Journal of Hyperstructures 3 *(1) (2014), 68-80.*
ISSN: 2251-8436 print/2322-1666 online

USING MODIFIED TWO-DIMENSIONAL BLOCK-PULSE FUNCTIONS FOR THE NUMERICAL SOLUTION OF NONLINEAR TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper, the Modified two-dimensional block-pulse functions (M2D-BFs) are used as a new set of basis functions for expanding two-dimensional functions. The main properties of M2D-BFs are determined and an operational matrix for integration obtained. M2D-BFs are used to solve nonlinear two-dimensional Volterra integral equations of the first kind. Some theorems are included to show convergence and advantage of the method. Finally, numerical examples is presented to show the efficiency and accuracy of the method.

Key Words: Nonlinear two-dimensional Volterra integral equations, Block-pulse functions, Operational matrix.

2010 Mathematics Subject Classification: Primary: 45D05; Secondary: 65D30, 65R20.

1. Introduction

Many phenomena in physics and engineering fields give rise to a nonlinear two-dimensional Volterra integral equation:

$$\int_0^x \int_0^y R(x,y,s,t,u(s,t))dtds = f(x,y); \ (x,y) \in D, \tag{1.1}$$

Received: 27 July 2013, Accepted: 15 January 2014. Communicated by D. Khojasteh Salkuyeh;

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where $u(s,t)$ is an unknown scalar valued function defined on district $D = [0,T_1) \times [0,T_2)$. The function $R(x,y,s,t,u)$ is given function defined on

$$W = \{(x,y,s,t,u) : 0 \leq s \leq x \leq T_1, 0 \leq t \leq y \leq T_2\}. \tag{1.2}$$

In this paper, we put

$$R(x,y,s,t,u) = k(x,y,s,t)[u(s,t)]^p, \tag{1.3}$$

where p is positive integer [7, 11].

Since any finite interval [a,b] can be transformed to $[0,1]$ by linear maps, without any loss of generality, we consider $[0,1)$ in replace of $[0,T_1)$ or $[0,T_2)$. While several numerical methods for approximating the solution of one-dimensional Volterra integral equations are known, for two-dimensional only a few are discussed in the literature. The numerical solution of equations of the type of (1.1) seems to have first been considered by Bel' tyukov and Kuznechikhina [2] where they proposed an explicit Rung-Kutta type method of order 3 without any convergence analysis. A bivariate cubic spline functions method of full continuity was obtained by Singh [15]. Brunner and Kauthen [3] introduced collocation and iterated collocation method for two-dimensional linear Volterra integral equations. An asymptotic error expansion of the iterated collocation solution for two-dimensional linear and nonlinear Volterra integral equations was obtained by Han and Zhang [6] and Guoqiang [4], respectively. Hadizadeh and Moatamedi [5] have investigated a differential transformation approach for nonlinear two-dimensional Volterra integral equations. Maleknejad et al. [9] used two-dimensional block-pulse functions to nonlinear integral equations. Babolian et al.[1] used two-dimensional triangular functions to nonlinear two-dimensional Volterra-Fredholm integral equations. Mirzaee and Rafei [11] used the block by block method for the numerical solution of the nonlinear two-dimensional Volterra integral equations.

Mirzaee and Hadadiyan [12] use the modified two-dimensional block-pulse functions method for the solutions mixed nonlinear Volterra- Fredholm type integral equations. In the present paper, we apply modification of block-pulse functions [12], to solve the nonlinear two-dimensional Volterra integral Eq. (1.1) with Eq. (1.2), and this is organized as follows: In Section 2, we will introduce M2D-BFs and its properties. In Section 3, theorems are proved for convergence analysis. In Section 4, we will apply these sets of M2D-BFs for approximating the solution of nonlinear Volterra integral equations. Numerical results are reported in Section 5. Finally , Section 6 concludes the paper .

