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ON CERTAIN GENERALIZED PRIME IDEALS IN BOOLEAN LIKE SEMIRING OF FRACTIONS

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Abstract. In this paper, we introduce the notions of semiprime ideals, 2-potent prime ideals, weakly prime ideals and weakly primary ideals in a Boolean like semiring of fractions. Further, we obtain various results concerning the notions.

Key Words: Prime, primary, weakly primary, ideals, Boolean like semirings.
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1. Introduction

The concept of Boolean like ring R was introduced by Foster in [3]. It is defined as a commutative ring with unity and of characteristic 2 in which $ab(1+a)(1+b) = 0$ for every $a, b \in R$. Later in [7] Venkateswarlu et. al. introduced the concept of Boolean like semirings by generalizing the concept of Boolean like ring of Foster. In fact, Boolean like semirings are special classes of near rings [4]. Venkateswarlu et al have made an extensive study of the class of Boolean like semirings in [8 ,9].
Recently in [5], Ketsela et al have introduced the notion of Boolean like semiring of fractions and proved that every Boolean like semiring of fraction is a Boolean like ring of Foster. However there are many interesting facts about the class of ideals in Boolean like semiring of fractions which do not subsume the properties of ideals in commutative rings. Some of the facts have been established in [6]. Now, in this paper we study the

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properties of special classes of ideals in Boolean like semiring of fractions.

This paper is divided into two sections of which the first section is devoted for collecting certain definitions and results concerning Boolean like semirings and as well as Boolean like semiring of fractions. In section 2, we introduce the notions of 2-potent prime, weakly prime , weakly primary, quasi primary and almost primary ideals of a Boolean like semiring of fractions. Also, we prove that $S^{-1}I$ is semiprime (resp., 2-potent prime, weakly prime, weakly primary, quasi prime) if I is semiprime (resp., 2-potent prime, weakly prime, weakly primary, quasi prime).

2. Preliminaries

In this section, we we recall certain definitions and results concerning Boolean like semirings from [5], [6], [7] and [8].

Definition 2.1. A non empty set R together with two binary operations $+$ and $\cdot$ satisfying the following conditions is called a Boolean like semiring;

(1) $(R, +)$ is an abelian group;
(2) $(R, \cdot)$ is a semi group;
(3) $a(b + c) = a.b + a.c$;
(4) $a + a = 0$;
(5) $ab(a + b + ab) = ab$ for all $a, b, c \in R$.

Theorem 2.2. *Let R be a Boolean like semiring. Then $a.0 = 0$ for every $a \in R$.*

Definition 2.3. A Boolean like semiring R is said to be weak commutative if $abc = acb$ for every $a, b, c \in R$.

Lemma 2.4. *Let R be weak commutative Boolean like semiring and let m and n be two integers. Then,*

(1) $a^m a^n = a^{m+n}$;
(2) $(a^m)^n = a^{mn}$;
(3) $(ab)^n = a^n b^n$ *for every* $a, b \in R$.

Definition 2.5. Let R be a weak commutative Boolean like semiring. A nonempty subset S of R is called multiplicatively closed whenever $a, b \in S$ implies $ab \in S$.

Theorem 2.6. *Let R be a weak commutative Boolean like semiring and S a multiplicatively closed subset of R. Define a relation $\sim$ on*

$R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists $s \in S$ such that $s(s_1 r_2 + s_2 r_1) = 0$. Then $\sim$ is an equivalence relation.

Lemma 2.7. *Let R be a weak commutative Boolean like semiring and S a multiplicatively closed subset of R. Then, for every $r \in R$ and $s, s', t \in S$ we have;*

(1) $\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{st}$;
(2) $\frac{rs}{s} = \frac{rs^{'}}{s^{'}}$;
(3) $\frac{s}{s} = \frac{s^{'}}{s^{'}}$.

