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HYPER JK-ALGEBRAS

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Abstract. In this paper, we introduce the notion of hyper JK-algebras and investigate these algebras properties. Moreover, we present relationships between hyper JK-algebras and pseudo hyper BCK-algebras and hyper pseudo MV-algebras under some conditions.

Key Words: Equality algebra, Hyper equality algebra, Hyper JK-algebra, Hyper pseudo MV-algebra, JK-algebra, pseudo hyper BCK-algebra.
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1. Introduction

JK-algebras are introduced by Dvurečenskij and Zahiri in [5], where they show that pseudo equality algebras that were defined in [10], are equality algebras. The notion of pseudo equality algebras is a generalization of equality algebras that introduced by Jenei in [8]. An equality algebra consisting of two binary operations meet and equivalence, and constant 1. An equality algebra $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an algebra of type $(2, 2, 0)$ such that, for all $x, y, z \in X$, the following axioms are fulfilled:

- (E1) $\langle X, \wedge, 1\rangle$ is a commutative idempotent integral monoid (i.e. $\wedge$-semilattice with top element 1).
- (E2) $x \sim y = y \sim x$.
- (E3) $x \sim x = 1$.
- (E4) $x \sim 1 = x$.
- (E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$.
- (E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$.

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(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is an equality operation. We write $x \leq y$ if and only if $x \wedge y = x$, as usual. Some valuable results related to equality algebras are obtained in [4, 6, 11, 12].

Definition 1.1. [5] A *JK-algebra* is an algebra $(X; \sim, \backsim, \wedge, 1)$ of type (2,2,2,0) that satisfies the following axioms, for all $a, b, c \in X$:
(F1) $(X; \wedge, 1)$ is a meet-semilattice with top element 1;
(F2) $a \sim a = 1 = a \backsim a$;
(F3) $a \sim 1 = a = 1 \backsim a$;
(F4) $a \leq b \leq c$ implies that $a \sim c \leq b \sim c,\ a \sim c \leq a \sim b,\ c \backsim a \leq c \backsim b$ and $c \backsim a \leq b \backsim a$;
(F5) $a \sim b \leq (a \wedge c) \sim (b \wedge c)$ and $a \backsim b \leq (a \wedge c) \backsim (b \wedge c)$;
(F6) $a \sim b \leq (c \sim a) \backsim (c \sim b)$ and $a \backsim b \leq (a \backsim c) \sim (b \backsim c)$;
(F7) $a \sim b \leq (a \sim c) \sim (b \sim c)$ and $a \backsim b \leq (c \backsim a) \backsim (c \backsim b)$.

Let H be a nonempty set and $\circ$ be a function from $H \times H$ to the nonempty power set of H, $P(H)^*$ that is $P(H) - \emptyset$, it means that $\circ : H \times H \to P(H)^*$. Then $\circ$ is said to be a *hyperoperation* on H. As a generalization of equality algebra, Cheng, Xin and Jun in [3] introduced hyper equality algebras as follows:

Definition 1.2. [3]
A hyper equality algebra $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a nonempty set H endowed with a binary operation $\wedge$, a binary hyperoperation $\sim$ and a top element 1 such that, for all $x, y, z \in H$, the following axioms are fulfilled:
(HE1) $\langle H, \wedge, 1\rangle$ is a meet-semilattice with top element 1.
(HE2) $x \sim y \ll y \sim x$.
(HE3) $1 \in x \sim x$.
(HE4) $x \in 1 \sim x$.
(HE5) $x \leq y \leq z$ implies $x \sim z \ll y \sim z$ and $x \sim z \ll x \sim y$.
(HE6) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$.
(HE7) $x \sim y \ll (x \sim z) \sim (y \sim z)$.
Where $x \leq y$ if and only if $x \wedge y = x$ and $A \ll B$ is defined by, for all $x \in A$, there exists $y \in B$ such that $x \leq y$. Define the following two derived operations, the implication and the equivalence operation of the hyper equality algebra $\langle H, \sim, \wedge, 1\rangle$ by

$$x \to y = x \sim (x \wedge y) \quad \text{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

Basic properties and definitions related to hyper equality algebras are given in [3] and their relations with the other hyperstructures are studied in [7].

