Journal of Hyperstructures 2 *(2)* (*2013*)*, 151-162.*
ISSN: 2251-8436 print/2322-1666 online

ON $(\odot, \vee)$-DERIVATIONS FOR BL-ALGEBRAS

LIDA TORKZADEH * AND LEILA ABBASIAN

ABSTRACT. In this paper, we define the concept of $(\odot, \vee)$-derivations for BL-algebras and discuss some related results. We study this derivation on boolean center $B(A)$ of a BL-algebra A. Finally, we investigate some properties of isotone $(\odot, \vee)$-derivations on a BL-algebra A and characterize the $(\odot, \vee)$-derivation on the Gödel structure $[0, 1]$.

Key Words: BL-algebra, $(\odot, \vee)$-derivation, isotone $(\odot, \vee)$-derivation.
2010 Mathematics Subject Classification: 06D35, 06F35, 03G25.

1. Introduction

BL-algebras are the algebraic structures for Hájek Basic Logic (BL-logic) [10], arising from the continuous triangular norms (t-norms), familiar in the frameworks of fuzzy set theory. BL-algebras rise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras do from classical logic or Lukasiewicz logic, respectively. The properties of a BL -algebra were presented in [4],[5],[7],[8],[10] and [16]. The notion of derivations, introduced from the analytic theory, is helpful for the research of structures and properties in algebraic systems. Several authors [1], [2], [3] and [13] have studied derivations in rings and near rings. After that Jun and Xin [12] applied the notion of derivations in ring and near ring theory to BCI-algebras. Szász introduced the concept of derivation on lattices in [14]. Recently some author [6], [9] and [17] studied the properties of derivations for lattices.

Received: 26 January 2013, Accepted: 16 August 2013. Communicated by M. Stefanescu
∗Address correspondence to Lida Torkzadeh; E-mail: ltorkzadeh@yahoo.com

In this paper, we define and study a derivation on BL-algebras which comes in analogy with Leibniz's formula for derivations in rings. The paper is organized as follows.
In section 2, the basic definitions and results are summarized. In section 3, we introduce $(\odot, \vee)$-derivations on BL -algebras and study their properties. We show they are not isotone in general. Some conditions are obtained such that $(\odot, \vee)$-derivations are isotone. Finally, we characterize $(\odot, \vee)$-derivations on the Gödel structure $[0, 1]$.

2. Preliminaries

A BL-algebra is an algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \to$ and two constants $0, 1$ such that:
$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
$(BL2)$ $(A, \odot, 1)$ is a commutative monoid,
$(BL3)$ $\odot$ and $\to$ form an adjoint pair i.e, $c \leq a \to b$ if and only if $a \odot c \leq b$, for all $a, b, c \in A$,
$(BL4)$ $a \wedge b = a \odot (a \to b)$, for all $a, b \in A$,
$(BL5)$ $(a \to b) \vee (b \to a) = 1$, for all $a, b \in A$.
Let A be a BL-algebra. We set $x^- = x \to 0$, for all $x \in A$.
We define the following operations known in A:
$x \oplus y := (x^- \odot y^-)^-$, $x \ominus y := x \odot y^-$ for any $x, y \in A$.
We denote the set of natural numbers by $\mathbb{N}$ and define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for $n \in \mathbb{N} - \{0\}$ and $a \in A$.

Theorem 2.1. *([4, 5, 8, 10]) In any BL-algebra A, the following properties hold for all $x, y, z \in A$:*
(1) $x \leq y$ if and only if $x \to y = 1$,
(2) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(3) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$ and $y^- \leq x^-$,
(4) $x, y \leq (y \to x) \to x$ and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(5) $x \odot y \leq x \wedge y$, $x \odot 0 = 0$ and $x \odot x^- = 0$,
(6) $1 \to x = x$, $x \to x = 1$, $x \leq y \to x$, $x \to 1 = 1$ and $0 \to x = 1$,
(7) $x \odot y = 0$ iff $x \leq y^-$,
(8) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$ and $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complemented elements in the lattice $L(A)$ (hence $B(A) = B(L(A))$) and it is called boolean center of A.

Theorem 2.2. [8, 10] *For $x \in A$, the following are equivalent:*
(i) $x \in B(A)$,
(ii) $x \odot x = x$ *and* $x^{--} = x$,
(iii) $x \odot x = x$, $x^- \to x = x$,
(iv) $x^- \vee x = 1$,
(v) $(x \to e) \to x = x$, *for any* $e \in A$.

