3. Main Results

Theorem 3.1. *Let $(F,\ \Phi,\ \Upsilon)$ represent a complete DMS, where Υ is an adjacency and Φ be a usual Euclidean metric on $\mathbb{Z}^n$. let J, K, L, M: $F \to F$ are four mappings such that $J(F) \subseteq M(F)$ and $K(F) \subseteq L(F)$ satisfy the following,*

$$\Phi(Ju,\ Kq) \leq \xi\,\Phi(Lu,\ Mq),\ \forall\ u,\ q \in F \text{ and } 0 < \xi < 1 \tag{3.1}$$

If L and M are continuous mappings and $\{J,\ L\}$ and $\{K,\ M\}$ are pairs of commutative mappings then there exists a unique common fixed point in F for all four mappings J, K, L, and M.

Proof. Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq M(F)$, let $u_1 \in F$ be chosen such that $Mu_1 = Ju_0$, and as $K(F) \subseteq L(F)$, let $u_2 \in F$ be chosen such that $Lu_2 = Ku_1$. So, in general, we construct sequences $\{u_n\}$ and $\{q_n\}$ in F such that,

$$\begin{aligned} q_{2n} &= Ju_{2n} = Mu_{2n+1} \\ q_{2n+1} &= Ku_{2n+1} = Lu_{2n+2}, \qquad \forall\ n = 0,\ 1,\ 2\ldots.. \end{aligned}$$

Now, by inequality (3.1), we have –

$$\begin{aligned} \Phi(q_{2n},\ q_{2n+1}) &= \Phi(Ju_{2n},\ Ku_{2n+1}) \\ &\leq \xi\,\Phi(Lu_{2n},\ Mu_{2n+1}) \\ &\leq \xi\,\Phi(q_{2n-1},\ q_{2n}) \end{aligned}$$

Similarly, it can be shown that,

$$\Phi(q_{2n+1},\ q_{2n+2}) \leq \xi\,\Phi(q_{2n},\ q_{2n+1})$$

Therefore, for all n, we have,

$$\Phi(q_{n+1},\ q_{n+2}) \leq \xi\,\Phi(q_n,\ q_{n+1}) \leq \ldots\ldots\ldots\ldots \leq \xi^{n+1}\ \Phi(q_0,\ q_1)$$

Now, for $n > m$ by, the triangle inequality property, we have-

$$\begin{aligned}\Phi(q_n, q_m) &\leq \Phi(q_n, q_{n+1}) + \Phi(q_{n+1}, q_{n+2}) + \ldots\ldots + \Phi(q_{m-1}, q_m)\\ &\leq [\xi^n + \xi^{n+1} + \ldots\ldots\ldots\ldots + \xi^{m-1}]\ \Phi(q_1, q_0)\\ &\leq \frac{\xi^m}{1-\xi}\Phi(q_1, q_0)\end{aligned}$$

$$\Phi(q_n, q_m) \leq \frac{\xi^m}{1-\xi}\Phi(q_1, q_0)$$

Since $0 < \xi < 1$, $\frac{\xi^m}{1-\xi}\ \Phi(q_1, q_0) \to 0$ as $n, m \to \infty$. This leads to the conclusion that $\{q_n\}$ is a Cauchy sequence. Also, we have (F, Φ, Υ) is a complete space. Thus, there must be a point $\sigma \in F$ such that,

$$\lim_{n\to\infty} Ju_{2n} = \lim_{n\to\infty} Mu_{2n+1} = \lim_{n\to\infty} Ku_{2n+1} = \lim_{n\to\infty} Lu_{2n+2} = \sigma \qquad (3.2)$$

Further, L is a continuous mapping, and $\{J, L\}$ is a pair of commutative mappings, then we have-

$$\lim_{n\to\infty} L^2u_{2n+2} = L\sigma \ \&\ \lim_{n\to\infty} LJu_{2n} = \lim_{n\to\infty} JLu_{2n} = L\sigma \qquad (3.3)$$

Now, put $u = Lu_{2n,}\ q = u_{2n+2}$ in (3.1) and using (3.2) and (3.3) we get,

$$\Phi(LJu_{2n}, Ku_{2n+1}) \leq \xi\Phi(L^2u_{2n}, Mu_{2n+1})$$

Taking the limit as $n \to \infty$, we obtain,

$$\Phi(L\sigma, \sigma) \leq \xi\Phi(L\sigma, \sigma)$$

Here, $0 < \xi < 1$ it follows that, $L\sigma = \sigma$.