2. Hyper JK-algebras

We commence with the following definition:

Definition 2.1. A *hyper JK-algebra* $\mathcal{X} = (X; \sim, \circledcirc, \wedge, 1)$ is a nonempty set X endowed with binary operations $\wedge$, $\sim$, $\circledcirc$ and a top element 1 such that, for all $x, y, z \in X$, the following axioms are fulfilled:
(HF1) $(X; \wedge, 1)$ is a meet-semilattice with top element 1;
(HF2) $1 \in x \sim x, 1 \in x \circledcirc x$, $x \sim y \ll y \sim x$ and $x \circledcirc y \ll y \circledcirc x$;
(HF3) $x \in (x \sim 1) \cap (1 \circledcirc x)$;
(HF4) $x \leq y \leq z$ implies that $x \sim z \ll y \sim z$, $x \sim z \ll x \sim y$, $z \circledcirc x \ll z \circledcirc y$ and $z \circledcirc x \ll y \circledcirc x$;
(HF5) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$ and $x \circledcirc y \ll (x \wedge z) \circledcirc (y \wedge z)$;
(HF6) $x \sim y \ll (x \sim z) \circledcirc (y \sim z)$ and $x \circledcirc y \ll (x \circledcirc z) \sim (y \circledcirc z)$;
(HF7) $x \sim y \ll (x \sim z) \sim (y \sim z)$ and $x \circledcirc y \ll (z \circledcirc x) \circledcirc (z \circledcirc y)$.
Where $x \leq y$ if and only if $x \wedge y = x$ and $A \ll B$ is defined by, for all $x \in A$, there exists $y \in B$ such that $x \leq y$.

We now give some examples of hyper JK-algebras:

Example 2.2. (i) Let $\mathcal{X} = (X; \sim, \wedge, 1)$ be a hyper equality algebra, then $\mathcal{X} = (X; \sim, \sim, \wedge, 1)$ becomes a hyper JK-algebra.
(ii) Let $\mathcal{X} = (X; \sim, \circledcirc, \wedge, 1)$ be a JK-algebra. For all $x, y \in X$, define $x \circ y = \{x \sim y\}$ and $x \bullet y = \{x \circledcirc y\}$. Then $\mathcal{Y} = (X; \circ, \bullet, \wedge, 1)$ is a hyper JK-algebra.
(iii) Let $X = [0, 1]$. For all $x, y \in X$, define $\wedge$, $\sim$ and $\circledcirc$ on X as follows: $x \wedge y = \min\{x, y\}$,

$$x \sim y = \begin{cases} [y, 1], & x = 1. \\ X, & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circledcirc y = \begin{cases} [x, 1], & y = 1. \\ X, & \text{otherwise.} \end{cases}$$

Then by routine calculations, $(H; \sim, \circledcirc, \wedge, 1)$ is a hyper JK-algebra.
(iv) Let $X = \{0, a, 1\}$ such that $0 < a < 1$. For any $x, y \in X$, define the operations $\wedge$, $\sim$ and $\circledcirc$ as follows: $x \wedge y = \min\{x, y\}$,

$\sim$	0	a	1
0	$\{1\}$	$\{a,1\}$	$\{a,1\}$
a	$\{a,1\}$	$\{0,a,1\}$	$\{a,1\}$
1	$\{0,1\}$	$\{a,1\}$	$\{1\}$

and

$\circledcirc$	0	a	1
0	$\{1\}$	$\{a,1\}$	$\{0,1\}$
a	$\{a,1\}$	$\{0,a,1\}$	$\{a,1\}$
1	$\{a,1\}$	$\{a,1\}$	$\{1\}$

Then $(X;\sim,\circledcirc,\wedge,1)$ is a hyper JK-algebra.

In any hyper JK-algebra $\mathcal{X}=(X;\sim,\circledcirc,\wedge,1)$, for any $x,y\in X$, we define the following derived binary operations on X as follows:

$$x\to y:=(x\wedge y)\sim x,\qquad \text{and}\qquad x\rightsquigarrow y=x\circledcirc(x\wedge y).$$

Proposition 2.3. *Let $\mathcal{X}=(X;\sim,\circledcirc,\wedge,1)$ be a hyper JK-algebra and, for any $x,y,z\in X$, consider*
(HF4a) $(x\wedge y\wedge z)\sim x\ll(x\wedge y)\sim x$ and $x\circledcirc(x\wedge y\wedge z)\ll x\circledcirc(x\wedge y)$.
(HF4aa) $x\to(y\wedge z)\ll x\to y$ and $x\rightsquigarrow(y\wedge z)\ll x\rightsquigarrow y$.
Then (HF4), (HF4a) and (HF4aa) are equivalent.

Proof. The statements (HF4a) and (HF4aa) are equivalent, according to their definitions. We show that (HF4) implies (HF4a) and vice versa. For any $x,y,z\in X$, we have $x\wedge y\wedge z\le x\wedge y\le x$. Then by (HF4), we get

$$(x\wedge y\wedge z)\sim x\ll(x\wedge y)\sim x,$$

and

$$x\circledcirc(x\wedge y\wedge z)\ll x\circledcirc(x\wedge y).$$

Now, suppose that (HF4a) holds and $x\le y\le z$. Then by (HF4a), we have

$$x\sim z=(x\wedge y\wedge z)\sim z\ll(z\wedge y)\sim z=y\sim z,$$

and by (HF5),

$$x\sim z=(z\wedge x)\sim z\ll(z\wedge x\wedge y)\sim(z\wedge y)=x\sim y.$$

Similarly, by (HF4a),

$$z\circledcirc x=z\circledcirc(x\wedge y\wedge z)\ll z\circledcirc(y\wedge z)=z\circledcirc z,$$

and finally, by (HF5), we have

$$z\circledcirc x=z\circledcirc(x\wedge z)\ll(z\wedge y)\circledcirc(x\wedge z\wedge y)=y\circledcirc x.$$