2. M2D-BFs and their properties

Definition. An $(m+1)^2$-set of M2D-BFs consists of $(m+1)^2$ functions which are defined over district D as follows:

$$\phi_{i_1,i_2}(x,y)=\begin{cases}1, \ (x,y)\in D_{i_1,i_2}, \ i_1,i_2=0(1)m\\ \\ 0, \ otherwise\end{cases}, \tag{2.1}$$

where

$$D_{i_1,i_2}=\{(x,y): x\in I_{i_1,\varepsilon}, y\in I_{i_2,\varepsilon}\}, \tag{2.2}$$

and

$$I_{\alpha,\varepsilon}=\begin{cases}[0,h-\varepsilon), \ \alpha=0\\ [\alpha h-\varepsilon,(\alpha+1)h-\varepsilon), \ \alpha=1(1)m\\ [1-\varepsilon,1), \ \alpha=m\end{cases}, \tag{2.3}$$

where m is arbitrary positive integer, and $h=\frac{1}{m}$.

From Eq. (2.1), it is clearly that the M2D-BFs can be expressed by the two modified one-dimensional block-pulse functions (M1D-BFs):

$$\phi_{i_1,i_2}(x,y)=\phi_{i_1}(x)\phi_{i_2}(y), \tag{2.4}$$

where $\phi_{i_1}(x)$ and $\phi_{i_2}(y)$ are the M1D-BFs related to variables x and y, respectively [9].

The M2D-BFs are disjointed with each other:

$$\phi_{i_1,i_2}(x,y)\phi_{j_1,j_2}(x,y)=\begin{cases}\phi_{i_1,i_2}(x,y), & i_1=j_1, i_2=j_2\\ 0, & otherwise\end{cases}, \tag{2.5}$$

and are orthogonal with each other:

$$\int_0^1\int_0^1 \phi_{i_1,i_2}(x,y)\phi_{j_1,j_2}(x,y)dydx=\begin{cases}\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon}), & i_1=j_1, i_2=j_2\\ 0, & otherwise\end{cases}, \tag{2.6}$$

where $(x,y)\in D$, $i_1,i_2,j_1,j_2=0(1)m$ and $\triangle(I_{i_1,\varepsilon})$ and $\triangle(I_{i_2,\varepsilon})$ are length of intervals $I_{i_1,\varepsilon}$ and $I_{i_2,\varepsilon}$, respectively.

2.1. **Vector forms.** We can also define $\Phi_{m,\varepsilon}(x,y)$, the M2D-BFs vector, as follows:

(2.7)
$$\Phi_{m,\varepsilon}(x,y) = [\phi_{0,0}(x,y), \dots, \phi_{0,m}(x,y), \dots, \phi_{m,0}(x,y), \dots, \phi_{m,m}(x,y)]^T,$$

were $(x,y) \in D$ and

$$\Phi_{m,\varepsilon}(x,y) = \Phi_{m,\varepsilon}(x) \otimes \Phi_{m,\varepsilon}(y), \tag{2.8}$$

and

$$\Phi_{m,\varepsilon}(x) = [\phi_0(x), \phi_1(x), \dots, \phi_m(x)]^T. \tag{2.9}$$

Also we have:

$$\int_0^x \int_0^y \Phi_{m,\varepsilon}(s,t)dtds = \int_0^x \int_0^y \Phi_{m,\varepsilon}(s) \otimes \Phi_{m,\varepsilon}(t)dtds =$$
$$\int_0^x \Phi_{m,\varepsilon}(s)ds \otimes \int_0^y \Phi_{m,\varepsilon}(t)dt = p_{m,\varepsilon} \otimes p_{m,\varepsilon} = P_{m,\varepsilon}, \tag{2.10}$$

where $p_{m,\varepsilon}$ is operational matrix of 1D-BFs defined over $[0,1)$, see [9].

