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SOME PROPERTIES OF FUZZY CONE SYMMETRIC SPACES

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ABSTRACT. In this work, we give a fuzzy analogy of cone symmetric spaces that we call fuzzy cone symmetric spaces. Since these structures are obtained by omitting the triangle inequality in fuzzy cone metric spaces, there are topological degenerations. After mentioning these degenerations, we investigate the relationship between cone (sym)metric and fuzzy cone (sym)metric spaces.

Key Words: Cone Symmetric, Fuzzy Cone Symmetric.
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1. Introduction

The fuzzy analogy of metric spaces has been introduced from a different point of view ([3, 15, 2, 6, 7]). Similarly, the fuzzy analogy of cone metric spaces defined by Bag [1] and Oner et al.[8] differently with the name of fuzzy cone metric spaces. For these structures, Bag followed the Kaleva et al. [6] type fuzzy metric spaces whereas Oner et al. used George et al. [3] type. In this work, we define fuzzy cone symmetric spaces by omitting triangle inequality in fuzzy cone metric spaces in sense of [8]. Of course, this modification causes some topological degenerations. For example, $M(x_1, x_2, \_) : int(P) \to [0, 1]$ may not be non-decreasing while it is non-decreasing in fuzzy cone metric spaces. As a result, open balls do not necessarily form a base in the induced topology (see Remark 3.11). We encounter second degeneration in the notion of a convergent sequence. Convergent sequences are generally defined by

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distance function in distance spaces. But in the symmetric setting, this definition does not coincide with the definition of convergent sequence in topology. Therefore, we will call convergence (and related definitions) in this structure as M-convergence to distinguish it with convergence in topology. M-convergence implies convergence. We also supply an example to show that an M-convergent sequence may have more than one limit (see Example 3.6) while it is impossible in a fuzzy cone metric. However, we can still characterize the M-convergence in these spaces with the convergence in $[0,1]$ (see Theorem 3.5). One of the purposes of this study is to give the connection between cone (sym)metric and fuzzy cone (sym)metric spaces. A similar idea was given in [3] for metric and fuzzy metric spaces. When a cone symmetric space is given, we observed that a fuzzy cone symmetric space can be constructed which we call standard fuzzy cone symmetric spaces (see Remark 3.7). Later we characterize the convergence in normal cone metric spaces with the convergence in induced standard fuzzy cone symmetric spaces (see Theorem 3.9). We also observed that for a normal cone metric space with $K = 1$, the induced standard fuzzy cone symmetric spaces is a fuzzy cone metric space (see Remark 3.8). Moreover, we prove that these two spaces induce the same topology (see Theorem 3.12) and one of them is complete if and only if the other is complete (see Corollary 3.13). After defining the contractions in these structures, we give a criterion for a contraction in a normal cone metric space with K to be a contraction in the standard fuzzy cone symmetric space. In [5], a contractive mapping was given to show that the set of contractions defined on cone metric and metric spaces are not the same. Similarly, this map is used to show that the set of contractions defined on fuzzy cone metric and fuzzy metric spaces are not the same (see Remark 3.16).

2. Preliminaries

Suppose that $X \neq \varnothing$, E is a real Banach space and $\theta \in E$ is the zero. Huang et al. [5] defined cones in E as follows: Let P be a subset of E such that nonempty, closed and $P \neq \{\theta\}$. If $a_1t_1 + a_2t_2 \in P$ and $t_1, -t_1 \in P$ implies $t_1 = \theta$ for every $a_1, a_2 \in \mathbb{R}$, $a_1, a_2 \geq 0$ and every $t_1, t_2 \in P$, then we say that P is a cone. Moreover, by using P, they defined a partial ordering [5]: $t_1 \preceq t_2 \Leftrightarrow t_2 - t_1 \in P$.