$\square$

We now give some properties of hyper JK-algebras as follows:

Proposition 2.4. *Let $\mathcal{X} = (X; \sim, \circledcirc, \wedge, 1)$ be a hyper JK-algebra, then for all $x, y, z \in X$ and $A, B, C \subseteq X$, we have*

(i) $x \to y \ll (x \wedge z) \to y$ *and* $x \rightsquigarrow y \ll (x \wedge z) \rightsquigarrow y$;
(ii) $x \leq y$ *implies* $z \to x \ll z \to y$ *and* $z \rightsquigarrow x \ll z \rightsquigarrow y$;
(iii) $A \ll B$ *implies* $C \to A \ll C \to B$ *and* $C \rightsquigarrow A \ll C \rightsquigarrow B$;
(iv) $x \sim y \ll x \to y$, $x \sim y \ll y \to x$, $x \circledcirc y \ll x \rightsquigarrow y$ *and* $x \circledcirc y \ll y \rightsquigarrow x$;
(v) $A \sim B \ll B \to A$ *and* $A \circledcirc B \ll A \rightsquigarrow B$;
(vi) $x \leq y$ *implies* $y \to x = x \sim y$ *and* $y \rightsquigarrow x = y \circledcirc x$;
(vii) $1 \in x \to 1$, $1 \in x \to x$, $1 \in x \rightsquigarrow 1$, $1 \in x \rightsquigarrow x$, $x \ll x \sim 1$ *and* $x \ll 1 \circledcirc x$;
(viii) $x \leq y$ *implies* $1 \in x \to y$ *and* $1 \in x \rightsquigarrow y$;
(ix) $x \leq y$ *implies* $x \sim 1 \ll y \sim 1$, $x \sim 1 \ll x \sim y$, $1 \circledcirc x \ll 1 \circledcirc y$ *and* $1 \circledcirc x \ll y \circledcirc x$;
(x) $x \to y \ll (x \wedge z) \to y$ *and* $x \rightsquigarrow y \ll (x \wedge z) \rightsquigarrow y$;
(xi) $x \to (y \wedge z) \ll (x \wedge z) \to y$ *and* $x \rightsquigarrow (y \wedge z) \ll (x \wedge z) \rightsquigarrow y$;
(xii) $x \to y = x \to (x \wedge y)$ *and* $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$;
(xiii) $x \to y \ll (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \ll (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$;
(xiv) $x \ll ((y \sim x) \circledcirc (y \sim 1)) \wedge ((1 \circledcirc y) \sim (x \circledcirc y))$;
(xv) *if, for all* $x \in X$, $x \sim 1 = x = 1 \circledcirc x$, *then* $x \ll ((y \sim x) \circledcirc y) \wedge (y \sim (x \circledcirc y))$;
(xvi) $x \ll y \to x$ *and* $x \ll y \rightsquigarrow x$;
(xvii) *if* $x \leq y$, *then* $x \ll y \sim x$ *and* $x \ll y \circledcirc x$;
(xviii) $y \ll (x \to y) \to y$ *and* $y \ll (x \rightsquigarrow y) \rightsquigarrow y$;
(xix) *if* $x \leq y$, *then* $y \ll (x \sim y) \sim y$ *and* $y \ll (x \circledcirc y) \circledcirc y$;
(xx) *if* $x \leq y$, *then* $y \to z \ll x \to z$ *and* $y \rightsquigarrow z \ll x \rightsquigarrow z$;
(xxi) *if* $A \ll B$, *then* $B \to C \ll A \to C$ *and* $B \rightsquigarrow C \ll A \rightsquigarrow C$;
(xxii) $(x \to y) \rightsquigarrow z \ll y \rightsquigarrow (x \rightsquigarrow z)$ *and* $(x \rightsquigarrow y) \to z \ll y \to (x \to z)$.
(xxiii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

Proof. (i) From (HF5), we have $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$. This means that $x \to y \ll (x \wedge z) \to y$. Similarly, by (HF5), we get that $x \circledcirc (x \wedge y) \ll (x \wedge z) \circledcirc (x \wedge y \wedge z)$. Hence, $x \rightsquigarrow y \ll (x \wedge z) \rightsquigarrow y$.

(ii) From $x \leq y$, one can write $z \to x = (x \wedge z) \sim z = (x \wedge y \wedge z) \sim z)$. Then by (HF4a) in Proposition 2.3, we have $z \to x \ll (y \wedge z) \sim z = z \to y$. Similarly, we have

$$z \rightsquigarrow x = z \circledcirc (x \wedge z) = z \circledcirc (x \wedge y \wedge z) \ll z \circledcirc (y \wedge z) = z \rightsquigarrow y.$$

(iii) The proof of is straightforward by (ii).

