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THE OPEN MONOPHONIC CHROMATIC NUMBER OF A GRAPH

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Abstract. A set P of vertices in a connected graph G is called *open monophonic chromatic set* if P is both an open monophonic set and a chromatic set. The minimum cardinality among the set of all open monophonic chromatic sets is called *open monophonic chromatic number* and is denoted by $\chi_{om}(G)$. Here properties of open monophonic chromatic number of connected graphs are studied. Open monophonic chromatic number of some standard graphs are identified. For $3 \leq m \leq n$, there is a connected graph G such that $\chi(G) = m$ and $\chi_{om}(G) = n$. For $3 \leq m \leq n$, there is a connected graph G such that $om(G) = m$ and $\chi(G) = \chi_{om}(G) = n$.Let r, d be two integers such that $r < d \leq 2r$ and suppose $k \geq 2$. Then there exists a connected graph G with *rad* $G = r$, *diam* $G = d$ and $\chi_{om}(G) = k$.

Key Words: Chromatic set, Chromatic number, Open Monophonic number, Open Monophonic chromatic number $\cdots$.
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1. Introduction

All the graphs considered here are undirected, connected and simple. For basic graph theoretic notation and terminology refer Buckley and Harary [3] and Chartrand and Zhang [4]. For any two vertices u and v in G the *distance d(u,v)* is the length of a shortest $u - v$ path in G. $u - v$ path of length $d(u, v)$ is called *u-v geodesic*. Let G be a graph and

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v be a vertex of G. The *eccentricity* of the vertex v is the maximum distance from v to any vertex. That is, $e(v) = max\{d(v, w) : w \in V(G)\}$. The *radius*, $rad\ (G)$ of G is the minimum eccentricity among the vertices of G. Therefore, $rad\ (G) = min\{e(v) : v \in V(G)\}$. The *diameter*, $diam\ (G)$ of G is the maximum eccentricity among the vertices of G. Thus, $diam\ (G) = max\{e(v) : v \in V(G)\}$ [8,9].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. That is the chord in a path $P : u_1, u_2, ..., u_n$ as an edge $u_i - u_j$ with $j \geq i + 2$. A path P is called *monophonic* if it is a chordless path. A *monophonic set* of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path of some pair of vertices of M. The *monophonic number* of a graph G is the cardinality of a minimum monophonic set of G[6,7].

A set M of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in M$, or v is an internal vertex of an $x - y$ monophonic path for some $x, y \in M$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* and this cardinality is the *open monophonic number*, $om(G)$ of G. An open monophonic set of cardinality $om(G)$ is called an *om-set* of G[12]. Detailed studies of monophonic numbers and open monophonic number are available in[1,2,11].

A *vertex coloring* or simply *coloring* of a graph G is a function $f : V(G) \to \mathbb{N}$ satisfying $f(u) = f(v) \Rightarrow \{u, v\}$ not belongs to $\phi(E(G))$ for all $u, v \in V(G)$; that means u and v are not adjacent in G. For $k \in \mathbb{N}$, a *k-vertex coloring* or a *proper k-vertex coloring* of G is a proper vertex coloring $c : V(G) \to \{1, 2, ..., k\}$. G is said to be k-colorable if G has a proper k- vertex coloring. The least $k \in \mathbb{N}$ such that G is k-vertex colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. If $\chi(G) = k$, then G is called *k-chromatic graph* [5].

Number of edges incident on a vertex v is the *degree of the vertex*, denoted by $deg(v)$. The *maximum degree* of G is the maximum degree among all the vertices of G and is denoted by $\Delta(G)$. The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices that are adjacent with v. A vertex v is an *extreme vertex* or *simplicial vertex* if the sub graph induced by its neighbourhood is complete. A vertex v in a connected graph G is a *cut-vertex* of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be *semi- extreme vertex* or *semi simplicial vertex* of G if $\Delta(< N(v) >) = |N(v) - 1|$. A graph G is said to be *semi-extreme graph* if every vertex of G is a semi extreme vertex.

Definition 1.1.

Let G be a k-chromatic graph and $V(G)$ the vertex set of G. A set $C \subseteq V(G)$ is called *chromatic set* if C contains all k vertices of different colors in G [10].