Theorem 2.8. *Let S be a multiplicatively closed subset in a weak commutative Boolean like semiring R. Define binary operations '+' and '.' on $S^{-1}R$ as follows:*

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} \quad \textit{and} \quad \frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

Then $(S^{-1}R, +, .)$ is a Boolean like ring.

Definition 2.9. A Boolean like ring R is a commutative ring with unity and is of characteristic 2 in which $ab(1+a)(1+b) = 0$ for every $a, b \in R$.

Theorem 2.10. $(S^{-1}R, +, .)$ *in Theorem 3 is a Boolean like ring.*

Definition 2.11. A subset I of a Boolean like semiring R is said to be an ideal of R if;

(1) $(I, +)$ is a subgroup of $(R, +)$;
(2) $ra \in R$ for every $a \in I, r \in R$;
(3) $(r + a)s + rs \in I$ for every $r, s \in R, a \in I$.

Definition 2.12. If R and $R^{'}$ are Boolean like semirings, a mapping $f : R \longrightarrow R^{'}$ is said to be a homomorphism of R into $R^{'}$ if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all a and b in R.

Definition 2.13. Let R and $R^{'}$ be two Boolean like semirings. If I is an ideal of $R^{'}$ and $f : R \to R^{'}$ is a homomorphism, then $f^{-1}(I)$ is an ideal of R, called the contraction of I and is denoted by I^c

Theorem 2.14. *Let R be a weak commutative Boolean like semiring, S a multiplicatively closed subset of R and I an ideal of R. Let $S^{-1}I = \{\frac{a}{s} | a \in I, s \in S\}$. Then $S^{-1}I$, called the extension of I and denoted by I^e, is an ideal of $S^{-1}R$.*

Theorem 2.15. *Let P be a prime ideal of a weak commutative Boolean like semiring R such that $P \cap S = \varnothing$. Then $S^{-1}P$ is a prime ideal of $S^{-1}R$.*

Theorem 2.16. *Let R be a weak commutative Boolean like semiring with right unity and let J be a prime ideal of $S^{-1}R$. Then J^c is a prime ideal of R and $J^c \cap S = \varnothing$*

Definition 2.17. A proper ideal P of a Boolean like semiring R is called primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P$ for every $x, y \in R$.

Definition 2.18. A proper ideal P of a Boolean like semiring R is called almost primary if $x \in P$ or $y^n \in P$ whenever $xy \in P \setminus P^2$ for every $x, y \in R$.

Remark 2.19. In a Boolean like semiring R, $a^n = a$ or a^2 or a^3 for evert $a \in R$.

Theorem 2.20. *Let S be a multiplicatively closed subset and I an ideal in a Boolean like semiring R. Then $\frac{r}{s}, r \in R, s \in S$; is in $S^{-1}I$ if and only if $mr \in I$ for some $m \in S$*

Theorem 2.21. *Let P be a primary ideal of a Boolean like semiring R and S be a multiplicatively closed subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a primary ideal of $S^{-1}R$.*

Theorem 2.22. *Let J be a primary ideal of $S^{-1}R$. Then J^c is also a primary ideal of R. Moreover $J^c \cap S = \emptyset$.*

Theorem 2.23. *Let P be an almost primary ideal of a Boolean like semiring R and S a multiplicatively closed subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is an almost primary ideal of $S^{-1}R$*

Theorem 2.24. *Let J be an almost primary ideal of $S^{-1}R$. Then J^c is also an almost primary ideal of R. Moreover $J^c \cap S = \varnothing$.*

Definition 2.25. An ideal I of a Boolean like semiring R is called semiprime if $x \in I$ whenever $x^2 \in I$ for every $x \in R$.

Definition 2.26. A proper ideal P of a Boolean like semiring R is called weakly prime if $0 \neq xy \in P$ implies $x \in P$ or $y \in P$ for every $x, y \in R$.

Definition 2.27. A proper ideal P of a Boolean like semiring R is called weakly primary if $0 \neq xy \in P$ implies $x \in P$ or $y^2 \in P$ for every $x, y \in R$.