(iv) From (HF2) and (HF5), we have $x \sim y \ll (x \wedge y) \sim (y \wedge y) = (x \wedge y) \sim y = y \to x$ and $x \circledcirc y \ll (x \wedge x) \circledcirc (x \wedge y) = x \circledcirc (x \wedge y) = x \rightsquigarrow y$.

(v) The proof by (iv) is clear.

(vi) Straightforward.

(vii) Apply the axioms (HF2) and (HF3).

(viii) Since $x \leq y$, from (HF2), we have $1 \in x \sim x = (x \wedge y) \sim x = x \to y$ and $1 \in x \circledcirc x = x \circledcirc (x \wedge y) = x \rightsquigarrow y$.

(ix) Since $x \leq y \leq 1$, the proof by (HF4) is clear.

(x) By (HF5), we can get $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$ and $x \circledcirc (x \wedge y) \ll (x \wedge z) \circledcirc (x \wedge z \wedge y)$. These imply (x).

(xi) By Proposition 2.3(HF4aa) and (x) the proof holds.

(xii) By (vi), $x \to y = x \sim (x \wedge y) = x \to (x \wedge y)$ and similarly $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$.

(xiii) By (HF5),

$$\begin{aligned}
x \to y &= (x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z) \qquad \text{by (HF7)}\\
&\ll ((x \wedge y \wedge z) \sim z) \sim ((x \wedge z) \sim z)\\
&= (z \to (x \wedge y)) \sim (z \to x) \qquad \text{by (v)}\\
&\ll (z \to x) \to (z \to (x \wedge y)) \qquad \text{by (ii)}\\
&\ll (z \to x) \to (z \to y).
\end{aligned}$$

Similarly, by (HF5), we have

$$\begin{aligned}
x \rightsquigarrow y &= x \circledcirc (x \wedge y) \ll (x \wedge z) \circledcirc (x \wedge y \wedge z) \qquad \text{by (HF7)}\\
&\ll (z \circledcirc (x \wedge z)) \circledcirc (z \circledcirc (x \wedge y \wedge z))\\
&= (z \rightsquigarrow x) \circledcirc (z \rightsquigarrow (x \wedge y)) \qquad \text{by (v)}\\
&\ll (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow (x \wedge y)) \qquad \text{by (ii)}\\
&\ll (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y).
\end{aligned}$$

(xiv) By (vii) and (HF6), we get

$$x \ll x \sim 1 \ll (y \sim x) \circledcirc (y \sim 1),$$

and

$$x \ll 1 \circledcirc x \ll (1 \circledcirc y) \sim (x \circledcirc y).$$

Thus, $x \ll ((y \sim x) \circledcirc (y \sim 1)) \wedge ((1 \circledcirc y) \sim (x \circledcirc y))$.

(xv) By (xiv), the proof is clear.

(xvi) By (vii), $x \ll x \sim 1$. Then by (iv), $x \ll x \sim 1 \ll 1 \to x$. Then by (x), we have $x \ll (1 \wedge y) \to x = y \to x$. Similarly, by (vii), (iv) and (x), we have $x \ll 1 \circledcirc x \ll 1 \rightsquigarrow x \ll (1 \wedge y) \rightsquigarrow x = y \rightsquigarrow x$.

(xvii) By (xvi) the proof is clear.

(xviii) By (xvi), (xvii) and (vi), we have

$$y \ll x \rightsquigarrow y \ll (x \rightsquigarrow y) \circledcirc y \ll (x \rightsquigarrow y) \rightsquigarrow y.$$

The case for $\to$ is similar.

(xix) By (xvii) and (xviii) the proof holds.

(xx) Apply (x) and for (xxi) apply (xx).

(xxii) By (xvi), we have $z \ll x \rightsquigarrow z$. Then by (xxi), $(x \to y) \rightsquigarrow z \ll (x \to y) \rightsquigarrow (x \rightsquigarrow z)$. Again by (xvi), we have $y \ll x \to y$. Then (xxi) together the above result, we have $(x \to y) \rightsquigarrow \ll y \rightsquigarrow (x \rightsquigarrow z)$. The other case is similar and the case (xxiii) is clear. □

Proposition 2.5. *Let $\mathcal{X} = (X; \sim, \circledcirc, \wedge, 1)$ be a hyper JK-algebra and $y \in X$ such that $y \sim 1 = y$. Then for any $x \in X$,*

(i) *if $x \le y$, then $x \ll (y \to x) \rightsquigarrow y$.*

(ii) *$x \ll y \to (y \rightsquigarrow x)$ and $x \ll y \rightsquigarrow (y \to x)$.*

Proof. (i) From Proposition 2.4(xiv),

$$\begin{aligned} x &\ll (y \sim x) \circledcirc y = (y \sim (x \wedge y)) \circledcirc y = (y \to x) \circledcirc y \\ &\ll (y \to x) \rightsquigarrow y \qquad \text{by Proposition 2.4(v).} \end{aligned}$$

(ii) According to (HF3), $1 \circledcirc y = y$. Then by Proposition 2.4(xiv), we have $x \ll y \sim (x \circledcirc y)$. By Proposition 2.4(iv), $x \circledcirc y \ll y \rightsquigarrow x$, then Proposition 2.4(iii) implies that $x \ll y \to (y \rightsquigarrow x)$.