Also, we write $t_1 \ll t_2$ if $t_2 - t_1 \in int\,(P)$ and $t_1 \prec t_2$ if $t_1 \neq t_2$ and $t_1 \preceq t_2$. If there is a $K > 0$ satisfying $\theta \preceq t_1 \preceq t_2 \rightarrow \|t_1\| \leq K\|t_2\|$

for any $t_1, t_2 \in E$, then we say that P is normal. We call the smallest number satisfying this inequality as the normal constant [5].

In [13], it was shown that for any $s \gg \theta$, $s \in E$, there is $\varepsilon > 0$ satisfying $x \in E, \|x\| < \varepsilon$ implies $x \ll s$ and for every $\theta \preceq t_1$ and $\theta \preceq t_2$, there is an element $\theta \preceq t_3$ such that $t_3 \preceq t_1$, $t_3 \preceq t_2$.

We assume that $int\,(P) \neq \varnothing$ for all cones in this paper.

Let $d : X \times X \to E$ be a map. Consider following conditions:
(C1) $d\,(x_1, x_2) = \theta \iff x_1 = x_2$ and $\theta \preceq d\,(x_1, x_2)$,
(C2) $d\,(x_1, x_2) = d\,(x_2, x_1)$,
(C3) $d\,(x_1, x_3) \preceq d\,(x_1, x_2) + d\,(x_2, x_3)$
for any $x_1, x_2, x_3 \in X$.

Definition 2.1 ([5])**.** If d satisfies the conditions (C1),(C2) and (C3) then we say that d is a cone metric on X and (X, d) is a cone metric space.

An open ball and the topology on this spaces are defined as follows [13]: Let $t \gg \theta$ and $x_1 \in X$. $B\,(x_1, t) = \{x_2 \in X : d\,(x_1, x_2) \ll t\}, \tau_d = \{U \subset X : \forall x_1 \in U, \exists B\,(x_1, t) \ni B\,(x_1, t) \subset U\}$

Definition 2.2 ([5])**.** Let (s_n) be a sequence in a cone metric space (X, d) and $s \in X$. We say that
i) (s_n) converge to s if for all $t \in int\,(P)$ there is $n_0 \in \mathbb{N}$ such that $d\,(s_n, s) \ll t$ whenever $n \geq n_0$ and it is denoted by $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ as $n \to \infty$.
ii) (s_n) is Cauchy sequence if for all $t \in int\,(P)$ there is $n_0 \in \mathbb{N}$ such that $d\,(s_n, s_m) \ll t$ whenever $n, m \geq n_0$.
iii) (X, d) is complete if all Cauchy sequence is convergent.

In [10], Radenovic and Kadelburg gave the following definition by omitting the condition (C3) in the Definition 2.1 and investigated the relations between them.

Definition 2.3 ([10])**.** If d satisfies the conditions (C1) and (C2) then we say that d is a cone symmetric on X and (X, d) is a cone symmetric space.

3. Fuzzy Cone Symmetric Spaces

Let P be a cone of E, M be a fuzzy set on $X^2 \times int\,(P)$ where $X \neq \varnothing$ and $*$ be a continuous t-norm. Consider following conditions:
(FC1) $M\,(x_1, x_2, t) > 0$,
(FC2) $M\,(x_1, x_2, t) = 1 \iff x_1 = x_2$,

(FC3) $M(x_1, x_2, t) = M(x_2, x_1, t)$,
(FC4) $M(x_1, x_3, t_1 + t_2) \geq M(x_1, x_2, t_1) * M(x_2, x_3, t_2)$,
(FC5) $M(x_1, x_2, \_) : int(P) \to [0,1]$ is continuous,
for any $t_1, t_2, t \in int(P)$ and $x_1, x_2, x_3 \in X$.

Definition 3.1 ([8])**.** If M satisfies the conditions (FC1),(FC2),(FC3), (FC4) and (FC5), then we say that M is fuzzy cone metric on X and $(X, M, *)$ is fuzzy cone metric space.

Now, we introduce the concept of fuzzy cone symmetric spaces by omitting the condition (FC4) in the Definition 3.1.