From Eqs. (2.5) and (2.7) we have:

(2.11)
$$\Phi_{m,\varepsilon}(x,y)\Phi^T_{m,\varepsilon}(x,y) = \begin{pmatrix} \phi_{0,0}(x,y) & 0 & \dots & 0 \\ 0 & \phi_{0,1}(x,y) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \phi_{m,m}(x,y) \end{pmatrix}.$$

Let X be a $(m+1)^2$-vector by using Eq. (2.7) we will have:

$$\Phi_{m,\varepsilon}(x,y)\Phi^T_{m,\varepsilon}(x,y)X = \widetilde{X}\Phi_{m,\varepsilon}(x,y), \tag{2.12}$$

where $\widetilde{X} = diag(X)$ is a $(m+1)^2 \times (m+1)^2$ diagonal matrix. The disjoint property of $\Phi_{m,\varepsilon}(x,y)$ also implies that for every $(m+1)^2 \times (m+1)^2$-matrix A, we have:

$$\Phi^T_{m,\varepsilon}(x,y)A\Phi_{m,\varepsilon}(x,y) = \widehat{A}^T\Phi_{m,\varepsilon}(x,y), \tag{2.13}$$

where $\widehat{A}^T$ is an $(m+1)^2$-vector with elements equal to the diagonal entries of matrix A.

2.2. **M2D-BFs expansions.** An arbitrary function $f(x,y)$ defined over district $L^2(D)$ can be expanded by the M2D-BFs as

$$f(x,y) \simeq f_{m,\varepsilon}(x,y) = \sum_{i_1=0}^{m}\sum_{i_2=0}^{m} f_{i_1,i_2}\phi_{i_1,i_2}(x,y)$$
$$= F^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y) = \Phi^T_{m,\varepsilon}(x,y)F_{m,\varepsilon}, \tag{2.14}$$

where

$$F_{m,\varepsilon} = [f_{0,0},\dots,f_{0,m},\dots,f_{m,0},\dots,f_{m,m}]^T, \tag{2.15}$$

and f_{i_1,i_2}, are obtained as:

$$f_{i_1,i_2} = \frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})}\int_{I_{i_1,\varepsilon}}\int_{I_{i_2,\varepsilon}} f(x,y)dydx. \tag{2.16}$$

Similarly an arbitrary function of four variables, $k(x,y,s,t)$, on district $L^2(D\times D)$ may be approximated with respect to M2D-BFs such as:

$$k(x,y,s,t) \simeq \Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t), \tag{2.17}$$

where $\Phi_{m,\varepsilon}(x,y)$ and $\Phi_{m,\varepsilon}(s,t)$ are M2D-BFs vector of dimension $(m+1)^2$, and $K_{m,\varepsilon}$ is the $(m+1)^2\times(m+1)^2$ M2D-BFs coefficients matrix.

3. Convergence analysis

In this sections, we show that the M2D-BFs method in the previous sections, is convergent and its order of convergence is $O(\frac{1}{km})$. For our purposes we will need the following theorems.

Theorem 1. Let

$$f_{m,\varepsilon}(x,y) = \sum_{i_1=0}^{m}\sum_{i_2=0}^{m} f_{i_1,i_2}\phi_{i_1,i_2}(x,y), \tag{3.1}$$

and for $i_1, i_2 = 0(1)(m)$ we have:

$$f_{i_1,i_2} = \frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})}\int_0^1\int_0^1 f(x,y)\phi_{i_1,i_2}(x,y)dxdy \quad . \tag{3.2}$$

Then the criterion of this approximation is that the mean square error between $f(x,y)$ and $f_{m,\varepsilon}(x,y)$ in the interval $(x,y)\in D$:

$$\epsilon = \int_0^1\int_0^1 (f(x,y) - f_{m,\varepsilon}(x,y))^2 dxdy, \tag{3.3}$$

reaches its minimum. Moreover, we have:

$$\int_0^1 \int_0^1 f^2(x,y)dxdy = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} f_{i_1,i_2}^2 ||\phi_{i_1,i_2}(x,y)||^2. \tag{3.4}$$

Proof. Proof is like similar theorem in [8].