3. Special Classes of Ideals

In this section, we study certain generalizations of prime ideals namely primary, almost primary, weakly prime, weakly primary ,semiprime, quasi prime and 2-potent ideals of Boolean like semiring of fractions. We begin with the following result.

Theorem 3.1. *Let I be an ideal of a weak commutative Boolean like semiring R and S be a multiplicatively closed subset of R. Then,*

(1) $I^{ec} = \cup(I : s) = \{r \in R | sr \in I$ *for some* $s \in S\}$.
(2) $I^e = S^{-1}R$ *if and only if* $I \cap S \neq \varnothing$.

Proof. 1. Let $r \in I^{ec}$. Then $f(r) = \frac{rs}{s} \in I^e$ for some $s \in S$. Then, $t(rs) \in I$ for some $t \in S$. Hence $(ts)r \in I$ since R is weak commutative. So, $r \in \cup(I : s)$ since $ts \in S$. To prove the other way, let $r \in \cup(I : s)$ so that $sr \in I$ for some s. And $f(r) = \frac{sr}{s}$ $\in I^e$ as a result, $r \in I^{ec}$

2. Suppose $I^e = S^{-1}R$ which implies $\frac{s}{s} \in I^e$ for $s \in S$. Hence $ts \in I$ for some t in S so that $I \cap S \neq \varnothing$ since $ts \in S$ Conversely, if $I \cap S \neq \varnothing$, let $s \in I \cap S$. Thus, $\frac{s}{s} \in I^e$ and let $\frac{r}{s} \in S^{-1}R$ so that $\frac{r}{s} = \frac{r}{s}\frac{s}{s} \in I^e$ since I^e is an ideal.

□

Now we study certain characterizations in which $I^{ec} = I$.

Definition 3.2. A proper ideal I of a Boolean like semiring is 2-potent prime if $x \in I$ or $y \in I$ whenever $xy \in I^2$ for every $x, y \in R$.

Theorem 3.3. *Let P be a 2-potent prime ideal of a weak commutative Boolean like semiring R and S a multiplicatively closed subset of R such that $P \cap S = \varnothing$. Then $P^{ec} = P$.*

Proof. Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$. So, $ty \in P$ for some $t \in S$. Hence, $[ty]^2 \in P^2$ which implies $t^2y^2 \in P^2$; so that $t^2 \in P$ or $y^2 \in P$ (since P is 2-potent prime). As a result, $y^2 \in P$ (since t^2 is not in P). Hence, $(y^2)^2 = y^2 \in P^2$. Therefore, $y \in P$. □

Remark 3.4. $P^{ec} = P$ for every prime ideal P of R disjoint from S since every prime ideal is 2-potent prime.

Now we introduce the notion of quasi prime ideals in Boolean like semirings. We begin with the following.

Definition 3.5. An ideal I of a Boolean like semiring R is called quasi prime if $x \in I$ whenever $x^3 \in I$ for every $x \in R$.

Lemma 3.6. *In Boolean like semiring, an ideal is semiprime if and only if it is quasi prime.*

Proof. Let P be a semiprime ideal of a Boolean like semiring R such that $x^3 \in P$ for some x in R. Then $x^2 = (x^3)(x^3) = x^6 = x^2x^4 = x^2x^2 = x^2 \in P$ implies $x \in P$ (since P is semiprime). Hence P is quasi prime. Conversely, let P be a quasi prime ideal such that $x^2 \in P$. Thus, $x^3 = x.x^2 \in P$. Hence $x \in P$ (since P is quasi prime). Therefore, P is semiprime. □

Corollary 3.7. *Let P be a primary ideal of a Boolean like semiring R such that $P \cap S = \varnothing$. Then $P^{ec} = P$ if P is semiprime or quasi prime.*

Proof. Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$. Then, $sy \in P$ for some $s \in S$. Hence, $s \in P$ or $y^2 \in P$ (since I is primary). So, $y^2 \in P$ since s is not in P. Thus, $y \in P$. □