Definition 3.2. If M satisfies the conditions (FC1),(FC2),(FC3) and (FC5), then we say that M is fuzzy cone symmetric on X and $(X, M, *)$ is fuzzy cone symmetric space.

Example 3.3. Let $E = \mathbb{R}, P = [0, \infty), X = (0, \infty), * = min$ and $M : X \times X \times int(P) \to [0,1]$ defined as $M(x_1, x_2, t) = \frac{min\{x_1,x_2\}+t}{max\{x_1,x_2\}+t}$. Then M is a fuzzy cone symmetric on X. On the other hand, since we have

$$\begin{aligned} M(x_1, x_2, t_1) * M(x_2, x_3, t_2) &= min\{\tfrac{min\{1,2\}+2}{max\{1,2\}+2}, \tfrac{min\{2,3\}+1}{max\{2,3\}+1}\} \\ &= min\{\tfrac{3}{4}, \tfrac{3}{4}\} = \tfrac{3}{4} \\ &> \tfrac{4}{6} = \tfrac{min\{1,3\}+3}{max\{1,3\}+3} = M(x_1, x_3, t_1 + t_2). \end{aligned}$$

for $x_1 = 1, x_2 = 2, x_3 = 3, t_1 = 2, t_2 = 1$, M is not a fuzzy cone metric on X.

It is clear that the set of fuzzy cone symmetric is larger then set of fuzzy cone metric spaces.

Now let's define the notion of convergence which does not coincide with the definition of convergent sequence in topology.

Definition 3.4. Let (s_n) be a sequence in a fuzzy cone symmetric space $(X, M, *)$ and $s \in X$. Then we say that
i) (s_n) M-converge to s if for all $t \in int(P)$ and all $r \in (0,1)$ there is $n_0 \in \mathbb{N}$ such that $1 - r < M(s_n, s, t)$ whenever $n \geq n_0$ and it is denoted by $s_n \underset{M}{\to} s$ as $n \to \infty$.
ii) (s_n) is M-Cauchy sequence if for $t \in int(P)$ and all $r \in (0,1)$ there is $n_0 \in \mathbb{N}$ such that $1 - r < M(s_n, s_m, t)$ whenever $n, m \geq n_0$.
iii) $(X, M, *)$ is M-complete if all M-Cauchy sequence is M-convergent.

Theorem 3.5. *Let (s_n) be a sequence in a fuzzy cone symmetric space $(X, M, *)$ and $s \in X$.*

1-) $s_n \underset{M}{\rightarrow} s \Longleftrightarrow M(s_n, s, t) \rightarrow 1$ *as* $n \rightarrow \infty$*, for any* $t \gg \theta$*.*
2-) (s_n) *is M-Cauchy* $\Longleftrightarrow M(s_n, s_m, t) \rightarrow 1$ *as* $n, m \rightarrow \infty$*, for any* $t \gg \theta$*.*

Proof. 1-) ($\Rightarrow$:) Let $s_n \underset{M}{\rightarrow} s$. For any $t \gg \theta$ and $r \in (0,1)$, there is $n_0 \in \mathbb{N}$ satisfying $1 - r < M(s_n, s, t)$ whenever $n \geq n_0$ which implies $M(s_n, s, t) \rightarrow 1$ as $n \rightarrow \infty$. ($\Leftarrow$:) Let $M(s_n, s, t) \rightarrow 1$ as $n \rightarrow \infty$. For any $t \gg \theta$ and $r \in (0,1)$, there is $n_0 \in \mathbb{N}$ satisfying $1 - M(s_n, s, t) < r$ whenever $n \geq n_0$. Hence $s_n \underset{M}{\rightarrow} s$ as $n \rightarrow \infty$.
2-) Similar. □

Following example shows that an M-convergent sequence may have more than one limit.

