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ON BI-IDEALS OF Γ-SEMIHYPERRINGS

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ABSTRACT. The concept of Γ-semihyperrings is a generalization of semirings, semihyperrings and Γ-semirings. The notion of bi-ideals and minimal bi-ideals in Γ-semihyperrings is introduced with several examples. We also made some ideal theoretic characterization of bi-ideals and minimal bi-ideals in Γ-semihyperrings. Then the notion of bi-simple Γ-semihyperrings is introduced and it is proved that "If R is a Γ-semihyperring without zero, then R is a bi-simple Γ-semihyperring if and only if $(k)_b = R$, for all $k \in R$, where $(k)_b$ is a bi-ideal generated by k."

Key Words: Γ-semihyperring, Bi-ideals, Minimal bi-ideals, bi-simple Γ-semihyperring.
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1. Introduction and preliminaries

The notion of a bi-ideal for a semigroup as a generalization of one sided ideal, two sided ideal and quasi-ideal of a semigroup was first introduced by Good and Hughes. Lajos and Szasz [6, 7] studied properties of bi-ideals in a semigroup. Lajos and Szasz [5], also introduced the concept of a bi-ideal for associative rings and proved some important properties in this respect. Shabir et al. [12] considered the concept of a bi-ideal of semirings and then furnished the relationship between bi-ideals and quasi-ideals of a semiring.

A Γ-semigroup is a generalization of a semigroup. The concept of a minimal bi-ideal, a 0-minimal bi-ideal, a bi-simple Γ-semigroup and a

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0-bi-simple Γ-semigroup for a Γ-semigroup were introduced by Iampan [2]. He proved various characterization of a bi-simple Γ-semigroup and a 0-bi-simple Γ-semigroup similar to characterization for semigroups given by Iampan [1]. Jagtap and Pawar [3] introduced the concepts of a bi-simple Γ-semiring and a 0-bi-simple Γ-semiring and proved some properties.

In 1934, Marty [8] introduced the notion of a hypergroup and then an algebraic hyperstructure were widely studied which are generalization of classical algebraic structure. In classical algebraic structure the composition of two elements is an element while in an algebraic hyperstructure composition of two elements is a set. Let H be a non-empty set then, the map $\circ : H \times H \to \wp^*(H)$ is called a hyperopertion where $\wp^*(H)$ is the family of all non-empty subsets of H and the couple $(H, \circ)$ is called a hypergroupoid. Moreover, the couple $(H, \circ)$ is called a semihypergroup if for every $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$.

The notion of a Γ-semihyperring as a generalization of semirings, semihyperrings and Γ-semirings was introduced by Dehkordi and Davvaz [9]. Pawar et al. [10] introduced regular (strongly regular) Γ-semihyperrings and gave it's characterization with the help of ideals of Γ-semihyperrings. The hyperstructure theory has a vast application in various branches of science. So it essential to study various concepts of classical algebraic theory in hyperstructure theory. We tried to study the concept of a bi-ideal for a Γ-semihyperring. The present paper is divided into four different sections. In section 2, the concept of bi-ideals of Γ-semihyperrings is introduced with examples. Some important characterization of bi-ideals in Γ-semihyperrings are done with the help of ideals of Γ-semihyperrings. It proved every quasi-ideals of Γ-semihyperrings is a bi-ideal of a Γ-semihyperring. In section 3, we introduced the notion of bi-simple Γ-semihyperrings and a 0-bi-simple Γ-semihyperring on the line of Iampan [2] and it is proved that "If R is a Γ-semihyperring without zero, then R is a bi-simple Γ-semihyperring if and only if $(k)_b = R$, for all $k \in R$, where $(k)_b$ is a bi-ideal generated by k." In section 4, we introduced the notion of a minimal bi-ideal in Γ-semihyperrings and characterized it with the help of minimal ideals in Γ-semihyperrings.

Here are some useful definitions and the readers are requested to refer [9], for more details

Definition 1.1. A semihyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following properties:

(1) $(R,+)$ is a commutative semihypergroup; that is, $(x+y)+z = x+(y+z)$ and $x+y=y+x$, for all $x,y,z \in R$.
(2) $(R,\cdot)$ is semihypergroup.
(3) The hyperoperation $\cdot$ is distributive with respect to hyperoperation $+$; that is $x\cdot(y+z) = x\cdot y + x\cdot z, (x+y)\cdot z = x\cdot z + y\cdot z$, for all $x,y,z \in R$.
(4) The element $0 \in R$ is an absorbing element; that is $x\cdot 0 = 0\cdot x = 0$, for all $x \in R$.