Similarly, since M is a continuous mapping and $\{K, M\}$ is a pair of commutative mappings, then we have-

$$\lim_{n\to\infty} M^2u_{2n+1} = M\sigma \ \&\ \lim_{n\to\infty} MKu_{2n+1} = \lim_{n\to\infty} KMu_{2n+1} = M\sigma \quad (3.4)$$

Now, put $u = u_{2n,}\ q = Mu_{2n+2}$ in (3.1) and using (3.2) and (3.4) we get,

$$\Phi(Ju_{2n}, KMu_{2n+1}) \leq \xi\Phi(Lu_{2n}, M^2u_{2n+1})$$

Taking the limit as $n \to \infty$, we obtain,

$$\Phi(\sigma, M\sigma) \leq \xi\Phi(\sigma, M\sigma)$$

Since $0 < \xi < 1$ it follows that, $M\sigma = \sigma$

Further, take $u = \sigma$ and $q = u_{2n+1}$ in inequality (3.1) we have,

$$\Phi(J\sigma, Ku_{2n+1}) \leq \xi\Phi(L\sigma, Mu_{2n+1})$$

$$\Phi(J\sigma,\, \sigma) \leq \xi\Phi\ (L\sigma,\, \sigma)$$

$$\Phi(J\sigma,\, \sigma) \leq \xi\Phi\ (\sigma,\, \sigma) \qquad (\because L\sigma = \sigma)$$

Implies that, $\Phi(J\sigma,\, \sigma) = 0$ i.e., $J\sigma = \sigma$.
Again, from inequality (3.1) we have,

$$\Phi(J\sigma,\, K\sigma) \leq \xi\Phi(L\sigma,\, M\sigma)$$

$$\Phi(J\sigma,\, K\sigma) \leq \xi\Phi(\sigma,\, \sigma), \qquad (\because L\sigma = \sigma \text{ and } M\sigma = \sigma)$$

Hence, $\Phi\ (J\sigma,\, K\sigma) = 0$ i.e., $J\sigma = K\sigma$.
Thus, we have proved that,

$$J\sigma = K\sigma = L\sigma = M\sigma = \sigma.$$

That means σ is a common fixed point of all four mappings J, K, L, and M.

Uniqueness: let σ_1 is another common fixed point of these mappings that is,

$$J\sigma_1 = K\sigma_1 = L\sigma_1 = M\sigma_1 = \sigma_1$$

Then, we have –

$$\begin{aligned} \Phi(\sigma,\, \sigma_1) &= \Phi(J\sigma,\, K\sigma_1) \\ &\leq \xi\Phi(L\sigma,\, M\sigma_1) \\ &\leq \xi\Phi(\sigma, \sigma_1) \\ &< \Phi(\sigma,\, \sigma_1), \qquad (\because \xi < 1) \end{aligned}$$

This is a contradiction. Hence $\sigma = \sigma_1$. Therefore, all four mappings J, K, L, and M have a unique common fixed point. □

Example 3.2. Let $(F,\, \Phi,\, \Upsilon)$ be a complete *DMS* with digital metric $\Phi(u,\, q) = |u - q|$ and J, K, L, and M are four mappings on F defined by,

$$J(u) = \tfrac{1}{2}\ ,\ \ K(u) = \tfrac{1}{2}\ ,\ L(u) = \tfrac{(u+1)}{3} \text{ and } M(u) = \tfrac{(2u+1)}{4},\ \forall\ u \in F$$

Then, it is easy to see that all the requirements and conditions which are given in Theorem 3.1 hold, and a unique common fixed point exists for all four mappings at $\sigma = \frac{1}{2}$ such that,

$$J\ \left(\tfrac{1}{2}\right) = K\ \left(\tfrac{1}{2}\right) = L\ \left(\tfrac{1}{2}\right) = M\ \left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

Remark 3.3. This theorem is an extension of the theorem of Rashmi Rani [11]. The result of Rashmi Rani [11] becomes a special case of this theorem if we take $J = K$ and $L = M$.