Example 3.6. Let $E = \mathbb{R}, P = [0, \infty), * = \cdot, X = [0,1] \cup \{2\}$ and $M : X \times X \times int(P) \rightarrow [0,1]$ defined as

$$M(x_1, x_2, t) = \begin{cases} \frac{t}{t + |x_1 - x_2|} & \text{if } x_1, x_2 \in [0,1] \\ \frac{t}{t + x_1} & \text{if } x_1 \in (0,1], x_2 = 2 \\ \frac{t}{t + x_2} & \text{if } x_2 \in (0,1], x_1 = 2 \\ \frac{t}{t+1} & \text{if } x_1 = 0, x_2 = 2 \text{ or } x_2 = 0, x_1 = 2. \end{cases}$$

Then $(X, M, *)$ is a fuzzy cone symmetric spaces. Since $M(1/n, 2, t) \rightarrow 1$ and $M(1/n, 0, t) \rightarrow 1$ as $n \rightarrow \infty$ for each t, $(1/n)$ is M-convergent and 0 and 2 are distinct limit points.

In [3], for a given metric space (X, d), authors constructed the standard fuzzy metric space and investigate the connection between metric and fuzzy metric spaces. Similarly, in the following, we give the connection between cone (sym)metric and fuzzy cone (sym)metric spaces.

Remark 3.7. A cone symmetric space (X, d) is given. If $a_1 * a_2 = a_1 \cdot a_2$ and $M_d(x_1, x_2, t) = \frac{\|t\|}{\|t\| + \|d(x_1, y_2)\|}$, then M_d is a fuzzy cone symmetric on X and $(X, M_d, \cdot)$ is said to be the standard fuzzy cone symmetric space induced by d.
1) Since for $t \gg \theta$ and $d(x, y) \gg 0$, we have $\|t\| > 0$ and $\|d(x_1, x_2)\| \geq 0$. Then it is obvious that $M_d(x_1, x_2, t) = \frac{\|t\|}{\|t\| + \|d(x_1, x_2)\|} > 0$.
2) For any $t \gg \theta$ and $x_1, x_2 \in X$, $M_d(x_1, x_2, t) = 1$ iff $\frac{\|t\|}{\|t\| + \|d(x_1, x_2)\|} = 1$ iff $\|d(x_1, x_2)\| = 0$ iff $d(x_1, x_2) = 0$ iff $x_1 = x_2$.
3) Since $d(x_1, x_2) = d(x_2, x_1)$, we can write $\|d(x_1, x_2)\| = \|d(x_2, x_1)\|$

and $M_d(x_1,x_2,t)=\frac{\|t\|}{\|t\|+\|d(x_1,x_2)\|}=\frac{\|t\|}{\|t\|+\|d(x_2,x_1)\|}=M_d(x_2,x_1,t)$
4) For any $x_1,x_2\in X$, we can think $M_d(x_1,x_2,\_)$ as $M_d(x_1,x_2,\_)=\|.\|_{int(P)}\circ f$ where $\|.\|:E\to[0,\infty)$, $t\mapsto\|t\|$ and $f:[0,\infty)\to[0,1]$, $f(r)=\frac{r}{r+\|d(x_1,x_2)\|}$. Since $\|.\|$ and f are continuous, $M_d(x_1,x_2,\_)$ is continuous.

Remark 3.8. If we are given a cone metric spaces with normal constant K, for $x_1,x_2,x_3\in X$ and $t\gg\theta, s\gg\theta$, we have $d(x_1,x_3)\preceq d(x_1,x_2)+d(x_2,x_3)$ and by normality $\|d(x_1,x_3)\|\le K\|d(x_1,x_2)+d(x_2,x_3)\|\le K\|d(x_1,x_2)\|+K\|d(x_2,x_3)\|$. Similarly, $t\preceq t+s$ implies $\|t\|\le K\|t+s\|$ and $\frac{K\|t+s\|}{\|t\|}\ge 1$, $s\preceq t+s$ implies $\|s\|\le K\|t+s\|$ and $\frac{K\|t+s\|}{\|s\|}\ge 1$. So we can write;

$$\|d(x_1,x_3)\|\le K^2\frac{\|t+s\|}{\|t\|}\|d(x_1,x_2)\|+K^2\frac{\|t+s\|}{\|s\|}\|d(x_2,x_3)\|$$

$$1+\frac{\|d(x_1,x_3)\|}{K^2\|t+s\|}\le 1+\frac{\|s\|\|d(x_1,x_2)\|+\|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$

$$\frac{K^2\|t+s\|+\|d(x_1,x_3)\|}{K^2\|t+s\|}\le\frac{\|s\|\|t\|+\|s\|\|d(x_1,x_2)\|+\|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$