If $e \in B(A)$, then $e \odot x = e \wedge x, \forall x \in A$.
We recall that a t-norm is a function $t : [0,1] \times [0,1] \to [0,1]$ such that (i) t is commutative and associative, (ii) $t(x,1) = x, \forall x \in [0,1]$, and (iii) t is nondecreasing in both components. If t is continuous, we define $x \odot_t y = t(x,y)$ and $x \to_t y = sup\{z \in [0,1] \mid t(z,x) \leq y\}$ for $x, y \in [0,1]$, then $I_t := ([0,1], min, max, \odot_t, \to_t, 0, 1)$ is a BL-algebra. There are three important continuous t-norms on $[0,1]$:
(i) Łukasiewicz: $Ł(x,y) = max\{x+y-1\}$ with $x \to_{Ł} y = min\{1-x+y, 1\}$, and also the n-element set $S_n = \{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$, for each integer $n \geq 2$, is a subalgebra of $[0,1]$,
(ii) Gödel: $G(x,y) = min\{x,y\}$ and $x \to_G y = 1$ if $x \leq y$, otherwise $x \to_G y = y$,
(iii) Product: $P(x,y) = xy$ and $x \to_P y = 1$ if $x \leq y$ and $x \to_P y = y/x$ otherwise.[10]

Definition 2.3. [17] Let L be a lattice and $d : L \to L$ be a function. Then d is called a derivation on L, if $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$.

3. $(\odot, \vee)$-derivations on BL-algebras

Definition 3.1. Let A be a BL-algebra and $d : A \to A$ be a function. We call d a $(\odot, \vee)$-derivation on A, if it satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$$

for all $x, y \in A$.

Now we give some examples and present some properties of $(\odot, \vee)$-derivations on BL-algebras.

Example 3.2. (i) Let A be a BL-algebra. We define a function $d : A \to A$ by $d(x) = 0$, for all $x \in A$. Then d is a $(\odot, \vee)$-derivation on A, which is called the zero $(\odot, \vee)$-derivation.
(ii) Let A be a BL-algebra. Then the identity function on A is a $(\odot, \vee)$-derivation on A.

(iii) Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\rightarrow$ as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra [11] and all of the $(\odot, \vee)$-derivations on A are:
(a_1) $d_1(0) = 0$, $d_1(a) = d_1(b) = d_1(1) = a$,
(a_2) $d_2(0) = d_2(1) = 0$, $d_2(a) = d_2(b) = a$,
(a_3) $d_3(0) = d_3(1) = d_3(b) = 0$, $d_3(a) = a$,
(a_4) $d_4(0) = 0$, $d_4(a) = a$, $d_4(b) = d_4(1) = b$,
(a_5) $d_5(0) = 0$, $d_5(a) = d_5(1) = a$, $d_5(b) = b$,
(a_6) $d_6(0) = d_6(1) = 0$, $d_6(a) = a$, $d_6(b) = b$,
(a_7) $d_7(0) = d_7(1) = d_7(a) = d_7(b) = 0$,
(a_8) $d_8(0) = 0$, $d_8(a) = a$, $d_8(b) = b$, $d_8(1) = 1$.

Proposition 3.3. *Let A be a BL-algebra and d be a $(\odot, \vee)$-derivation on A. Then the following hold for all $x, y \in A$:*
(1) $d(0) = 0$,
(2) $d(x^n) = x^{n-1} \odot d(x)$, $\forall n \geq 1$,
(3) *If* $x \leq y$, *then* $d(x) \leq y^{--}$. *Hence* $x \odot y = 0$ *implies* $d(x) \odot y = 0$,
(4) $d(x) \leq x^{--}$, *and moreover* $x \in B(A)$ *implies* $d(x) \leq x$,
(5) $d(x) = d(x) \vee (x \odot d(1))$ *and so* $x \odot d(1) \odot (d(x))^{-} = 0$,
(6) *If* $d(1) = 1$, *then* $d(B(A)) = B(A)$,
(7) $d(x^{-}) \leq (d(x))^{-}$,
(8) $d(x \odot y) \leq d(x) \vee d(y) \leq d(x) \oplus d(y)$,
(9) $d(x) = 1$ *implies* $x^{-} = 0$,
(10) $d(1) = 1$ *if and only if* $x \leq d(x)$, *for all* $x \in A$.