Remark 1.2.

In view of the definition 1.1 , chromatic number of G is the minimum cardinality among all the chromatic sets of G. That is $\chi(G) = min\{C_i, C_i \text{ is a chromatic set of } G\}$.

Definition 1.3.

A set $C \subseteq V(G)$ is called *monophonic chromatic set* if C is both a monophonic set and a chromatic set. The minimum cardinality among all monophonic chromatic sets is called *monophonic chromatic number* and is denoted by $\chi_m(G)$ [10].

Example 1.4. Consider the graph G given in Figure 2.

Here G is a connected graph with chromatic number 3. The set $C_1 = \{v_1, v_2, v_3\}$ is a minimum chromatic set. But it is not a monophonic set. The set $C_2 = \{v_2, v_8\}$ is a minimum monophonic set, but it is not a chromatic set. Here $C_3 = \{v_2, v_3, v_8,\}$ is a minimum monophonic chromatic set. Therefore $\chi_m(G) = 3$.

Theorem 1.5. *Every open monophonic set of a graph G contains it's extreme vertices. Also if the set of all extreme vertices of G is an open monophonic set, it is the unique minimum open monophonic set of G[12].*

2. Open Monophonic Chromatic Number

Definition 2.1.

A set P of vertices in a connected graph G is called *open monophonic chromatic set* if P is both an open monophonic set and a chromatic set. The minimum cardinality among the set of all open monophonic chromatic sets is called *open monophonic chromatic number* and is denoted by $\chi_{om}(G)$.

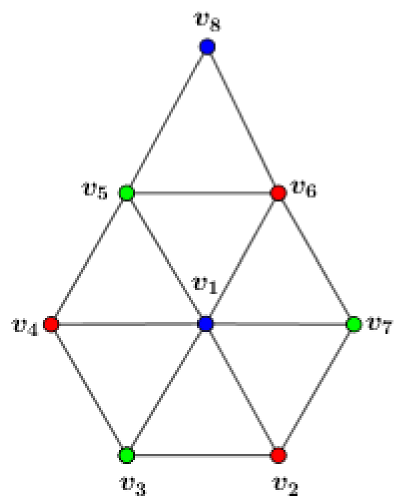

FIGURE 1. Graph G with monophonic chromatic number 3

Example 2.2.

Consider the graph G given in Figure 2. Here the set $M_1 = \{v_1, v_4\}$ is a minimum monophonic chromatic set. That is $\chi_m(G) = 2$. The set $\{v_1, v_4, v_8\}$ is a minimum open monophonic chromatic set. Hence $\chi_{om}(G) = 3$.

Theorem 2.3. *For any connected graph G of order n, $2 \leq \chi_{om}(G) \leq n$.*

Proof: Since open monophonic set needs at least two vertices, every open monophonic chromatic set contains at least two vertices. Therefore $\chi_{om}(G) \geq 2$. Now the set of all vertices of G is an open monophonic chromatic set, $\chi_{om}(G) \leq n$.

Remark 2.4.

The bounds in this theorem are sharp. For complete graph, $\chi_{om}(G) = n$. For path graph P_n with even number of vertices, $\chi_{om}(P_n) = 2$.

Since every open monophonic set is a monophonic set, every open monophonic chromatic set is also a monophonic chromatic set. Combining with Theorem 2.1 we have the following theorem:

Theorem 2.5. *For any connected graph G of order n, $2 \leq \chi_m(G) \leq \chi_{om}(G) \leq n$.*

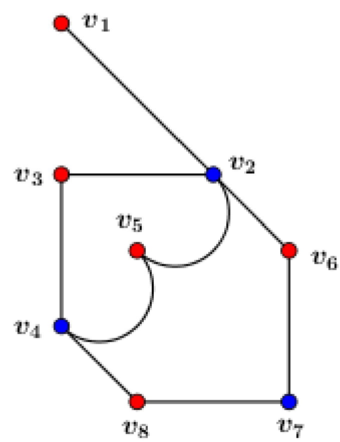

FIGURE 2. Graph G with $\chi(G) = 2, m(G) = 2$ and $\chi_{om}(G) = 3$.

Since every open monophonic chromatic set is a monophonic chromatic set we have the following result from Theorem 1.1.