Since $\|d(x_1,x_2)\|$, $\|d(x_2,x_3)\|\ge 0$,

$$\frac{K^2\|t+s\|+\|d(x_1,x_3)\|}{K^2\|t+s\|}\le\frac{\|s\|\|t\|+\|s\|\|d(x_1,x_2)\|+\|t\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$

$$\frac{K^2\|t+s\|+\|d(x_1,x_3)\|}{K^2\|t+s\|}\le\frac{\|s\|\|t\|+\|s\|\|d(x_1,x_2)\|+\|t\|\|d(x_2,x_3)\|+\|d(x_1,x_2)\|\|d(x_2,x_3)\|}{\|s\|\|t\|}$$

$$\begin{aligned}\frac{\|s\|\|t\|}{\|s\|\|t\|+\|t\|\|d(x_1,x_2)\|+\|t\|\|d(x_2,x_3)\|+\|d(x_1,x_2)\|\|d(x_2,x_3)\|}&\le\frac{K^2\|t+s\|}{K^2\|t+s\|+\|d(x_1,x_3)\|}\\ \left(\frac{\|t\|}{\|t\|+\|d(x_1,x_2)\|}\right)\left(\frac{\|s\|}{\|s\|+\|d(x_2,x_3)\|}\right)&\le\frac{K^2\|t+s\|}{K^2\|t+s\|+\|d(x_1,x_3)\|}\\ &\le\frac{K^2\|t+s\|}{\|t+s\|+\|d(x_1,x_3)\|}\\ M_d(x_1,x_2,t)\cdot M_d(x_2,x_3,s)&\le K^2M_d(x_1,x_2,t+s)\end{aligned}$$

Hence, the induced standard fuzzy cone symmetric space $(X,M_d,\cdot)$ is "almost " a fuzzy cone metric space. In particular, if $K=1$, the induced standard fuzzy cone symmetric space $(X,M_d,\cdot)$ is a fuzzy cone metric space.

Following characterizations can be given for convergent and Cauchy sequences.

Theorem 3.9. *Let (s_n) be a sequence in a normal cone metric space (X,d) with K, $s\in X$ and $(X,M_d,\cdot)$ be the standard fuzzy cone symmetric space.*

1-) $s_n \to s$ *in* $(X,d) \Longleftrightarrow s_n \underset{M}{\to} s$ *in* $(X, M_d, \cdot)$.
2-) (s_n) *is Cauchy in* $(X,d) \Longleftrightarrow (s_n)$ *is M-Cauchy in* $(X, M_d, \cdot)$

Proof. 1-) In Lemma 1 of [5], it was proven that $(s_n) \to s$ in (X,d) iff $\|d(s_n,s)\| \to 0$ as $n \to \infty$. By Theorem 3.5, $s_n \underset{M}{\to} s$ in $(X, M_d, \cdot)$ iff for any $t \gg \theta$, $M_d(s_n, s, t) \to 1$ as $n \to \infty$. Then $\frac{\|t\|}{\|t\|+\|d(s_n,s)\|} \to 1$ as $n \to \infty$ for any $t \gg \theta$ iff $\|d(s_n,s)\| \to 0$ as $n \to \infty$. Therefore $s_n \to s$ in (X,d).
2-) Similar. □

Definition 3.10. An open ball with center x_1, radius $r \in (0,1)$ and $t \gg \theta$ in a fuzzy cone symmetric space $(X, M, *)$ is given by $B(x_1, r, t) = \{x_2 \in X : 1 - r < M(x_1, x_2, t)\}$.