Now, we establish some "digital common fixed point theorems" in *DMS* with weakly compatible mappings which satisfy certain contractive conditions.

Theorem 3.4. *Let $(F,\ \Phi,\ \Upsilon)$ represent a complete DMS, where Υ is an adjacency and Φ be a usual Euclidean metric on $\mathbb{Z}^n$. let J, K: $F \to F$ are two self-mappings such that $J(F) \subseteq K(F)$ satisfying the following,*

$$\Phi(Ju,\ Jq) \leq \xi\,\Phi(Ku,\ Kq),\ \forall\ u,\ q \in F \text{ and } 0 < \xi < 1 \tag{3.5}$$

If $K(F) \subseteq F$ is complete and $\{J,\ K\}$ is a pair of weakly compatible mappings then there exists a unique common fixed point in F for mappings J and K.

Proof. Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq K(F)$, let $u_1 \in F$ be chosen such that $q_0 = Ju_0 = Ku_1$. Continuing this procedure having chosen $u_n \in F$, we chose $u_{n+1} \in F$ such that,

$$q_n = Ju_n = Ku_{n+1}, \qquad \forall\ n = 0,\ 1,\ 2,\ldots$$

Now, by inequality (3.5), we have –

$$\begin{aligned}\Phi(q_n,\ q_{n-1}) &= \Phi(Ju_n,\ Ju_{n-1})\\ &\leq \xi\,\Phi(Ku_n,\ Ku_{n-1})\\ &\leq \xi\,\Phi(q_{n-1}, q_{n-2})\end{aligned}$$

Implies that-

$$\Phi(q_n,\ q_{n-1}) \leq \xi\,\Phi(q_{n-1},\ q_{n-2}) \leq \ldots\ldots\ldots\ldots \leq \xi^{n-1}\ \Phi(q_1,\ q_0)$$

Now, for $n > m$ by, the triangle inequality property, we have-

$$\begin{aligned}\Phi(q_n,\ q_m) &\leq \Phi(q_n,\ q_{n-1}) + \Phi(q_{n-1},\ q_{n-2}) + \ldots\ldots\ldots + \Phi(q_{m+1},\ q_m)\\ &\leq [\xi^{n-1} + \xi^{n-2} + \ldots\ldots\ldots\ldots + \xi^m]\ \Phi(q_1,\ q_0)\\ &\leq \frac{\xi^m}{1-\xi}\ \Phi(q_1, q_0)\end{aligned}$$

$$\Phi(q_n, q_m) \leq \tfrac{\xi^n}{1-\xi}\,\Phi(q_1, q_0)$$

Since $0 < \xi < 1$, $\frac{\xi^m}{1-\xi}\Phi(q_1, q_0) \to 0$ as $n, m \to \infty$. This leads to the conclusion that $\{q_n\}$ is a Cauchy sequence. Also, we have $K(F) \subseteq F$ is complete. Then there must be a point σ in $K(F)$ such that $q_n \to \sigma$ as $n \to \infty$. Subsequently, we can find $\eta \in F$ such that $K(\eta) = \sigma$. Further, from inequality (3.5), we have,

$$\begin{aligned}\Phi(Ku_n, J\eta) &= \Phi(Ju_{n-1}, J\eta)\\ &\leq \xi\Phi(Ku_{n-1}, K\eta)\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain,

$$\Phi(K\eta, J\eta) \leq \xi\Phi(K\eta, K\eta)$$

Implies that, $\Phi(K\eta, J\eta) = 0$.
Therefore, we get $K\eta = J\eta$. Hence, $K\eta = J\eta = \sigma$ that is σ is the point of coincidence of mappings J and K. We will now show that the uniqueness of point of coincidence σ. For this, let $\sigma_1 \in F$ is another point of coincidence of mappings J and K such that, $K\eta_1 = J\eta_1 = \sigma_1$. Now,

$$\begin{aligned}\Phi(K\eta_1, K\eta) &= \Phi(J\eta_1, J\eta) \qquad (\because J\eta_1 = K\eta_1 \text{ and } J\eta = K\eta)\\ &\leq \xi\Phi(K\eta_1, K\eta)\end{aligned}$$