Again by Proposition 2.4(xiv), $x \ll (y \sim x) \circledcirc y$. Then by (HF2), $x \ll y \circledcirc (y \sim x)$. By Proposition 2.4(v), $x \ll y \rightsquigarrow (y \sim x)$. Then by Proposition 2.4(iv) and (iii), we have $x \ll y \rightsquigarrow (y \to x)$. □

3. Relationship with the other pseudo hyper algebras

In this section we investigate the existence of a relationship between pseudo hyper JK-algebras with a special version of pseudo hyper algebras, i. e., pseudo hyper BCK-algebras and pseudo hyper MV-algebras.

3.1. Pseudo hyper BCK-algebras.

We recall pseudo hyper BCK-algebras from [1].

Definition 3.1. A *hyper pseudo BCK-algebra* is a structure $(H, \circ, *, 1)$, where "$*$" and "$\circ$" are hyperoperations on H and "1" is a constant element, that satisfies the following:

(PHK1) $(x \circ z) \circ (y \circ z) \ll x \circ y,\ (x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$,

(PHK3) $x \circ H \ll \{x\},\ x * H \ll \{x\}$,

(PHK4) $x \le y$ and $y \le x$ imply $x = y$,
for all $x, y, z \in H$, where $x \le y$ if and only if $1 \in x \circ y$ if and only if $1 \in x * y$ and for any $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \le b$

We now give the following definition:

Definition 3.2. Let $(H, \circ, *, 1)$ be a hyper pseudo BCK-algebra. We call $(H, \circ, *, 1)$ a *hyper pseudo BCK-meet-semilattice* if $(H, \le)$ is a meet $(\wedge)$-semilattice.

Theorem 3.3. *Let $\mathcal{X} = (X; \sim, \circledcirc, \wedge, 1)$ be a hyper JK-algebra such that $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$, for all $x, y, z \in H$. Then $(X, \circ, *, 1)$ is a hyper pseudo BCK-meet-semilattice, where for any $x, y \in X$, $x \circ y = y \to x$ and $x * y = y \rightsquigarrow x$.*

Proof. Define $1 \in x \circ y$ if and only if $1 \in x * y$ if and only if $x \le' y$. Moreover, for any $A, B \subseteq X$, we define $A \ll' B$ if and only if for any $x \in A$ there exists $y \in B$ such that $x \le' y$. By Proposition 2.4(xiii), $y \to x \ll (z \to y) \to (z \to x)$. This implies that there are $a \in y \to x$ and $b \in (z \to y) \to (z \to x)$ such that $a \le b$. Then Proposition 2.4(viii) implies that $1 \in a \to b$. Hence, $1 \in b \circ a$ and this holds if and only if $b \le' a$ if and only if $(x \circ z) \circ (y \circ z) \ll x \circ y$. By a similar argument we have $(x * z) * (y * z) \ll x * y$. So, the axiom (PHK1) holds.

The axiom (PHK2) by the assumption $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$ holds.

(PHK3) By Proposition 2.4(xvi), we have $x \ll y \to x$, for all $x, y \in X$. Thus, $x \ll X \to x$. This means that $x \circ X \ll' \{x\}$. Similarly, one can show that $x * X \ll' \{x\}$. The axiom (PHK4), by Proposition 2.4(xxiii) holds. Thus, $(X, \circ, *, 1)$ is a hyper pseudo BCK-meet-semilattice. $\square$

Question 3.4. Let $\mathcal{B} = (H, \circ, *, \wedge, 1)$ be a hyper pseudo BCK-meet-semilattice. Under which conditions is a $\mathcal{B}$ becomes a hyper JK-algebra?