Theorem 2. Assume $f(x,y)$ is continuous and is differentiable over district $[-h, 1+h]\times[-h, 1+h]$, and $f_{m,\varepsilon_i}(x,y); \varepsilon_i = \frac{ih}{k}$, for $i = 0(1.1)(k-1)$, are correspondingly M2D-BFs(ε_0)=2D-BFs, M2D-BFs (ε_1), $\cdots$, M2D-BFs(ε_{k-1}) expansions of $f(x,y)$ base on $(m+1)^2$ M2D-BFs over district D and

$$\bar{f}_{m,k}(x,y) = \frac{1}{k} \sum_{i=0}^{k-1} f_{m,\varepsilon_i}(x,y), \tag{3.5}$$

then for sufficient large m we have:

$$||f(x,y) - \bar{f}_{m,k}(x,y)||_\infty \le \frac{1}{k} \max_{\varepsilon_i} ||f(x,y) - f_{m,\varepsilon_i}(x,y)||_\infty. \tag{3.6}$$

Proof. see [12]

Theorem 3. Let the representation error between $f(x,y)$ and its two-dimensional block-pulse functions, $f_m(x,y) = f_{m,\varepsilon_0}(x,y)$ (M2D-BFs(ε_0)= 2D-BFs), over the district D, as follows :

$$e(x,y) = f(x,y) - f_m(x,y). \tag{3.7}$$

Then $||e(x,y)|| = O(\frac{1}{m})$ and

$$\lim_{m\to+\infty} f_m(x,y) = \lim_{m\to+\infty} f_{m,\varepsilon_0}(x,y) = f(x,y). \tag{3.8}$$

Proof. Proof is like similar theorem in [10].

Theorem 2 and 3 conclude that error estimation for M2D-BFs is $||e(x,y)|| = O(\frac{1}{km})$.

Suppose that $f(x,y)$ is approximated by

$$f_{m,\varepsilon_i}(x,y) = \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} f_{i_1,i_2} \phi_{i_1,i_2}(x,y), \tag{3.9}$$

from [12] we have:

$$\lim_{m\to+\infty} f_{m,\varepsilon_i}(x,y) = f(x,y). \tag{3.10}$$

4. Method of solution

In this section, we solve two-dimensional nonlinear Volterra integral equations of the first kind of the form Eq. (1.1) with Eq. (1.3) by using M2D-BFs.

We now approximate functions $u(x,y), f(x,y), [u(x,y)]^p$ and $k(x,y,s,t)$ with respect to M2D-BFs by the way mentioned in Section 2 as

$$\begin{cases} u(x,y) \simeq U_{m,\varepsilon}^T \Phi_{m,\varepsilon}(x,y), \\ f(x,y) \simeq F_{m,\varepsilon}^T \Phi_{m,\varepsilon}(x,y), \\ [u(x,y)]^p \simeq \Phi_{m,\varepsilon}^T(x,y) U_{m,\varepsilon,p}, \\ k(x,y,s,t) \simeq \Phi_{m,\varepsilon}^T(x,y) K_{m,\varepsilon} \Phi_{m,\varepsilon}(s,t), \end{cases} \tag{4.1}$$

where $\Phi_{m,\varepsilon}(x,y)$ is defined in Eq. (2.4), the vectors $U_{m,\varepsilon}, F_{m,\varepsilon}, U_{m,\varepsilon,p}$, and matrix $K_{m,\varepsilon}$ are M2D-BFs coefficients of $u(x,y), f(x,y), [u(x,y)]^p$ and $k(x,y,s,t)$, respectively.

Lemma 1. Let $(m+1)^2$-vectors $U_{m,\varepsilon}$ and $U_{m,\varepsilon,p}$ be M2D-BFs coefficients of $u(x,y)$ and $[u(x,y)]^p$, respectively. If

$$U_{m,\varepsilon} = [u_{0,0}, \ldots, u_{0,m}, \ldots, u_{m,0}, \ldots, u_{m,m}]^T, \tag{4.2}$$

then

$$U_{m,\varepsilon,p} = [u_{0,0}^p, \ldots, u_{0,m}^p, \ldots, u_{m,0}^p, \ldots, u_{m,m}^p]^T, \tag{4.3}$$

where $p \geq 1$, is a positive integer.