As $0 < \xi < 1$, we get $\Phi(K\eta_1, K\eta) = 0$ i.e., $K\eta_1 = K\eta$. This implies,

$$K\eta_1 = K\eta = J\eta = J\eta_1 = \sigma = \sigma_1$$

Hence, by Proposition 2.15, it is clear that mappings J and K have a unique common fixed point. □

Theorem 3.5. *Let (F, Φ, Υ) represent a complete DMS, where Υ is an adjacency and Φ be a usual Euclidean metric on $\mathbb{Z}^n$. let J, K: $F \to F$ are two self-mappings such that $J(F) \subseteq K(F)$ satisfying the following,*

$$\Phi(Ju,Jq) \leq \xi(\Phi(Ju, Ku)+\Phi(Jq, Kq)), \forall\ u, q \in F \text{ and } \xi\in(0, \tfrac{1}{2}) \quad (3.6)$$

If $K(F) \subseteq F$ is complete and $\{J, K\}$ is a pair of weakly compatible mappings then there exists a unique common fixed point in F for mappings J and K.

Proof. Let $u_0 \in F$ be an arbitrary point. Since $J(F) \subseteq K(F)$, let $u_1 \in F$ be chosen such that $q_0 = Ju_0 = Ku_1$. Continuing this procedure having chosen $u_n \in F$, we chose $u_{n+1} \in F$ such that,

$$q_n = Ju_n = Ku_{n+1}, \qquad \forall\ n = 0,\ 1,\ 2, \ldots\ldots$$

Now, by inequality (3.6), we have -

$$\begin{aligned} \Phi(q_n,\ q_{n-1}) &= \Phi\ (Ju_n,\ Ju_{n-1}) \\ &\leq \xi(\Phi(Ju_n,\ Ku_n) + \Phi(Ju_{n-1},\ Ku_{n-1})) \\ &\leq \xi(\Phi(q_n, q_{n-1}) + \Phi(q_{n-1}, q_{n-2})) \end{aligned}$$

i.e., $\quad \Phi(q_n,\ q_{n-1}) \leq \frac{\xi}{1-\xi}\ \Phi(q_{n-1},\ q_{n-2})$

or, $\quad \Phi(q_n,\ q_{n-1}) \leq w\,\Phi(q_{n-1},\ q_{n-2}), \quad$ where $w = \frac{\xi}{1-\xi} \in (0,1)$

Implies that,

$$\Phi(q_n,\ q_{n-1}) \leq w\,\Phi(q_{n-1},\ q_{n-2}) \leq \ldots\ldots\ldots\ldots \leq w^{n-1}\,\Phi(q_1,\ q_0)$$

Now, for $n > m$ by, the triangle inequality property, we have -

$$\begin{aligned} \Phi(q_n,\ q_m) &\leq \Phi(q_n,\ q_{n-1}) + \Phi(q_{n-1},\ q_{n-2}) + \ldots\ldots\ldots + \Phi(q_{m+1},\ q_m) \\ &\leq [w^{n-1} + w^{n-2} + \ldots\ldots\ldots\ldots + w^m]\ \Phi(q_1,\ q_0) \\ &\leq \frac{w^m}{1-w}\Phi(q_1, q_0) \end{aligned}$$