Proof. (1) Putting $x = y = 0$ in Definition 3.1, we have $d(0) = d(0 \odot 0) = (d(0) \odot 0) \vee (0 \odot d(0)) = 0$.
(2) Setting $x = y$ in Definition 3.1, we get $d(x^2) = d(x \odot x) = (x \odot d(x)) \vee (d(x) \odot x) = x \odot d(x)$. Then (2) can be easily proved by induction.
(3) Let $x \leq y$. Then $x \odot y^{-} = 0$, so $0 = d(0) = (d(x) \odot y^{-}) \vee (x \odot d(y^{-}))$. Thus, $d(x) \odot y^{-} = 0$, that is $d(x) \leq y^{--}$ by Theorem 2.1.
(4) It follows from (3).
(5) By Definition 3.1, we obtain $d(x) = d(x \odot 1) = (d(x) \odot 1) \vee (x \odot d(1)) = d(x) \vee (x \odot d(1))$. So $x \odot d(1) \leq d(x)$, by Theorem 2.1, we get that

$x \odot d(1) \odot d(x)^{-} = 0$.
(6) If $d(1) = 1$, then by part (5), $d(x) = d(x) \vee x$, so $x \leq d(x)$ for all $x \in A$. Also $x \in B(A)$ implies that $d(x) \leq x$, by part (4). Therefore $d(x) = x$, for all $x \in B(A)$, that is $d(B(A)) = B(A)$.
(7) By part (4), we have $x^{---} \leq (d(x))^{-}$ and $d(x^{-}) \leq x^{---}$. Thus $d(x^{-}) \leq (d(x))^{-}$.
(8) By Theorem 2.1, we can conclude that $x \vee y \leq x \oplus y$, for all $x, y \in A$. So $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) \leq d(x) \vee d(y) \leq d(x) \oplus d(y)$.
(9) It follows from (4).
(10) Let $d(1) = 1$. Then by part (5) we have $x = x \odot d(1) \leq d(x)$, for all $x \in A$. The proof of the converse is easy. $\square$

In the following example, we show that the condition $d(1) = 1$ is necessary in Proposition 3.3 part(6).

Example 3.4. Let $A = \{0, a, b, c, d, 1\}$, where $0 < c, d < a < 1$, $0 < c < b < 1$. Define $\odot$ and $\rightarrow$ as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra[16]. Define the function d on A by:

$d(0) = d(1) = d(c) = 0$, $d(d) = d$, $d(b) = c$, $d(a) = d$.

Then d is a $(\odot, \vee)$-derivation on A, while $d(B(A)) = \{0, c, d, 1\} \neq \{0, b, d, 1\} = B(A)$. Also d is not a lattice derivation defined in Definition 2.3, since $d(a \wedge b) = d(c) = 0 \neq c = c \vee 0 = (a \wedge c) \vee (d \wedge b) = (a \wedge d(b)) \vee (d(a) \wedge b)$.

Proposition 3.5. *Let d be a $(\odot, \vee)$-derivation on a BL-algebra A. Then the following hold for all $x, y \in B(A)$:*
(1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$,
(2) $d(x) \odot d(y) \leq d(x \odot y)$,
(3) $(d(x))^n \leq d(x^n)$, for all $n \geq 1$,
(4) $d(x) = x \odot d(x)$.

Proof. (1) By Definition 3.1, for $x, y \in B(A)$, we have:

$d(x \wedge y) = d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) = (d(x) \wedge y) \vee (x \wedge d(y))$.
(2) Let $x, y \in B(A)$. By Proposition 3.3 part(4) we have :
$d(x) \odot d(y) \leq d(x) \odot y \leq (d(x) \odot y) \vee (x \odot d(y)) = d(x \odot y)$.
The proofs of (3) and (4) follow from (2) and Proposition 3.3 part(2), respectively. □

In the following example, we give a $(\odot, \vee)$-derivation d such that $d(1) = 1$ and $d(x) \nleq x$, for some $x \in A$.