3.2. **Hyper pseudo MV-algebras.** We recall the definition of hyper pseudo MV-algebras from [2] as follows:

Definition 3.5. A hyper pseudo MV-algebra is a non-empty set M with a binary hyperoperation $+$, two unary operations $'$, $*$ and two constants 0, 1 satisfying the following conditions, for all $x, y, z \in M$,
(HSMV1) $x + (y + z) = (x + y) + z$,
(HSMV2) $1 \in (x + 1) \cap (1 + x)$,
(HSMV3) $1^* = 1' = 0$,

(HSMV4) $(x' + y')^* = (x^* + y^*)'$,
(HSMV5) $x + (x^* \odot y) = y + (y^* \odot x) = (x \odot y') + y = (y \odot x') = x$,
(HSMV6) $x \odot (x' + y) = (x + y^*) \odot y$,
(HSMV7) $(x')^* = x$,
(HSMV8) $1 \in (x + x^*) \cap (x' + x)$,
(HSMV9) $1 \in (x' + y) \cap (y' + x)$ implies $x = y$,
(HSMV10) $1 \in x' + y$ if and only if $1 \in y + x^*$,
where $y \odot x = (x' + y')^*$, $A' = \{a' : a \in A\}$, $A^* = \{a^* : a \in A\}$, $A \odot B = \bigcup\{a \odot b : a \in A, b \in B\}$ and $A + B = \{a + b : a \in A, b \in B\}$, for any $A, B \subseteq M$.

Proposition 3.6. [2] *Let $\mathcal{M} = (M; +,', *, 0, 1)$ be a hyper pseudo MV-algebra. The following properties, for any $x, y, z \in M$ hold:*

(i) $(x^*)' = x$.
(ii) $x \leq 1$ *and* $0 \leq x$.
(iii) $x \in (0 + x) \cap (x + 0)$.
(iv) $x \leq y$ *if and only if* $y' \leq x'$ *if and only if* $y^* \leq x^*$.
(v) $x \leq y$ *implies that* $x + z \leq y + z$ *and* $z + x \leq z + y$.
(vi) $x \ll x + y$ *and* $y \ll x + y$.
(vii) $x \leq y$ *if and only if* $1 \in y + x^*$.

Theorem 3.7. *Let $\mathcal{M} = (M; +, *,', 0, 1)$ be a linearly ordered hyper pseudo MV-algebra such that for any $x, y, z \in M$,*
(i) $x \in 1 + x^*$.
(ii) $y + x^* \ll (x + z^*)' + (y + z^*)$ *and* $y \ll (z + y^*)'$.
(iii) $x' + y \ll (z' + y) + (z' + x)^*$ *and* $y \ll (y' + z)^*$.
(iv) $y + x^* \ll (z + x^*) + (z + y^*)^*$ *and* $x^* \ll (x + z^*)^*$.
(v) $x' + y \ll (y' + z)' + (x' + z)$ *and* $x' \ll (z' + x)'$.
Then it is a hyper JK-algebra.

Proof. Let $\mathcal{M} = (M; +, *,', 0, 1)$ be a linearly ordered hyper pseudo MV-algebra with the order $\leq$. For any $x, y \in M$, we define the operations $\sim$ and $\circledcirc$ on M as follow:

$$x \sim y = x \leftrightarrow y = (x \to y) \wedge (y \to x)$$

and

$$x \circledcirc y = x \leftrightsquigarrow y = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$$

where $x \to y = y + x^*$ and $x \rightsquigarrow y = x' + y$. Now, we show that $\mathcal{X} = (M; \sim, \circledcirc, \wedge, 1\rangle$ is a JK-algebra. By Proposition 3.6(ii), $(M, \leq)$ is a $\wedge$-semilattice with top element 1. Hence we have (HF1).

(HF2) Let $x, y \in M$. Then

$$\begin{aligned} x \sim y &= (x \to y) \wedge (y \to x) \\ &= (y + x^*) \wedge (x + y^*) \\ &= (x + y^*) \wedge (y + x^*) \\ &= (y \to x) \wedge (x \to y) \\ &= y \sim x \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} x \circledcirc y &= (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \\ &= (x' + y) \wedge (y' + x) \\ &= (y' + x) \wedge (x' + y) \\ &= (y \rightsquigarrow x) \wedge (x \rightsquigarrow y) \\ &= y \circledcirc x. \end{aligned} \tag{3.2}$$

Moreover, $x \sim x = x + x^*$ and $x \circledcirc x = x' + x$. Then by (HSMV8), we have $1 \in x \sim x, x \circledcirc x$. Thus, (HF2) holds.

(HF3) By Proposition 3.6(iii), for any $x \in M$, we have $x \in x + 0$. Moreover, since $1 \in 1 + x^*$, (HSMV10) implies that $1 \in x' + 1$. Then by $x \in 1 + x^*, x' + 1$, definitions of $\sim$ and $\circledcirc$, the axiom (HF3) holds.