$$\Phi(q_n, q_m) \leq \frac{w^n}{1-w}\ \Phi(q_1, q_0)$$

Since $0 < w < 1$, $\frac{w^m}{1-w}\ \Phi(q_1,\ q_0) \to 0$ as $n,\ m \to \infty$. This leads to the conclusion that $\{q_n\}$ is a Cauchy sequence. Also, we have $K(F) \subseteq F$ is complete. Then there must be a point σ in $K(F)$ such that $q_n \to \sigma$ as $n \to \infty$. Subsequently, we can find $\eta \in F$ such that $K(\eta) = \sigma$. Further, from inequality (3.6), we have,

$$\begin{aligned} \Phi(Ku_n,\ J\eta) &= \Phi(Ju_{n-1},\ J\eta) \\ &\leq \xi(\Phi(Ju_{n-1},\ Ku_{n-1}) + \Phi(J\eta,\ K\eta)) \\ &\leq \xi(\Phi(Ku_n,\ Ku_{n-1}) + \Phi(J\eta,\ K\eta)) \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain,

$$\Phi(K\eta,\ J\eta) \leq \xi(\Phi(K\eta,\ K\eta) + \Phi(J\eta,\ K\eta))$$

Therefore, $\Phi(K\eta,\ J\eta) \leq \xi\Phi(J\eta,\ K\eta)$

Since $0 < \xi < 1$, we get $\Phi(K\eta,\ J\eta) = 0$. Hence, $K\eta = J\eta = \sigma$ that is, σ is a point of coincidence of J and K. We will now show that the uniqueness of the point of coincidence σ. For this, let $\sigma_1 \in F$ is another point of coincidence of mappings J and K such that, $K\eta_1 = J\eta_1 = \sigma_1$. Now,

$$\begin{aligned}\Phi(K\eta_1,\, K\eta) &= \Phi(J\eta_1,\, J\eta)\\ &\leq \xi(\Phi(J\eta_1,\, K\eta_1) + \Phi(J\eta,\, K\eta)\\ &\leq \xi(0+0) \qquad (\because K\eta_1 = J\eta_1 \text{ and } K\eta = J\eta)\end{aligned}$$

Which gives $\Phi(K\eta_1,\, K\eta) = 0$ i.e., $K\eta_1 = K\eta$. This implies,

$$K\eta_1 = K\eta = J\eta = J\eta_1 = \sigma = \sigma_1$$

Hence, by proposition 2.15, it is clear that mappings J and K have a unique common fixed point. □

4. Conclusion

This paper is aimed at introducing the perception of weakly compatible and commutative mappings in digital metric space and by using these mappings and their variants, establish some digital common fixed point theorems. Our results broaden and extend many prevailing known results in the literature. These results are applications in fixed point theory. Which can be used to compress digital images and can be beneficial in processing and redefining image storage.

References

[1] A. Rani, K. Jyoti and A. Rani, *Common fixed point theorems in digital metric spaces*, Int. j. sci. eng. Res, **7** (2016), 1704-1716.

[2] G. Jungck, *Commuting mappings and fixed point*, Amer. Math. Monthly, **83** (1976), 261- 263.

[3] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math Math Sci., **9** (1986), 771-779.

[4] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math Math Sci., **11** (1988), 285-288.

[5] L. Boxer, *A classical construction for the digital fundamental group*, J. Math. Imaging V, **10** (1999), 51-62.

[6] L. Boxer, *digitally continuous functions,*, Pattern Recognit Lett., **15** (1994), 833-839.

[7] O. Ege and I. Karaca, *Banach fixed point theorem for digital images*, J. of Non. Sci. and Appli., **8** (2015), 237-245.

[8] R. Rani, *Common fixed point theorem in digital spaces using contraction mappings*, int. j. stat. appl. Math, **3** (2018), 140-142.

[9] Rosenfeld, *Digital topology*, Amer. Math. Monthly, **86** (1979), 76-87.

[10] S. Dalal, *Common fixed point results for weakly compatible map in digital metric spaces*, Sch J. Phys. Math. Stat., **4** (2017), 196-201.

[11] S. Kumar, D. Jain, O. Ege, D. Y. Shin and C. Park, *Commuting and compatible mappings in digital metric spaces*, J. Fixed Point Theory Appl., **22** (2020), 1-16.

[12] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math, **32** (1982), 149-153.

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