Proof.(By induction) When $p = 1$, Eq. (4.3) follows at once from $[u(x,y)]^p = u(x,y)$. Suppose that Eq. (4.3) holds for p, we shall deduce it for $(p+1)$. Since $[u(x,y)]^{p+1} = u(x,y)[u(x,y)]^p$, from Eqs. (4.1), (2.12) it follows that

$$\begin{aligned} [u(x,y)]^{p+1} = u(x,y)[u(x,y)]^p &\simeq U_{m,\varepsilon}^T \Phi_{m,\varepsilon}(x,y) \Phi_{m,\varepsilon}^T(x,y) U_{m,\varepsilon,p} \\ &= U_{m,\varepsilon}^T \widetilde{U}_{m,\varepsilon,p} \Phi_{m,\varepsilon}(x,y). \end{aligned} \tag{4.4}$$

Now by using Eq. (4.3) we obtain

$$U_{m,\varepsilon}^T \widetilde{U}_{m,\varepsilon,p} = [u_{0,0}^{p+1}, \ldots, u_{0,m}^{p+1}, \ldots, u_{m,0}^{p+1}, \ldots, u_{m,m}^{p+1}]^T, \tag{4.5}$$

therefore Eq. (4.3) holds for $(p+1)$, and the lemma is established.□

To approximate the integral part in Eq. (1.1), from Eq. (4.1) we get

$$(4.6)\quad \int_0^x \int_0^y k(x,y,s,t)[u(s,t)]^p dtds \simeq$$
$$\int_0^x \int_0^y \Phi^T_{m,\varepsilon}(x,y) K_{m,\varepsilon} \Phi_{m,\varepsilon}(s,t) \Phi^T_{m,\varepsilon}(s,t) U_{m,\varepsilon,p} dtds =$$
$$\Phi^T_{m,\varepsilon}(x,y) K_{m,\varepsilon} \left(\int_0^x \int_0^y \Phi_{m,\varepsilon}(s,t) \Phi^T_{m,\varepsilon}(x,y) U_{m,\varepsilon,p} dtds \right) =$$
$$\Phi^T_{m,\varepsilon}(x,y) K_{m,\varepsilon} \int_0^x \int_0^y \widetilde{U}_{m,\varepsilon,p} \Phi_{m,\varepsilon}(s,t) dtds =$$
$$\Phi^T_{m,\varepsilon}(x,y) K_{m,\varepsilon} \widetilde{U}_{m,\varepsilon,p} \int_0^x \int_0^y \Phi_{m,\varepsilon}(s,t) dtds.$$

Now by using Eq. (2.10), we have:

$$(4.7)\quad \int_0^x \int_0^y k(x,y,s,t)[u(s,t)]^p dtds \simeq \Phi^T_{m,\varepsilon}(x,y) K_{m,\varepsilon} \widetilde{U}_{m,\varepsilon,p} P_{m,\varepsilon} \Phi_{m,\varepsilon}(x,y),$$

in which $K_{m,\varepsilon} \widetilde{U}_{m,\varepsilon,p} P_{m,\varepsilon}$ is an $(m+1)^2 \times (m+1)^2$ matrix. By using Eq. (2.13) we have:

$$(4.8)\quad \int_0^x \int_0^y k(x,y,s,t)[u(s,t)]^p dtds \simeq \widehat{U}^T_{m,\varepsilon,p} \Phi_{m,\varepsilon}(x,y),$$

where $\widehat{U}_{m,\varepsilon,p}$ is and $(m+1)^2$-vector with elements equal to the diagonal entries of matrix $K_{m,\varepsilon} \widetilde{U}_{m,\varepsilon,p} P_{m,\varepsilon}$. So, the ith component of the column vector $\widehat{U}_{m,\varepsilon,p}$ will be

$$(4.9)\quad \sum_{j=1}^{i} p_{ji} k_{ij} v_j; \quad i = 1(1)(m+1)^2,$$

where $p_{ij},\ k_{ij}$ and v_j are the elements of $P_{m,\varepsilon}, K_{m,\varepsilon}, U_{m,\varepsilon,p}$, respectively, and

$$v_j = (u_j)^p.$$

Applying Eqs. (4.1) and (4.6) in Eq. (1.1) with Eq. (1.3), we get

$$(4.10)\quad \widehat{U}^T_{m,\varepsilon,p} \Phi_{m,\varepsilon}(x,y) \simeq F^T_{m,\varepsilon} \Phi_{m,\varepsilon}(x,y).$$

