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PURE SUBMODULES OF BCK- MODULES

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Abstract. In this paper by considering the notion of BCK-module, we introduce pure BCK- submodules and we prove some results by it. In particular, we show that if X is a BCK- algebra, M is a cyclic BCK-module and N a prime BCK- submodule of M, then N is a pure BCK-submodule of M.

Key Words: BCK- algebra, BCK- module, multiplication BCK- module, prime BCK-submodule, pure BCK- submodule.
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1. Introduction

In 1966, Imai and Iseki [5, 8] introduced BCK-algebras. This notion was originated from two different ways: (1) set theory, and (2) classical and no classical propositional calculi. Certain algebraic structures, for example Boolean- algebra, MV-algebras, are introduced as BCK-algebras [7]. Every module is an action of ring on certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. BCK-module is an action of BCK-algebra on commutative group. In 1994, the notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem [2]. They established isomorphism theorems and studied some properties of BCK-modules. The theory of BCK-modules was further developed by Z. Perveen and M. Aslam [12]. Now, in this paper we introduce the concept of pure BCK- submodules and we prove some results by it. In particular, we show that if X is a BCK- algebra, M a cyclic BCK-module and N

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a prime BCK-submodule of M, then N is a pure BCK-submodule of M.

2. Preliminaries

Let us to begin this section with the definition of a BCK-algebra.

Definition 2.1.[9] Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BCK- algebra if it satisfies the following axioms:

(BCK1)$((x * y) * (x * z)) * (z * y) = 0$,

(BCK2) $(x * (x * y)) * y = 0$,

(BCK3) $x * x = 0$,

(BCK4) $0 * x = 0$,

(BCK5) $x * y = y * x = 0$ imply that $x = y$, for all $x, y, z \in X$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

If there is an element 1 of a BCK- algebra X, satisfying $x * 1 = 0$, for all $x \in X$, the element 1 is called unit of X. A BCK- algebra with unit is called to be bounded.

Definition 2.2.[9] Let $(X, *, 0)$ be a BCK- algebra and X_0 be a nonempty subset of X. Then X_0 is called to be a subalgebra of X, if for any $x, y \in X_0$, $x * y \in X_0$ i.e., X_0 is closed under the binary operation $*$ of X.

Definition 2.3.[9] A BCK- algebra $(X, *, 0)$ is said to be commutative , if it satisfies, $x * (x * y) = y * (y * x)$, for all x, y in X.

Definition 2.4.[9] A BCK- algebra $(X, *, 0)$ is called implicative, if $x = x * (y * x)$, for all x, y in X.

Definition 2.5.[9] A nonempty subset A of BCK- algebra $(X, *, 0)$ is called an *ideal* of X if it satisfies the following conditions:

(i) $0 \in A$,
(ii) $(\forall x \in X)(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

Definition 2.6.[9] Suppose A is an ideal of BCK- algebra $(X, *, 0)$. For any x, y in X, we denote $x \sim y$ if and only if $x * y \in A$ and $y * x \in A$. It is easy to see that, $\sim$ is an equivalence relation on X.

Denote the equivalence class containing x by C_x and $\frac{X}{A} = \{C_x : x \in X\}$. Also we define $C_x * C_y = C_{x*y}$, for all x, y in X .

Definition 2.7.[9] Let X be a lower BCK- semilattice and A be a proper ideal of X. Then A is said to be prime if $a \wedge b = b * (b * a) \in A$ implies that $a \in A$ or $b \in A$, for any a, b in X.

Lemma 2.8.[9] In a lower BCK- semilattice $(X, *, 0)$ the following are equivalent:

(i) I is a prime ideal,

(ii) I is an ideal and satisfies that for any $A, B \in I(X)$, $A \subseteq I$ or $B \subseteq I$ whenever $A \cap B \subseteq I$.

Definition 2.9.[1] Let $(X, *, 0)$ be a BCK-algebra, M be an abelian group under $+$ and let $(x, m) \longrightarrow x \cdot m$ be a mapping of $X \times M \longrightarrow M$ such that

(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,

(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,

(iii) $0 \cdot m = 0$,

for all $x, y \in X, m_1, m_2 \in M$,where $x \wedge y = y * (y * x)$. Then M is called a left X-module.
If X is bounded, then the following additional condition holds:
(iv) $1 \cdot m = m$.

A right X-module can be defined similarly.

Lemma 2.10.[1] Every bounded implicative BCK-algebra is a module.