Theorem 3.8. *If I is a semiprime ideal of a weak commutative Boolean like semiring R and S is a multiplicatively closed subset of R, then I^e is a semiprime ideal of $S^{-1}R$.*

Proof. Let I be a semiprime ideal and $\frac{r}{s} \in S^{-1}R$ such that $\frac{r}{s} \notin I^e$. Then $tr \notin I$ $\forall t \in S$ so that $(tr)^2 = t^2r^2 \notin I$ since I is semiprime. Thus, $\frac{t^2r^2}{t^2s^2} \notin I^e$. Therefore, $\frac{r^2}{s^2} \notin I^e$. □

Theorem 3.9. *If J is a semiprime ideal of $S^{-1}R$, then J^c is a semiprime ideal of R.*

Proof. Let x be in R such that $x^2 \in J^c$. Thus, $f(x^2) \in J$ which implies $f(x)f(x) \in J$. So, $[f(x)]^2 \in J$. Hence, $f(x) \in J$ since J is semiprime Thus, $x \in J^c$. □

Remark 3.10.

a) Every 2-potent prime ideal is semiprime.

b) I^e is a 2-potent prime ideal of $S^{-1}R$ if I is a 2-potent prime ideal of R.

c) J^c is a 2-potent prime ideal of R if J is a 2-potent prime ideal of $S^{-1}R$.

d) I^{ec} is a semiprime ideal of R if I is a semiprime.

Theorem 3.11. *Let P be a weakly primary ideal of a Boolean like semiring R and S be a multiplicative subset of R. Then P^e is a weakly primary ideal of $S^{-1}R$.*

Proof. Let P be a weakly primary ideal of R such that $0 \neq \frac{r_1r_2}{s_1s_2} \in P^e$. So, $s[r_1r_2] \in P$ and $s(r_1r_2) \neq 0$ $\forall s \in S$. Which implies, $[sr_1]r_2 \in P$

. Hence $sr_1 \in P$ or $r_2^2 \in P$. If $sr_1 \in P$ then $\frac{r_1}{s} = \frac{sr_1}{ss} \in P^e$ and if $(r_2)^2 \in P$, then $[\frac{r_2}{s_2}]^2 = \frac{r_2^2}{s_2^2} \in P^e$. In any case, the result holds. □

Remark 3.12.
i) P^e is a weakly prime ideal of $S^{-1}R$ whenever P is weakly prime.
ii) The contraction of a weakly primary (and hence weakly prime) ideal of a Boolean like semiring R is weakly primary (weakly prime) if R is a domain.

Theorem 3.13. *If Q is a primary ideal of a Boolean like semiring R such that $P = r(Q)$ and Q is disjoint from a multiplicative subset S of R, then $S^{-1}P$ is a prime ideal of $S^{-1}R$*

Proof. Since the radical of a primary ideal is prime, P is prime. Thus the extension $S^{-1}P$ of a prime ideal is also prime in $S^{-1}R$. □

Lemma 3.14. *If Q is a primary ideal of a Boolean like semiring R disjoint from S, then $[r(Q)]^{ec} = r(Q)$.*

Proof. The proof is straightforward. □

Lemma 3.15. *Let I and J be ideals of a Boolean like semiring R. Then $S^{-1}I \subseteq S^{-1}J$ if $I \subseteq J$ and the converse will be true if J is a 2-potent prime ideal of R disjoint from S.*

Theorem 3.16. *Let S be a multiplicatively closed subset of a Boolean like semiring R and I and J be ideals of R. Then,*

a) $S^{-1}(I + J) = S^{-1}I + S^{-1}J$
b) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$
c) $S^{-1}(I \cap J) = (S^{-1}I) \cap (S^{-1}J)$
d) $S^{-1}r(I) = r(S^{-1}I)$