Example 3.6. Let $A = \{-\infty, \ldots, -3, -2, -1, 0\}$, where $-\infty < \ldots < -3 < -2 < -1 < 0$. Define $\odot$ and $\to$ on A as follows:

$\to$	$-\infty$	...	-3	-2	-1	0
$-\infty$	0	...	0	0	0	0
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	0	0	0	0
-2	$-\infty$	...	-1	0	0	0
-1	$-\infty$	...	-2	-1	0	0
0	$-\infty$	...	-3	-2	-1	0

$\odot$	$-\infty$	a	-3	-2	-1	0
$-\infty$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	-6	-5	-4	-3
-2	$-\infty$	...	-5	-4	-3	-2
-1	$-\infty$	...	-4	-3	-2	-1
0	$-\infty$	...	-3	-2	-1	0

Then $(A, \wedge, \vee, \odot, \to, -\infty, 0)$ is a BL-algebra [11]. We define the function d on A by $d(x) = x + 1$, for all $x \neq -\infty$, $d(-\infty) = -\infty$ and $d(0) = 0$. It is easy to check that d is a $(\odot, \vee)$-derivation on A. Also $d(x) \nleq x$, $\forall x \in A - \{0, -\infty\}$.

Consider the derivation d_1 in Example 3.2 part (iii). It is clear that $d_1(1) = a \notin B(A)$. So the condition $d(1) \in B(A)$ may fail for a general derivation d.

Proposition 3.7. *Let d be a $(\odot, \vee)$-derivation on A. If $d(1) \in B(A)$, then $d(d(1)) = d(1)$.*

Proof. Let $d(1) \in B(A)$. Then $d(d(1)) \leq d(1)$ by Proposition 3.3 part(4). Also $d(1) \in B(A)$ implies that $d(1) \odot d(1) = d(1)$ and so by Proposition 3.3 part(5), we get that $d(d(1)) = d(d(1)) \vee d(1)$, that is $d(1) \leq d(d(1))$. Hence $d(d(1)) = d(1)$. $\square$

Proposition 3.8. *Let A be a non-trivial BL-algebra and $0 \neq a \in B(A)$. Then the map $d : A \to A$ defined by $d(0) = 0$, $d(x) = a$, for all $x \in A \setminus \{0\}$, is not a $(\odot, \vee)$-derivation on A.*

Proof. Let $a \in B(A) \setminus \{0\}$ and d be a $(\odot, \vee)$-derivation on A. If $a \neq 1$, then $0 = d(a \odot a^-) = (d(a) \odot a^-) \vee (a \odot d(a^-)) = (a \odot a^-) \vee (a \odot a) = 0 \vee a^2 = a^2 = a$. Thus $a = 0$, which is a contradiction.
Now, let $a = 1$. Then $d(x \odot x) = d(x) \odot x = x$, since d is a $(\odot, \vee)$-derivation. On the other hand, since A is a non-trivial BL-algebra, so there exists $x \in A \setminus \{0, 1\}$. If $x \odot x = 0$, then $x = 0$, which is a contradiction. If $x \odot x \neq 0$, we have $d(x \odot x) = 1$, thus $x = 1$, which is a contradiction. The proof is complete. $\square$

Proposition 3.9. *Let A be a BL-algebra and $a \notin B(A)$. If the map $d : A \to A$ defined by $d(0) = 0$, $d(x) = a$, for all $x \in A$, is a $(\odot, \vee)$-derivation on A, then $a \leq x$ or $a \leq x^-$, for $x \in A \setminus \{0\}$.*

Proof. For any $x \in A \setminus \{0\}$, we have $d(x \odot x) = d(x) \odot x = a \odot x$. Now we have two cases:
i) If $x \odot x \neq 0$, then $a \odot x = a$, so $a \leq x$.
ii) If $x \odot x = 0$, then $a \odot x = 0$ and thus $a \leq x^-$.
Therefore $a \leq x$ or $a \leq x^-$, for all $x \in A \setminus \{0\}$. $\square$

The following example shows that the converse of the above proposition is not true in general.

Example 3.10. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\to$ as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra[11] and $a \leq x$, for every $x \in A \setminus \{0\}$. Also the map d defined by $d(0) = 0$, $d(x) = a$, for all $x \in A$, is not a $(\odot, \vee)$-derivation on A, otherwise we have $a = d(a) = d(b \odot b) = b \odot a = 0$, that is not true.

A function $d : A \to A$ is called an isotone function if, for every $x, y \in A$, $x \leq y$ implies that $d(x) \leq d(y)$.