Theorem 2.6. *Every open monophonic chromatic set of graph G contains it's extreme vertices. Also if the set of all extreme vertices of G is an open monophonic chromatic set, it is the unique minimum open monophonic chromatic set of G.*

Corollary 2.7.

For complete graph K_n with n vertices, $\chi_{om}(K_n) = n$.

Remark 2.8.

Converse of Corollary 1 need not be true. For cycle graph C_4 with four vertices, $\chi_{om}(C_4) = 4$, which is not complete.

Theorem 2.9. *Let G be a connected graph and non-trivial. If G has no extreme vertices, then* $\chi_{om}(G) \geq 3$.

Proof: For a connected graph having no extreme vertex contains at least four vertices. Suppose P is an open monophonic chromatic set. Let $x \in P$. Then by definition, there exist two vertices u and v such that x lies in an internal vertex of $u - v$ monophonic path. Hence $\{u, v, x\}$ lies in P shows that $\chi_{om}(G) \geq 3$.

Theorem 2.10. *Let $G = C_n$, the cycle graph of n vertices. Then*

$$\chi_{om}(C_n) = \begin{cases} 3, & for\ n \neq 4,5 \\ 4, & for\ n = 4,5 \end{cases}$$

Proof: For $n \neq 4,5$ the set of vertices $\{v_2, v_4, v_6\}$ form a minimum open monophonic chromatic set. For $n = 4,5$ the set of vertices $\{v_1, v_2, v_3, v_4\}$ is a minimum open monophonic chromatic set. For $n = 3$, G is a complete graph and the result follows from Corollary 1.

Theorem 2.11. *If $G = P_n$ is the path graph with n vertices, then*

$$\chi_{om}(P_n) = \begin{cases} 2, & for\ n\ is\ even \\ 3, & for\ n\ is\ odd \end{cases}$$

Proof: Suppose $P : u_1, u_2, ..., u_n$ is the path graph with end vertices u_1 and u_n. When n is even, the vertices u_1 and u_n have different colors. Thus the set $M = \{u_1, u_n\}$ is a minimum chromatic set and is a minimum open monophonic set. Hence M is an open monophonic chromatic set. Therefore $\chi_m(G) = 2$. If n is odd, the vertices in M are of same color. Therefore $M \cup \{u_{n-1}\}$ is a minimum open monophonic chromatic set. That is $\chi_{om}(G) = 3$.

Theorem 2.12. *If $G = K_{m,n}$ is the complete bipartite graph with particians m and n, then $\chi_{om}(K_{m,n}) = 4$, $2 \leq m \leq n$.*

Proof: Consider the partician sets $X = \{u_1, u_2, ...u_m\}$ and $Y = \{v_1, v_2, ...v_n\}$ of G. Now chromatic number of a complete bipartite graph is 2. X is not a chromatic set since the vertex u_i are of same color. Similarly each vertex v_j are of same color. It is easily verified that no 3-element subset of vertices of G is an open monophonic set of G so that $om(G) \geq 4$. Let P be any set of four vertices formed by taking two vertices from each of X and Y. Combining all these arguments it is clear that P is an open monophonic chromatic set of G so that $\chi_{om}(K_{m,n}) = 4$.

Theorem 2.13. *Let $W_n = K_1 + C_{n-1}$ be the wheel graph of n vertices. Then*

$$\chi_{om}(W_n) = \begin{cases} 4, & for\ n \neq 5,6 \\ 5, & for\ n = 5,6. \end{cases}$$

Proof: Given $W_n = K_1 + C_{n-1}$. Let u be the center vertex and $C = \{v_1, v_2, ...v_{n-1}\}$ the vertices in C_{n-1}. Since u is adjacent with all

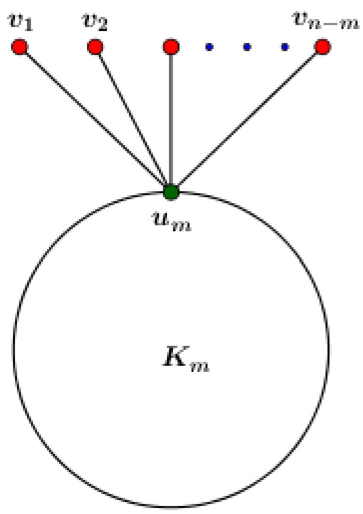

FIGURE 3. Graph G with $\chi(G) = m$ and $\chi_{om}(G) = n$.