(HF4) Let $x, y, z \in M$ such that $x \leq y \leq z$. By Proposition 3.6(iv), $z^* \leq y^* \leq x^*$. Then by (HSMV8) and Proposition 3.6(v), we have $1 \in z + z^* \ll z + y^* \ll z + x^*$. Thus, for any $A \subseteq M$,

$$(z + x^*) \wedge A = (z + y^*) \wedge A. \tag{3.3}$$

Also, by Proposition 3.6(v), since $x \leq y$, we have

$$x + z^* \ll y + z^*. \tag{3.4}$$

Then, for any $x, y, z \in M$, by (3.3) and (3.4), we have

$$\begin{aligned} x \sim z &= (x \to z) \wedge (z \to x) \\ &= (z + x^*) \wedge (x + z^*) \\ &\ll (z + x^*) \wedge (y + z^*) \\ &= (z + y^*) \wedge (y + z^*) \\ &= y \sim z. \end{aligned}$$

Similarly, by Proposition 3.6(iv), $z' \leq y' \leq x'$. Then by (HSMV8) and Proposition 3.6(v), we have $1 \in z' + z \ll y + z \ll x' + z$. Thus, for any $A \subseteq M$,

$$(x' + z) \wedge A = (y' + z) \wedge A. \tag{3.5}$$

Also, by Proposition 3.6(v), since $x \leq y$, we have

$$z' + x \ll z + y. \tag{3.6}$$

Then, for any $x, y, z \in M$, by (3.5) and (3.6), we have

$$\begin{aligned} x \circledcirc z &= (x \rightsquigarrow z) \wedge (z \rightsquigarrow x) \\ &= (x' + z) \wedge (z' + x) \\ &\ll (x' + z) \wedge (z' + y) \\ &= (y' + z) \wedge (z' + y) \\ &= y \circledcirc z. \end{aligned}$$

Since $z^* \leq y^*$, by Proposition 3.6(v), we have

$$x + z^* \ll x + y^*. \tag{3.7}$$

Moreover, by (HSMV8) and Proposition 3.6(v), since $x \leq y \leq z$, we get that

$$1 \in x + x^* \ll y + x^* \ll z + x^*.$$

So, for any $A \subseteq M$,

$$(z + x^*) \wedge A = (y + x^*) \wedge A. \tag{3.8}$$

Then, for any $x, y, z \in M$, by (3.7), (3.8) and Proposition 3.6(v), we have

$$\begin{aligned} x \sim z &= (x \rightarrow z) \wedge (z \rightarrow x) \\ &= (z + x^*) \wedge (x + z^*) \\ &\ll (z + x^*) \wedge (x + y^*) \\ &= (y + x^*) \wedge (x + y^*) \\ &= x \sim y. \end{aligned}$$

On the other hand, since $z' \leq y'$, by Proposition 3.6(v), we have

$$z' + x \ll y' + x. \tag{3.9}$$

Moreover, by (HSMV8) and Proposition 3.6(v), we get that

$$1 \in x' + x \ll x' + y \ll x' + z.$$

So, for any $A \subseteq M$,

$$(x' + z) \wedge A = (x' + y) \wedge A. \tag{3.10}$$

Then, for any $x, y, z \in M$, by (3.9), (3.10) and Proposition 3.6(v), we have

$$\begin{aligned} x \circledcirc z &= (x \rightsquigarrow z) \wedge (z \rightsquigarrow x) \\ &= (x' + z) \wedge (z' + x) \\ &\ll (x' + z) \wedge (y' + x) \\ &= (x' + y) \wedge (y' + x) \\ &= x \circledcirc y. \end{aligned}$$

The above arguments show that (HF4) holds.

Since $\mathcal{M}$ is a linearly ordered hyper pseudo MV-algebra, clearly (HF5) holds.

(HF6) By Proposition 3.6(vi), we have $x^* \ll z+x^*$. Then by condition (ii) i.e., $y \ll (z + y^*)'$, we have

$$y + x^* \ll (z + y^*)' + (z + x^*). \tag{3.11}$$

Then, for any $x, y, z \in M$, the condition (ii) and (3.11) imply that

$$x \to y \ll (z \to x) \rightsquigarrow (z \to y) \tag{3.12}$$

and

$$x \to y \ll (y \to z) \rightsquigarrow (x \to z). \tag{3.13}$$

Without loss of generality, suppose $x, y, z \in M$ such that $x \leq y \leq z$. Since $\mathcal{M}$ is linearly ordered. Then by Proposition 3.6(vii), $1 \in z + x^* = x \to z$ and $1 \in z + y^* = y \to z$. Then, by (3.12), we have

$$\begin{aligned} x \to y &\ll (z \to x) \rightsquigarrow (z \to y) \\ &\ll ((z \to x) \wedge (x \to z)) \rightsquigarrow ((z \to y) \wedge (y \to z)) \\ &= (x \sim z) \rightsquigarrow (y \sim z) \end{aligned} \tag{3.14}$$

and similarly by (3.13), we get that

$$x \to y \ll (y \sim z) \rightsquigarrow (x \sim z). \tag{3.15}$$

Thus, by (3.20) and (3.21), for all $x, y, z \in M$, we obtain that

$$x \to y \ll ((x \sim z) \rightsquigarrow (y \sim z)) \wedge ((y \sim z) \rightsquigarrow (x \sim z)) = (x \sim z) \circledcirc (y \sim z).$$