Example 2.11.[1] Let A be a nonempty set and $X = P(A)$ be the power set of A. Then X is a bounded commutative BCK-algebra with $x \wedge y = x \cap y$, for all $x, y \in X$. Define $x + y = (x \cup y) \cap (x \cap y)^{'}$, the symmetric difference. Then $M = (X, +)$ is an abelian group with empty set $\emptyset$ as an identity element and $x + x = \emptyset$. Define $x \cdot m = x \cap m$, for any $x, m \in X$. Then simple calculations show that :

(i) $(x \wedge y) \cdot m = (x \cap y) \cap m = x \cap (y \cap m) = x \cdot (y \cdot m)$,

(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,

(iii) $0 \cdot m = \emptyset \cap m = \emptyset = 0$,

(iv) $1 \cdot m = A \cap m = m$. Thus X itself is an X-module.

Definition 2.12.[1] Let M_1, M_2 be X-modules. A mapping $f : M_1 \longrightarrow M_2$ is called a BCK- homomorphism, if for any $m_1, m_2 \in M_1$, we have :

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(ii) $f(x \cdot m_1) = x \cdot f(m_1)$, for all $x \in X$.

Ker(f) and Im(f) have usual meaning.

Theorem 2.13.[9] Let X be bounded implicative and M be an X-module. If S is a $\wedge$-closed subset of X, then the submodules of $M_s = \{\frac{m}{s} : m \in M, s \in S\}$ are on the form N_s where $N = \{n \in M : \frac{n}{1} \in N_s\}$.

Definition 2.14.[10] Let X be a BCK-algebra and M be a group. Then M is called a multiplication BCK-module if for each BCK-submodule N of M, there exists a BCK-ideal I of X, such that $N = I.M$.

Definition 2.15.[1] Let M be a left BCK- module over X and N be a BCK- submodule of M. Then we define $[N : M] = \{x \in X \mid x \cdot M \subseteq N\}$. Also $[N : M]$ is an ideal of X.

Theorem 2.16.[10] Let X be a BCK-algebra, and M be a group. Then M is a multiplication BCK-module if and only if for each submodule K of M, $K = [K : M].M$.

Definition 2.17.[11] Let M be a left BCK- module over X and N be a submodule of M. Then N is said to be prime BCK-submodule of M, if $N \neq M$ and $x \cdot m \in N$, implies that $m \in N$ or $x.M \subseteq N$, for any x in X and any m in M.

Example 2.18.[11] Let $X = P(A = \{1, 2, ..., n\})$, $B_i = \{1, 2, ..., n\} - \{i\}$, for $i \in \{1, 2, ..., n\}$. Then $P(B_i)$ is a prime BCK- submodule of $P(A)$.

3. Pure BCK- submodule

The notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem in 1994 [2]. In this section we define pure BCK-submoduls and we obtain some theorems.

Definition 3.1. Let X be a BCK-algebra. X-submodule N of X-module M is called pure if $I.N = N \bigcap I.M$, for every ideal I of X.

Example 3.2. Assume $A = \{1, 2\}$ and $X = P(A)$. Simple calculations and Example 2.11 show that all X-submodules of $P(A)$ and all BCK-ideals of $P(A)$ are $\{\emptyset\}$, $\{\emptyset, \{1\}\}$, $\{\emptyset, \{2\}\}$, $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Definition 3.1 shows the pure BCK-submodules of $P(A)$ in this example are all X-submodules.

Theorem 3.3. Let X be a BCK-algebra, M a cyclic X-module and N a prime X-submodule of M. Then N is pure.

Proof: Assume that I is an ideal of X. As $I.N \subseteq N \bigcap I.M$ is trivial, we shall prove the reverse inclusion. Let $n \in N \bigcap I.M$. Now since M is a cyclic X-module, then there exists $m \in M$ such that $M = X.m$. Therefore for $i = 1, 2, ..., k$ there exist $x_i \in I$ and $x \in X$ such that $\sum_{i=1}^{k} x_i.(x.m) = \sum_{i=1}^{k} (x_i \wedge x).m$. Since $x_i \wedge x \leq x_i$ and $x_i \in I$ we get $x_i \wedge x \in I$. Hence $(x_i \wedge x).m \in I.m \Rightarrow n = \sum_{i=1}^{k} (x_i \wedge x).m \in I.m$.
So there exists $x' \in I$ such that $n = x'.m$ But $n = x'.m \in N$ and N is

prime, so $m \in N$ (it follows that $n = x'.m \in I.N$) or $x' \in (N : M)$, Hence $n = x'.m = (x' \wedge x').m = x'.(x'.m) \in I.N$.