Remark 3.11. $M(x_1, x_2, \_) : int(P) \to [0,1]$ is non-decreasing in fuzzy cone metric spaces [8]. But this is not valid for fuzzy cone symmetric spaces since the triangle angle inequality is omitted. As we know from symmetric structures, open balls do not necessarily form a base in fuzzy cone symmetric spaces while they are a base in fuzzy cone metric spaces (Theorem 2.6 in [8]). Even if they satisfies the following condition for a given topology on this space:

$$U \subset X \text{ open iff } \forall x, \text{ there is } t \gg \theta \text{ and } r \in (0,1) \text{ with } B(x,r,t) \subset U$$

, an open ball is not necessarily open. Moreover, this topology may not be Hausdorff and first countable. On the other hand, when a normal cone metric space (X,d) with $K = 1$ is given, since $(X, M_d, *)$ is a fuzzy cone metric space, open balls form a base for the topology

$$\tau_{M_d} = \{U \subset X : \forall x, \text{ there is } t \gg \theta \text{ and } r \in (0,1) \text{ with } B(x,r,t) \subset U\}$$

on X and clearly, an open ball is open. In the following, we show that τ_{M_d} is the same with τ_d induced by corresponding cone metric.

Theorem 3.12. *If* (X,d) *is a normal cone metric space with* $K = 1$ *and* $(X, M_d, \cdot)$ *is the standard fuzzy cone metric space induced by* d*, then* $\tau_{M_d} = \tau_d$.

Proof. Let $B(x_1, c) = \{x_2 \in X : d(x_1, x_2) \ll c\}$ and $B(x_1, r, t) = \{x_2 \in X : M_d(x_1, x_2, t) > 1 - r\} = \{x_2 \in X : \frac{\|t\|}{\|t\|+\|d(x_1,x_2)\|} > 1 - r\}$ be open balls with the center x. For an arbitrary open ball $B(x_1, r, t)$, if we choose $c = \dfrac{r}{1-r}t$, then for $x_2 \in B(x_1, c)$, we have $d(x_1, x_2) \ll c = \frac{r}{1-r}t$.

Therefore we have $\|d(x_1,x_2)\| < \frac{r}{(1-r)}\|t\| \Rightarrow \frac{\|t\|}{\|t\|+\|d(x_1,x_2)\|} > 1-r \Rightarrow M_d(x_1,x_2,t) > 1-r$. Hence $x_2 \in B(x_1,r,t)$ and $\tau_{M_d} \subset \tau_d$. Conversely, let $B(x_1,c)$ be an arbitrary open ball. Then there exist $\epsilon > 0$ such that $z \ll c$ whenever $\|z\| < \epsilon$. If we choose $r \in (0,1)$ and $t \gg \theta$ such that $\epsilon > \dfrac{r\|t\|}{1-r}$, for $x_2 \in B(x_1,r,t)$, we have $M(x_1,x_2,t) = \frac{\|t\|}{\|t\|+\|d(x_1,x_2)\|} > 1-r$. Therefore we have $\|d(x_1,x_2)\| < \frac{r\|t\|}{(1-r)} < \epsilon \Rightarrow d(x_1,x_2) \ll c$ means $x_2 \in B(x_1,c)$ and $\tau_d \subset \tau_{M_d}$. □

Corollary 3.13. *If (X,d) is a normal cone metric space with $K=1$ and $(X,M_d,\cdot)$ is the standard fuzzy cone metric space induced by d, then (X,d) is complete if and only if $(X,M_d,\cdot)$ is complete.*

Proof. From Theorem 3.9 and Remark 3.11. □

Definition 3.14. Let $f: X \to X$ be a mapping on a fuzzy cone symmetric space $(X,M,*)$. If there is a $k \in (0,1)$ satisfying $\frac{1}{M(f(x_1),f(x_2),s)} - 1 \leq k\left(\frac{1}{M(x_1,x_2,s)} - 1\right)$ for any $s \gg \theta$ and $x_1,x_2 \in X$, then f is said to be fuzzy cone contractive and k is said to be the contractive constant of f.