other vertices, u belongs to any chromatic set of W_n. For $n \neq 5, 6$, assign four different colors to the vertices u, v_1, v_3 and v_5. Using these colors we can assign one color to each vertex in W_n in a non adjacent way. Hence chromatic number of the graph is ≤ 4. Now the vertex set $\{v_1, v_3, v_5\}$ form a minimum open monophonic set. Combining these results we have $\{u, v_1, v_3, v_5\}$ is a minimum open monophonic chromatic set. Thus $\chi_{om}(W_n) = 4$. For $n = 4$, W_4 is complete graph and by Corollary 1, $\chi_{om}(W_n) = 4$.

For $n = 5, 6$, no three vertices of C form a minimum open monophonic set. $D = \{v_1, v_2, v_3, v_4\}$ is a minimum open monophonic set. Thus $D \cup \{u\}$ is a minimum open monophonic chromatic set. That is $\chi_{om}(W_n) = 5$.

3. Realisation Results

Theorem 3.1. *For $3 \leq m \leq n$, there is a connected graph G such that $\chi(G) = m$ and $\chi_{om}(G) = n$.*

Proof: Consider a complete graph K_m of m vertices $\{u_1, u_2, ..., u_m\}$. Add $n - m$ pendant vertices with the vertex u_m. This is the graph G (See Figure 3). Since each vertex $u_1, u_2...u_m$ is of degree at least $m - 1$, they belongs to every m-colorable set. Each vertex $v_1, v_2, ..., u_{n-m}$ is a pendant vertex so that they can color any one of their non adjacent vertex u_i. Thus there are exactly m colors. Therefore $\chi(G) = m$.

Now the vertices $v_1, v_2, ..., v_{n-m}$ are extreme vertices and $u_1, u_2, ...,$ u_{m-1} are semi-extreme vertices which belong to every minimum open monophonic set. Since u_m is adjacent with all other vertices so that it belongs to any chromatic set. In fact, the set $\{u_1, u_2, ..., u_m, v_1, v_2, ..., v_{n-m}\}$ is a minimum open monophonic chromatic set. Therefore $\chi_{om}(G) = (n-m) + m = n$.

Theorem 3.2. *For a triplet of integers (m,n,t) with* $3 \leq m \leq n < t$ *and* $t = m + n + 1$*, there is a connected graph* G *of order* t *such that* $om(G) = m$ *and* $\chi(G) = \chi_{om}(G) = n$.

Proof: Consider a cycle graph of $n+1$ vertices with vertex set $\{v_1, v_2, v_3, ...,$ $v_{n+1}\}$. Join each vertex v_i with all other vertices $v_j, j \neq i$ except two. Let it be v_1 and v_3. Add m pendant vertices $u_1, u_2..., u_m$ at $v_1, v_2..., v_s (m \leq n)$ respectively. This is the graph G(See Figure 4). This graph contains $t = m + n + 1$ vertices.

Each vertex u_i is an extreme vertex and belongs to every minimum open monophonic set. In fact the set $P_1 = \{u_1, u_2..., u_s\}$ is a minimum open monophonic set of G. Therefore $om(G) = m$.

Now consider the set $P_2 = \{v_1, v_2, v_4, ..., v_{n+1}\}$ of n vertices. Since each vertex of P_2 is of at least $n-1$ degree $\chi(G) \geq n$. Assign the color of v_{i+1} to u_i for $1 \leq i \leq m$ since they are non adjacent, the set of vertices $P = \{u_1, u_2, ...u_m, v_{m+2}, v_{m+3}..., v_n, v_{n+1}\}$ is a minimum open monophonic chromatic set with cardinality n. That is $\chi_{om}(G) = n$.

Theorem 3.3. *Let* r, d *be two integers such that* $r < d \leq 2r$ *and suppose* $k \geq 2$*. Then there exists a connected graph* G *with rad* $G = r$*, diam* $G = d$ *and* $\chi_{om}(G) = k$.