Lemma 3.4. Let X be bounded implicative, M be a X-module and S be a $\wedge$-closed subset of X. Then if J is a pure submodule of M_s, then there exists a pure submodule N of M such that $J = N_s$.

Proof: By Theorem 2.13, we get that $J = N_s$ is a submodule of M_s. Now we show that if $J = N_s$ be a pure submodule of M_s, then N is a pure submodule of M. As $I.N \subseteq N \bigcap I.M$ is trivial, we shall prove the reverse inclusion. Let $n \in N \bigcap I.M$. Then there exist $x_i \in I$ and $m_i \in M$ such that $n = \sum_{i=1}^{k} x_i.m_i$. If $n = \sum_{i=1}^{k} x_i.m_i \in N \bigcap I.M$, then $\frac{\sum_{i=1}^{k} x_i.m_i}{1} \in N_s \bigcap (I.M)_s$. Hence $\frac{\sum_{i=1}^{k} x_i.m_i}{1} \in (I.N)_s$ (because N_s is pure). So by Theorem 2.17, we get that $\sum_{i=1}^{k} x_i.m_i = n \in I.N$ and the proof is complete.

Theorem 3.5. Let M be a left BCK-module over X. Then P is a pure BCK-submodule in M containing N if and only if $\frac{P}{N}$ is a pure BCK-submodule in $\frac{M}{N}$.

Proof: **Necessity**. Assume that I is an ideal of X. As $I.\frac{P}{N} \subseteq \frac{P}{N} \bigcap I.\frac{M}{N}$ is trivial, we shall prove the reverse inclusion. Let $p + N \in \frac{P}{N} \bigcap I.\frac{M}{N}$. Then there exist $x_i \in I$ and $m_i \in M$ such that $p + N = \sum_{i=1}^{k} x_i.(m_i + N) = \sum_{i=1}^{k}(x_i.m_i) + N = (\sum_{i=1}^{k}(x_i.m_i)) + N \in \frac{I.M}{N} = I.\frac{M}{N}$. So $p \in I.M \cap P$. Then by purity of P, we get $p \in I.P$, hence $p + N \in I.\frac{P}{N}$.
Sufficiency. Assume that I is an ideal of X. As $I.P \subseteq P \bigcap I.M$ is trivial, we shall prove the reverse inclusion. Let $p \in P \bigcap I.M$. Then $p + N \in \frac{P}{N} \bigcap I.\frac{M}{N}$, and by purity of $\frac{P}{N}$, $p + N \in I.\frac{P}{N}$, so $p \in I.P$ and the proof is complete.

Theorem 3.6. Let M_1 and M_2 be left BCK- modules over X and ϕ be a BCK- epimorphism from M_1 to M_2. Also N be a pure BCK-submodule of M_1. Then $\phi(N)$ is a pure BCK- submodule of M_2.

Proof: Assume that I is an ideal of X. N is pure submodule of M_1, then $I.N = N \bigcap I.M_1$. So $\phi(I.N) = \phi(N \bigcap I.M_1)$. Hence $I.\phi(N) = \phi(N) \bigcap I.\phi(M_1)$. Since ϕ is epimorphism, we get $\phi(M_1) = M_2$. So $\phi(N)$ is pure submodule of M_2.

Corollary 3.7. Let X be a BCK-algebra, M be a left X-module and N, be a pure X- submodule of M. Then $\frac{M}{N}$ is a pure X -submodule of M.

Definition 3.8. We will say that a submodule N of M is idempotent in M if $N = [N : M].N$.

Example 3.9. In Example 3.2, by simple calculations we get $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ is idempotent in $P(A)$.

Theorem 3.10. Let X be a BCK-algebra, M a X-module and N be a submodule of M. If N is a pure submodule of M, then N is idempotent in M.

Proof: Since N is pure in M, we have that $[N : M].N = N \bigcap [N : M].M = N$, and hence N is idempotent in M.

Theorem 3.11. Let X be a BCK-algebra, M a multiplication X-module and N a submodule of M. If $[N : M]$ is an idempotent ideal, then N is idempotent in M.

Proof: Obviously we get $N = [N : M].M = [N : M]^2.M = [N : M].N$. So N is idempotent in M.

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