Now we give a criterion for a contraction in a normal cone metric space with K to be a contraction in the standard fuzzy cone symmetric space.

Proposition 3.15. *If (X,d) is a normal cone metric space with K and $T: X \to X$ is a contraction on (X,d) satisfying $d(T(x_1),T(x_2)) \preceq kd(x_1,x_2)$ for any $x_1,x_2 \in X$ where $0<k<1$ and $kK<1$, then T is fuzzy cone contractive on $(X,M_d,\cdot)$.*

Proof. By the normality, $d(T(x_1),T(x_2)) \preceq kd(x_1,x_2)$ implies $\|d(T(x_1),T(x_2))\| \leq kK\|d(x_1,x_2)\|$. Then we have $\frac{\|d(T(x_1),T(x_2))\|}{\|s\|} \leq kK\frac{\|d(x_1,x_2)\|}{\|s\|}$

$$\Rightarrow \frac{\|s\|+\|d(T(x_1),T(x_2))\|}{\|s\|} - 1 \leq kK\left(\frac{\|s\|+\|d(x_1,x_2)\|}{\|s\|} - 1\right)$$

$$\Rightarrow \frac{1}{\frac{\|s\|}{\|s\|+\|d(T(x_1),T(x_2))\|}} - 1 \leq kK\left(\frac{1}{\frac{\|s\|}{\|s\|+\|d(x_1,x_2)\|}} - 1\right)$$

$\Rightarrow \frac{1}{M_d(T(x_1),T(x_2),s)} - 1 \leq kK\left(\frac{1}{M_d(x_1,x_2,s)} - 1\right)$. Hence T is fuzzy cone contractive on $(X,M_d,\cdot)$. □

Finally, a contractive mapping is given as an example to show that set of contractions defined on fuzzy cone metric and fuzzy metric spaces are not the same.

Remark 3.16. Let $E = \mathbb{R}^2$ and $P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \subset E$. Here P is a normal cone with $K = 1$. On the other hand let $X = \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in [0,1]\} \cup \{(0, y_1) \in \mathbb{R}^2 : y_1 \in [0,1]\}$ and $d : X \times X \to E$ be a mapping given by

$$d((x_1, 0), (x_2, 0)) = \left(\tfrac{4}{3}|x_1 - x_2|, |x_1 - x_2|\right),$$
$$d((0, y_1), (0, y_2)) = \left(|y_1 - y_2|, \tfrac{2}{3}|y_1 - y_2|\right),$$
$$d((x_1, 0), (0, y_1)) = d((0, y_1), (x_1, 0)) = \left(\tfrac{4}{3}x_1 + y_1, x_1 + \tfrac{2}{3}y_1\right).$$

In [5], it is noted that (X, d) is a complete cone metric space and moreover, the mapping T defined on X given by $T((a, 0)) = (0, a)$, $T((0, a)) = \left(\frac{1}{2}a, 0\right)$ is a contraction on (X, d) satisfying the contractive condition $d(T((x_1, y_1)), T((x_2, y_2))) \preceq kd((x_1, y_1), (x_2, y_2))$ for any (x_1, y_1), $(x_2, y_2) \in X$ with constant $k = \frac{3}{4}$ which $(0, 0) \in X$ is the unique fixed point however in Euclidean metric on X, T is not a contraction. On the other hand, by Remark 3.8, the standard fuzzy cone symmetric space $(X, M_d, \cdot)$ induced by d is fuzzy cone metric space. Hence by Proposition 3.15, T is also a fuzzy cone contractive on $(X, M_d, \cdot)$ with contractive constant $\frac{3}{4}$ which has $(0, 0) \in X$ as the unique fixed point. But if we consider the Euclidean metric on X and $(X, M, \cdot)$ is considered as the standard fuzzy metric space induced by this metric, by Proposition 3.7 in [4], T is not a contraction on $(X, M, \cdot)$

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