Consequently we will have

$$(4.11)\quad \widehat{U}_{m,\varepsilon,p} = F_{m,\varepsilon}.$$

After solving the above nonlinear system by using Newton-Raphson method, we can find $U_{m,\varepsilon}$ and then

$$u_{m,\varepsilon}(x,y) = U^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y). \tag{4.12}$$

Then

$$u(x,y) \simeq \bar{u}_{m,k}(x,y) = \frac{1}{k}\sum_{i=0}^{k-1} u_{m,\varepsilon_i}(x,y), \tag{4.13}$$

where $\varepsilon_i = \frac{ih}{k}$, $i = 0(1)(k-1)$ is the estimation of the solution of two-dimensional Volterra integral equation of the first kind.

5. Numerical examples

In this section, the example is given to certify the convergence and error bound of the presented method. All results are computed by using a program written in the Matlab. The numerical experiments are carried our for the selected grid point which are proposed as $(2^{-l}\ ;\ l = 1,2,3,4,5,6)$ and m terms and k times of modifications of the M2D-BFs series. The following problems have been tested.

Example 1. Consider the following linear two-dimensional Volterra integral equation [10]:

$$\int_0^x \int_0^y (sin(y+s)+sin(x+t)+3)u(s,t)dtds = f(x,y);\ (x,y) \in D, \tag{5.1}$$

and $f(x,y)$ is selected so that $u(x,y) = cos(x+y)$ is the exact solution. Furthermore, Table 1 and Figures 1-2 illustrates the numerical results for this example.

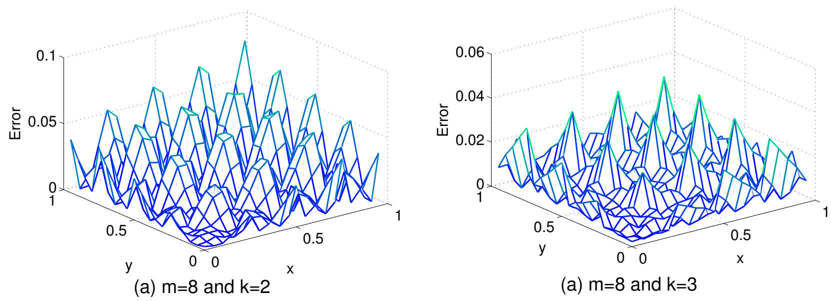

Figure 1. Absolute value of error, Example 1 with $m = 8$ and $k = 2, 3$

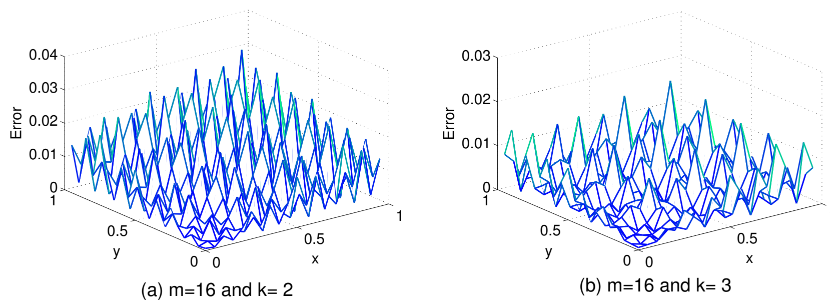

Figure 2. Absolute value of error, Example 1 with $m = 16$ and $k = 2, 3$

Table 1: Numerical results of Example 1 with M2D-BFs

Nodes (x,y)	Error for m=8			Error for m=16		
(x,y)=2^{-l}	k=1(Ref.[10])	k=2	k=3	k=1(Ref.[10])	k=2	k=3
$l = 1$	0.085303	0.048609	0.035575	0.042396	0.023996	0.017469
$l = 2$	0.052873	0.029492	0.021487	0.025605	0.014216	0.010271
$l = 3$	0.045225	0.019710	0.012312	0.013869	0.007682	0.005582
$l = 4$	0.003428	0.007026	0.000477	0.011432	0.004991	0.003125
$l = 5$	0.002421	0.000779	0.000313	0.000863	0.001769	0.000118
$l = 6$	0.003886	0.002244	0.001778	0.000602	0.000193	0.000077