Proof. a) Since $I \subseteq I + J$ and $J \subseteq I + J$, and $I + J$ is an ideal, we have $S^{-1}I \subseteq S^{-1}(I + J)$ and $S^{-1}J \subseteq S^{-1}(I + J)$. Thus we have $S^{-1}I + S^{-1}J \subseteq S^{-1}(I + J)$. Conversely, let $\frac{a}{s} \in S^{-1}(I + J)$. Then we have, $ta \in I + J$ for some t in S. So, $ta = a_1 + a_2$ where $a_1 \in I$ and $a_2 \in J$. And hence, $\frac{a}{s} = \frac{ta}{ts} = \frac{a_1+a_2}{ts} = \frac{a_1}{ts} + \frac{a_2}{ts} \in S^{-1}I + S^{-1}J$.

b) Let $m \in S^{-1}(IJ)$. So that $m = \frac{x}{s}$ for some $x \in IJ$. Hence, $x = \sum a_i b_i$ for some a_i in I and b_i in J. So, $\frac{a_i}{s} \in S^{-1}I$ and $\frac{b_i}{s} \in S^{-1}J$. Thus, $m = \frac{x}{s} = \frac{\sum a_i b_i}{s} = \sum \frac{a_i b_i}{s} = \sum \frac{a_i b_i s}{s^2} = \sum (\frac{a_i}{s}).(\frac{b_i}{s} s) \in (S^{-1}I)(S^{-1}J)$ Hence $S^{-1}(IJ) \subseteq (S^{-1}I)(S^{-1}J)$.

To show that $(S^{-1}I)(S^{-1}J) \subseteq S^{-1}(IJ)$, let $y \in (S^{-1}I)(S^{-1}J)$. Then $y = \sum \frac{a_i}{s_i}\frac{b_i}{n_i}$ for some $\frac{a_i}{s_i} \in S^{-1}I$ and $\frac{b_i}{n_i} \in S^{-1}J$. So $y = \sum \frac{a_i b_i}{q_i}$ where $q_i = s_i n_i$. Thus, $y = \frac{\sum a_i b_i}{q}$ where $q = \prod q_i$. As a result, $y \in S^{-1}(IJ)$.

c) Clearly $S^{-1}(I \cap J) \subseteq (S^{-1}I) \cap (S^{-1}J)$. Let $\frac{x}{s} \in (S^{-1}I) \cap (S^{-1}J)$. Then, $\frac{x}{s} \in (S^{-1}I)$ and $\frac{x}{s} \in (S^{-1}J)$. This implies, $\exists t_1, t_2 \in S \ni t_1 x \in I, t_2 x \in J$. Hence, $(t_1 x)t_2 \in I, t_1(t_2 x) \in J$ since I and J are ideals. So, $(t_1 t_2)x \in I, (t_1 t_2)x \in J$ since R is weak commutative. Thus,$(t_1 t_2)x \in I \cap J$. As a result, $\frac{x}{s} \in S^{-1}(I \cap J)$.

d) Let $\frac{x}{s} \in S^{-1}r(I)$. Then, $tx \in r(I), t \in S$. So that, $(tx)^n \in I$ for some natural number n. Hence, $t^n x^n \in I$ so that $\frac{x^n}{s^n} = \frac{t^n x^n}{t^n s^n} \in S^{-1}I$. Thus, $[\frac{x}{s}]^n \in S^{-1}I$. Consequently, $\frac{x}{s} \in r(S^{-1}I)$. Conversely, let $\frac{x}{s} \in r(S^{-1}I)$. So that $[\frac{x}{s}]^k \in S^{-1}I$ for some natural number k. Hence, $\frac{x^k}{s^k} \in S^{-1}I \Rightarrow mx^k \in I$ for some m in S. $m^k x^k \in I$ since I is ideal. As a result, $(mx)^k \in I$. So, $mx \in r(I)$. Thus, $\frac{mx}{ms} \in S^{-1}r(I)$. Consequently, $\frac{x}{s} \in S^{-1}r(I)$. □

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