Example 3.11. Consider the Example 3.2 part(iii). One can see that d_1 is an isotone $(\odot, \vee)$-derivation, while d_2 is not an isotone $(\odot, \vee)$-derivation.

Lemma 3.12. *Let A be a BL-algebra and $a \in A$. Then the map $d_a : A \to A$ defined by $d_a(x) = a \odot x$, for all $x \in A$, is an isotone $(\odot, \vee)$-derivation on A.*

Proof. Let $x, y \in A$. Then $d_a(x \odot y) = a \odot x \odot y = (a \odot x \odot y) \vee (a \odot x \odot y) = (d_a(x) \odot y) \vee (x \odot d_a(y))$ and so d_a is a $(\odot, \vee)$-derivation.
Now let $x \leq y$. Then $x \odot a \leq y \odot a$ and so $d_a(x) \leq d_a(y)$. □

In the following example we show that every isotone $(\odot, \vee)$-derivation is not necessarily of the form of d_a for some $a \in A$.

Example 3.13. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $\odot$ and $\to$ as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra [11]. Define the function d on A by $d(0) = 0$, $d(x) = a$, for all $x \in A$. Then d is an isotone $(\odot, \vee)$-derivation on A, while there is no $t \in A$ such that $d(x) = t \odot x$, for all $x \in A$.

Lemma 3.14. *Let d be a $(\odot, \vee)$-derivation on a BL-algebra A. If $d(1) = 1$, then A is an isotone $(\odot, \vee)$-derivation.*

Proof. Let $x \leq y$. Then, by (BL4) and Proposition 3.3 part (10), we have $d(x) = d(x \wedge y) = d(y \odot (y \to x)) = (d(y) \odot (y \to x)) \vee (y \odot d(y \to x)) \leq d(y) \vee y = d(y)$. □

The converse of the above lemma is not true in general. The $(\odot, \vee)$-derivation d_1 in Example 3.2 part *(iii)* is isotone, while $d_1(1) = a \neq 1$

Proposition 3.15. *Let A be a BL-algebra. If d is an isotone $(\odot, \vee)$-derivation on A such that $d(x) \leq x$ and $d(x) = d(x) \odot d(x)$, for all $x \in A$, then the following hold for all $x, y \in A$:*
(i) $d(x) = d(1) \odot x$,
(ii) $d(x \odot y) = d(x) \odot d(y)$,
(iii) $d(x \wedge y)) = d(x) \wedge d(y)$,
(iv) $d(x \vee y)) = d(x) \vee d(y)$,

(v) $d(d(x)) = d(x)$,
(vi) $d(x \to y) \leq d(x) \to d(y)$.

Proof. (i) By Proposition 3.3 part(5), we have $(x \odot d(1)) \leq d(x)$, for all $x \in A$. Since $d(x) \leq x$ and $d(x) \leq d(1)$, we get that $d(x) = d(x) \odot d(x) \leq x \odot d(1)$, for all $x \in A$. Therefore $d(x) = x \odot d(1)$.
(ii) By (i), we have: $d(x \odot y) = d(1) \odot (x \odot y) = d(1) \odot d(1) \odot x \odot y = (d(1) \odot x) \odot (d(1) \odot y) = d(x) \odot d(y)$.
(iii) By (i) and Theorem 2.1, we have $d(x \wedge y) = d(1) \odot (x \wedge y) = (d(1) \odot x) \wedge (d(1) \odot y) = d(x) \wedge d(y)$.
(iv) The proof is similar to that of (iii).
(v) By (i),(ii) we obtain, $d(d(x)) = d(x) \odot d(1) = d(x \odot 1) = d(x)$.
(vi) By (ii) and (BL4), we have: $d(x) \odot d(x \to y) = d(x \odot (x \to y)) = d(x \wedge y) \leq d(y)$, thus $d(x \to y) \leq d(x) \to d(y)$. $\square$

If an isotone $(\odot, \vee)$-derivation d satisfies the conditions of Proposition 3.15, d may not be a homomorphism, since $d(x \to y) = d(x) \to d(y)$ may fail. Consider the isotone $(\odot, \vee)$-derivation d_1 in Example 3.2 part *(iii)*. One can see that $d_1(x) \leq x$ and $d_1(x) = d_1(x) \odot d_1(x)$, for all $x \in A$, while $d_1(a \to b) = d_1(1) = a \neq 1 = d_1(a) \to d_1(b)$.