Example 2. Consider the following nonlinear two-dimensional Volterra integral equation [10]:

$$\int_0^x \int_0^y u^2(s,t)dtds = f(x,y); \ (x,y) \in D, \tag{5.2}$$

where

$$f(x,y) = \frac{1}{45}xy(9x^4 + 10x^2y^2 + 9y^4). \tag{5.3}$$

The exact solution is $u(x,y) = x^2 + y^2$. Furthermore, Table 2 and Figures 3-4 illustrates the numerical results for this example.

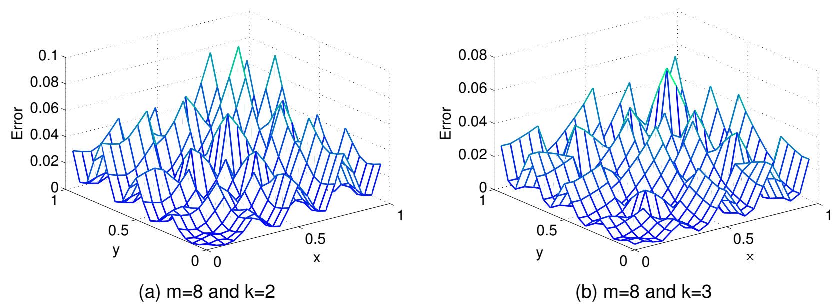

Figure 3. Absolute value of error, Example 2 with $m = 8$ and $k = 2, 3$

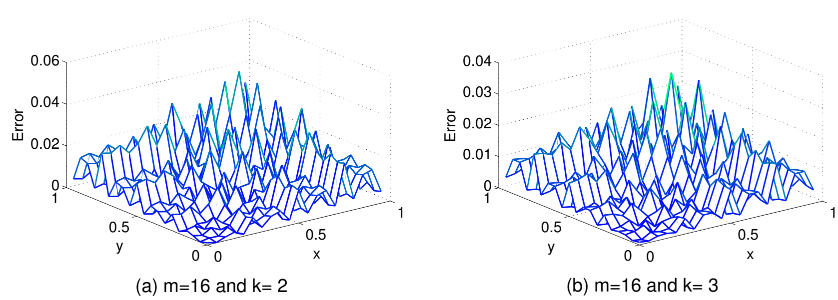

Figure 4. Absolute value of error, Example 2 with $m = 16$ and $k = 2, 3$

Table 2: Numerical results of Example 2 with M2D-BFs

Nodes (x,y)	Error for m=8			Error for m=16		
(x,y)=2^{-l}	k=1(Ref.[10])	k=2	k=3	k=1(Ref.[10])	k=2	k=3
$l=1$	0.266615	0.017766	0.003043	0.286467	0.007733	0.002964
$l=2$	0.173809	0.016814	0.005273	0.162543	0.004442	0.000761
$l=3$	0.216501	0.005137	0.003526	0.078379	0.004203	0.001318
$l=4$	0.039305	0.006534	0.002632	0.098525	0.000188	0.000230
$l=5$	0.017610	0.003268	0.002545	0.020213	0.000085	0.000658
$l=6$	0.047430	0.004000	0.003277	0.009606	0.000817	0.000636

6. Conclusion

In this paper we have worked out a computational method for approximate solution of nonlinear two-dimensional Volterra integral equations of the first kind, based on the expansion of the solution as series of M2D-BFs. This method converts a nonlinear two-dimensional Volterra integral equation whose answer are the coefficients of M2D-BFs expansion of the solution of nonlinear two-dimensional Volterra integral equation. Note that the find system extracted from the nonlinear equations will be nonlinear and proper technique such Newton-Raphson method could be applied. This method can be easily extended and applied to nonlinear two-dimensional Volterra integral equations of the second kind and nonlinear two-dimensional Fredholm integral equations.

Acknowledgments

The authors wish to thank the referee for his (her) helpful comments

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