Proof: Construct a graph G as follows: Let C_{2r} be a cycle of order $2r$ with vertex set $\{x_1, x_2, ..., x_{2r}\}$ and let P_{d-r+1} be a path of order $d-r+1$ with vertex set $\{y_1, y_2, ...y_{d-r}\}$. Identify y_0 with x_0. Add $k-2$ new vertices $z_1, z_2, ...z_{k-2}$ and join each vertex z_i for $(1 \leq i \leq k-2)$ with the vertex y_{d-r-1}. Join the edge $x_r - x_{r+2}$. This is the graph G in Figure 5. Then *rad* $G = r$ due to the vertex x_r and *diam* $G = d$ due to the vertex x_{r+1}.

Now the vertices $P = \{z_1, z_2, ...z_{k-2}, y_{d-r}\}$ are end vertices and x_{r+1} is an extreme vertex. By definition they belongs to every minimum

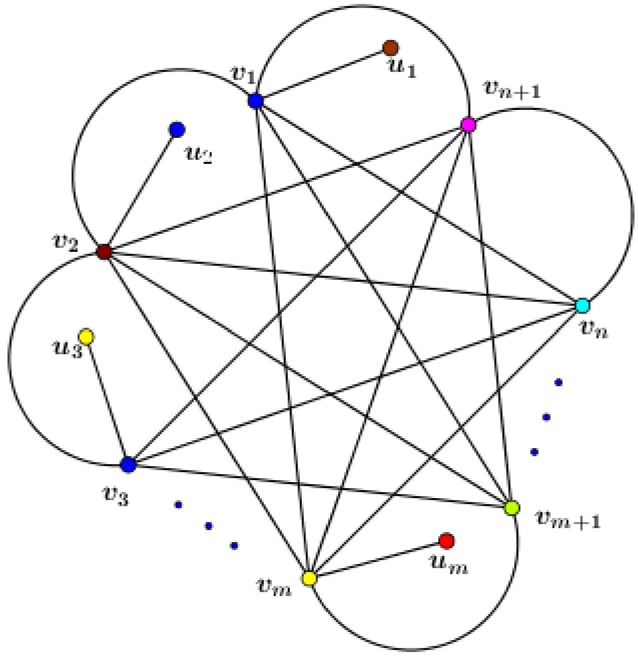

FIGURE 4. Graph G with $om(G) = m$ and $\chi(G) = \chi_{om}(G) = n$.

open monophonic chromatic set. Infact $P \cup \{x_{r+1}\}$ is a minimum open monophonic chromatic set with cardinality k. That is $\chi_{om}(G) = k$.

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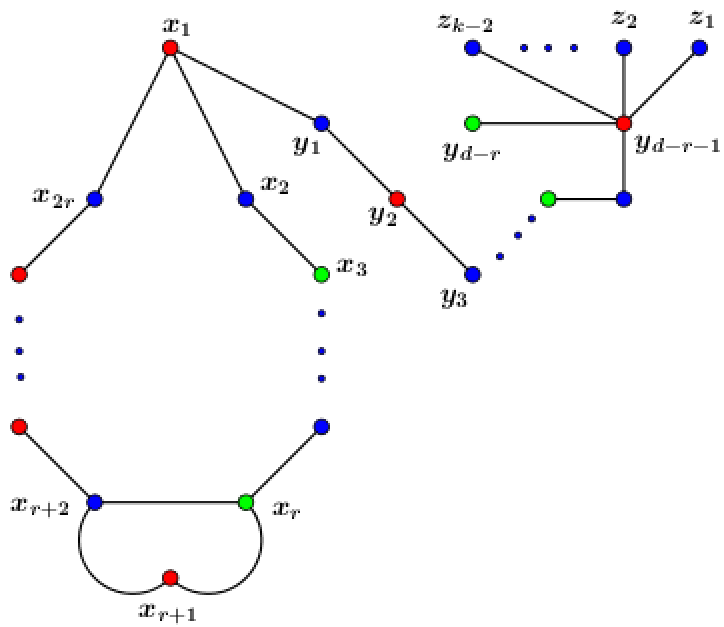

FIGURE 5. Graph G with $rad\ G = r$, $diam\ G = d$ and $\chi_{om}(G) = k$.

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