Proposition 3.16. *Let d be a $(\odot, \vee)$-derivation on the Gödel BL-algebra A. Then the following hold for every $x, y \in A$:*
(i) $d(x) \leq x$,
(ii) if $x \leq d(1)$, *then* $d(x) = x$,
(iii) if $x \geq d(1)$, *then* $d(1) \leq d(x)$,
(iv) if $x \leq y$, *then* $d(x) = x$ *or* $d(y) \leq d(x)$,
(v) $d(d(1)) = d(1)$.

Proof. *(i)* For $x \in A$, we have $d(x) = d(x \odot x) = d(x) \odot x = min\{d(x), x\}$ and so $d(x) \leq x$.
(ii) Let $x \leq d(1)$. Then $d(x) = d(x \odot 1) = d(x) \vee (x \odot d(1)) = d(x) \vee (min\{x, d(1)\} = d(x) \vee x$ and so $x \leq d(x)$. By *(i)*, we get that $d(x) = x$.
(iii) Let $x \geq d(1)$. Then similar to the proof *(ii)*, we have $d(x) = d(x) \vee d(1)$, thus $d(1) \leq d(x)$.
(iv) Let $x \leq y$. Then by *(i)*, $d(x) \leq y$, and so $d(x) = d(x \odot y) = d(x) \vee (x \odot d(y))$. Now we have tow cases:
(1) If $x \leq d(y)$, then $d(x) = d(x) \vee x$, therefore $d(x) = x$.
(2) If $d(y) \leq x$, then $d(x) = d(x) \vee d(y)$ and so $d(y) \leq d(x)$.
(v) Since $d(1) \leq d(1)$, then by (ii), $d(d(1)) = d(1)$. $\square$

Proposition 3.17. *Let A be a Gödel BL-algebra and $a \in A$. The map $d : A \to A$ defined by $d(x) = a$ if $x > a$ and $d(x) = x$ if $x \leq a$ is a $(\odot, \vee)$-derivation on A.*

Proof. For $x, y \in A$, we have the following cases:
Case 1, $x, y \leq a$. Then $x \odot y \leq a = d(x \odot y) = x \odot y = (d(x) \odot y) \vee (x \odot d(y))$.
Case 2, $x, y > a$. Then $x \odot y > a$ and thus $d(x \odot y) = a = a \vee a = (d(x) \odot y) \vee (x \odot d(y))$.
Case 3, $x \leq a$ and $y > a$. Then $x < y$ and so $d(x \odot y) = x$. On the other hand,$(d(x) \odot y) \vee (x \odot d(y)) = (x \odot y) \vee (x \odot a) = x$. Thus $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$.
Case 4, $x > a$ and $y \leq a$. Similar to (c), we get that $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$. □

Proposition 3.18. *The map $d : S_n \to S_n$, for $n \geq 2$, defined by $d(x) = 0$ if $x \neq \frac{n-1}{n}$ and $d(x) = \frac{1}{n}$ if $x = \frac{n-1}{n}$ is a $(\odot, \vee)$-derivation on S_n.*

Proof. For every $x, y \in S_n$, we have the following cases:
Case 1, If $x = 0$ or $y = 0$, then $d(x \odot y) = 0 = (d(x) \odot y) \vee (x \odot d(y))$.
Case 2, If $x = 1$ and $y = \frac{k}{n}$, then $d(x \odot y) = d(1 \odot \frac{k}{n}) = d(\frac{k}{n})$. On the other hand, $(d(x) \odot y) \vee (x \odot d(y)) = 0 \vee d(\frac{k}{n}) = d(\frac{k}{n})$. Hence $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$.
Case 3, If $x = y = \frac{n-1}{n}$, then $d(x \odot y) = d(\frac{n-1}{n} \odot \frac{n-1}{n}) = d(max\{\frac{n-1}{n} + \frac{n-1}{n} - \frac{n}{n}, 0\}) = d(max\{\frac{n-2}{n}, 0\})$. Since $\frac{n-2}{n} < \frac{n-1}{n}$, so $d(x \odot y) = 0$. On the other hand, $(d(x) \odot y) \vee (x \odot d(y)) = d(\frac{n-1}{n}) \odot \frac{n-1}{n} = \frac{1}{n} \odot \frac{n-1}{n} = max\{\frac{1}{n} + \frac{n-1}{n} - \frac{n}{n}, 0\} = 0$. Thus $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$.
Case 4, If $x = \frac{n-1}{n}$ and $y = \frac{k}{n} < \frac{n-1}{n}$, then $d(x \odot y) = d(max\{\frac{k-1}{n}, 0\}) = 0 = 0 \vee (\frac{1}{n} \odot \frac{k}{n}) = (x \odot d(y)) \vee (d(x) \odot y)$.
Case 5, If $x = \frac{m}{n} < \frac{n-1}{n}$ and $y = \frac{k}{n} < \frac{n-1}{n}$, then $d(x \odot y) = 0 = (d(x) \odot y) \vee (x \odot d(y))$. □