This shows that

$$x \sim y \ll (x \sim z) \circledcirc (y \sim z). \tag{3.16}$$

By Proposition 3.6(v)-(vi) and the condition (iii) i.e., $y \leq (y' + z)^*$, we have

$$x' + y \ll (x' + z) + (y' + z)^*. \tag{3.17}$$

Then, for any $x, y, z \in M$, the condition (iii) and (3.17) imply that

$$x \rightsquigarrow y \ll (z \rightsquigarrow x) \to (z \rightsquigarrow y) \tag{3.18}$$

and

$$x \rightsquigarrow y \ll (y \rightsquigarrow z) \to (x \rightsquigarrow z). \tag{3.19}$$

Again, suppose that $x, y, z \in M$ such that $x \leq y \leq z$. Then by Proposition 3.6(vii), $1 \in z + x^*$ and $1 \in z + y^*$. Then, by (HSMV10), $1 \in x' + z = x \rightsquigarrow z$ and $1 \in y' + z = y \rightsquigarrow z$ (3.12). Then by (3.18), we have

$$\begin{aligned} x \rightsquigarrow y &\ll (z \rightsquigarrow x) \to (z \rightsquigarrow y) \\ &\ll ((z \rightsquigarrow x) \wedge (x \rightsquigarrow z)) \to ((z \rightsquigarrow y) \wedge (y \rightsquigarrow z)) \\ &= (x \circledcirc z) \to (y \circledcirc z) \end{aligned} \tag{3.20}$$

and similarly by (3.19), we get that

$$x \rightsquigarrow y \ll (y \circledcirc z) \to (x \circledcirc z). \tag{3.21}$$

Thus, by (3.20) and (3.21), for all $x, y, z \in H$, we obtain that

$$x \rightsquigarrow y \ll ((x \circledcirc z) \to (y \circledcirc z)) \wedge ((y \circledcirc z) \to (x \circledcirc z)) = (x \circledcirc z) \sim (y \circledcirc z).$$

Hence,

$$x \circledcirc y \ll (x \circledcirc z) \sim (y \circledcirc z). \tag{3.22}$$

Then (3.16) and (3.22) imply (HF6).

(HF7) By the condition (iv) i.e., $y + x^* \ll (z + x^*) + (z + y^*)^*$, $x^* \ll (x + z^*)^*$ and by Proposition 3.6(v)-(vi), we have

$$x \to y \ll (y \to z) \to (x \to z) \tag{3.23}$$

and

$$x \to y \ll (z \to x) \to (z \to y). \tag{3.24}$$

By a similar argument that we have discussed for the axiom (HF6), by (3.23) and (3.24), one can see that

$$x \sim y \ll (x \sim z) \sim (y \sim z). \tag{3.25}$$

By the condition (v) and by Proposition 3.6(v)-(vi), we have

$$x \rightsquigarrow y \ll (y \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow z) \tag{3.26}$$

and

$$x \rightsquigarrow y \ll (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y). \tag{3.27}$$

So, similarly, by applying (3.26) and (3.27), we have

$$x \circledcirc y \ll (x \circledcirc z) \circledcirc (y \circledcirc z). \tag{3.28}$$

Hence, (3.24) and (3.28) imply (FH7). Thus, $\mathcal{X} = (M; \sim, \circledcirc, \wedge, 1)$ is a hyper JK-algebra. □

Example 3.8. [2] Let $M = \{0, a, b, c, 1\}$ be a set such that $0 \leq a \leq b \leq c \leq 1$ and define the operations $\sim$ and $\circledcirc$ on M as follow:

$+$	0	a	b	c	1
0	$\{0\}$	$\{0,a\}$	$\{0,b\}$	$\{0,c\}$	M
a	$\{0,a\}$	$\{0,a\}$	$\{a,b\}$	M	M
b	$\{0,b\}$	M	$\{0,b\}$	$\{b,c\}$	M
c	$\{0,c\}$	$\{a,c\}$	M	$\{0,c\}$	M
1	M	M	M	M	M

and

$'$	0	a	b	c	1
	1	b	c	a	0

$*$	0	a	b	c	1
	1	c	a	b	0

Then $\mathcal{M} = (M; +, *, ', 0, 1)$ is a hyper pseudo MV-algebra that satisfies the conditions of Theorem 3.7. Thus, $\mathcal{X} = (M; \sim, \circledcirc, \wedge, 1)$ is a hyper JK-algebra.

Question 3.9. How one can describe the converse of Theorem 3.7? i.e., if we have a hyper JK-algebra, under which condition it becomes a hyper pseudo MV-algebra?

4. Conclusions and Future Works

In this work we introduced a new version of hyperstructures that we called it hyper JK-algebra and we have given some properties of these new algebras. Moreover, we show that under some conditions any hyper JK-algebra is a pseudo BCK-algebra and by adding some conditions on a hyper pseudo MV-algebra we have obtained a hyper JK-algebra. The states and homomorphisms on hyper JK-algebras, some results on quotient structure and filter theory and positive implicative hyper equality algebras could be topics for our next task.

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