4. Conclusion and Future Research

In this paper, we applied the notion of derivation in rings to BL-algebras and we have introduced the notion of a $(\odot, \vee)$- derivation. We have also presented many important properties of this derivation on BL-algebras. We showed that a $(\odot, \vee)$- derivation d may not be isotone and found some conditions under which d is isotone. Finally, we characterized $(\odot, \vee)$- derivations on the the Gödel structure $[0, 1]$.

Since BL-algebras and residuated lattices are closely related, we will use the results of this paper to study derivations on residuated lattices and related algebraic systems. Some important issues for future works are: (i) developing the properties of a derivation, (ii) defining new derivations which are related to given derivations on BL-algebras, (iii) finding useful results on the other algebraic structures.

Acknowledgments

The authors wish to thank the referee for many helpful comments.

References

[1] E. Albas, *On ideals and orthogonal generalized derivations of semiprime ring*, Math. J. Okayama Univ., **49** (2007), 53-58.

[2] H. E. Bell and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar., **53** (1989), 339-346.

[3] H. E. Bell and G. Mason, *On derivations in near-rings and near-fields*, North-Holland Math. Studies, **137** (1987), 31-35.

[4] D. Busneag and D. Piciu, *BL-algebra of fractions relative to an $\wedge$-closed system,* An. St. Univ. Ovidius Constanta, **11** (1) (2003), 31-40.

[5] D. Busneag and D. Piciu, *On the lattice of deductive systems of a BL-algebra,* Cent. Eur. J. Math., **1** (2)(2003), 221-238

[6] Y. Ceven and M. A. Ozturk, *On f-derivations of lattices,* Bull. Korean Math. Soc., **45**(4) (2008), 701-707.

[7] R. Cignoli, F. Esteva, L. Godo and A. Torrens, *Basic fuzzy logic is the logic of continuous t-norms and their residua*, Soft Comput., **4**(2000), 106-112 .

[8] A. Di Nola, G. Georgescu and A. Iorgulescu, *Pseudo BL-algebra: Part I*, Mult.-Valued Log., **8**(5-6) (2002), 673-714 .

[9] L. Ferrari, *On derivations of lattices*, Pure Math. Appl., **12** (2001), 365-382.

[10] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht (1998).

[11] A. Iorgulescu, *Algebras of logic as BCK-algebras*, Bucharest University of Economics Bucharest, Romania (2008) .

[12] Y. B. Jun and X. L. Xin, *On derivations of BCI-algebras*, Inform. Sci., **159** (2004), 167-176.

[13] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.

[14] G. Szász, *Derivations of lattices*, Acta Sci. Math. (Szeged), **37** (1975), 149-154.

[15] E. Turunen, *BL-algebras of basic fuzzy logic*, Mathware Soft Comput., **6** (1999), 49-61 .

[16] E. Turunen, *Boolean deductive systems of BL-algebras*, Arch. Math. Logic, **40** (2001), 467-473 .

[17] X .L. Xin, T. Y. Li and J.H. Lu, *On derivations of lattices*, Inform. Sci. , **178**(2008), 307-316.

[18] J. Zhan and Y. Lin Liu, *On f-derivations of BCI-algebras*, Int. J. Math. Math. Sci., **11** (2005), 1675-1684.

L. Torkzadeh
Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran
Email: ltorkzadeh@yahoo.com

L. Abbasian
Department of Statistics, Kerman Branch, Islamic Azad University, Kerman, Iran
Email: leila_abbasian6@yahoo.com