Definition 1.2. A semihyperring $(R,+,\cdot)$ is called commutative if and only if $a\cdot b = b\cdot a$, for all $a,b \in R$.

Definition 1.3. Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-semihyperring if there is a map $R\times\Gamma\times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$, for all $a,b\in R$ and $\alpha\in\Gamma$) and $\wp^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(1) $a\alpha(b+c) = a\alpha b + a\alpha c$
(2) $(a+b)\alpha c = a\alpha c + b\alpha c$
(3) $a(\alpha+\beta)c = a\alpha c + a\beta c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a,b,c\in R$ and for all $\alpha,\beta\in\Gamma$.

In the above definition, if R is a semigroup, then R is called a multiplicative Γ-semihyperring.

Definition 1.4. A Γ-semihyperring R is called commutative if $a\alpha b = b\alpha a$, for all $a,b\in R$ and $\alpha\in\Gamma$.

Definition 1.5. A Γ-semihyperring R with zero, if there exists $0\in R$ such that $a\in a+0$ and $0\in 0\alpha a, 0\in a\alpha 0$, for all $a\in R$ and $\alpha\in\Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x\in R$, then

$$A+B = \{x \mid x\in a+b, a\in A, b\in B\}$$
$$A\Gamma B = \{x \mid x\in a\alpha b, a\in A, b\in B, \alpha\in\Gamma\}.$$

Definition 1.6. A non-empty subset R_1 of Γ-semihyperring R is called a Γ-sub semihyperring if it is closed with respect to the multiplication and addition, that is, $R_1+R_1\subseteq R_1$ and $R_1\Gamma R_1\subseteq R_1$.

Definition 1.7. A right (left) ideal I of a Γ-semihyperring R is an additive sub semihypergroup of $(R,+)$ such that $I\Gamma R\subseteq I(R\Gamma I\subseteq I)$. If I is both right and left ideal of R, then we say that I is a two sided ideal or simply an ideal of R.

2. Bi-ideals in Γ-semihyperrings

In this section, the concept of a bi-ideal of a Γ-semihyperring is introduced analogously with the definition of a bi-Γ-ideal of Γ-semiring given by Kaushik and Khan [4]. We proved some important properties accordingly.

Definition 2.1. A non-empty set B of a Γ-semihyperring R is a bi-ideal of R if B is a Γ-subsemihyperring of R and $B\Gamma R\Gamma B \subseteq B$.

Definition 2.2. [10] A subset E of a Γ-semihyperring R is an idempotent set if there exists $\Gamma_1 \subseteq \Gamma$ such that $E\Gamma_1 E = E$. It is referred as E is a Γ_1-idempotent.

Definition 2.3. [11]An element e of a Γ-semihyperring R is said to be a left (right) identity of R if $r \in e\alpha r (r \in r\alpha e)$ for all $r \in R$ and $\alpha \in \Gamma$. An element e of a Γ-semihyperring R is said to be a two sided identity or simply an identity if e is both left and right identity, that is $r \in e\alpha r \cap r\alpha e$ for all $r \in R$ and $\alpha \in \Gamma$.

Example 2.4. [10] Consider the following:

$$R = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{R}\}$$
$$\Gamma = \{z \mid z \in \mathbb{Z}\}$$
$$A_\alpha = \{\begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \mid a, b \in \mathbb{R}, \alpha \in \Gamma\}.$$

Then, R is a Γ-semihyperring under the the matrix addition with hyperoperation $M\alpha N \to MA_\alpha N$, for all $M, N \in R$ and $\alpha \in \Gamma$.

Then

$$B = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0, a, b \in \mathbb{R}\}$$

is a bi-ideal of a Γ-semihyperring R.

Example 2.5. [10] Let $R = Q^+, \Gamma = \{z \mid z \in \mathbb{Z}\}$ and $A_\alpha = \alpha\mathbb{Z}^+$, we define $x\alpha y \to xA_\alpha y, \alpha \in \Gamma$ and $x, y \in R$. Then R is a Γ-semihyperring under ordinary addition and multiplication.

Then $\mathbb{Z}$ is a bi-ideal of a Γ-semihyperring R.

Definition 2.6. [11] A subsemihypergroup Q of $(R, +)$ is said to be a quasi-ideal of a Γ-semihyperring R if $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$.

Theorem 2.7. *Any one sided (two sided) ideal of a* Γ*-semihyperring* R *is a bi-ideal of* R*.*

Proof. Let A be a left ideal of a Γ-semihyperring R. Then A is a Γ-subsemihyperring of R. As A is a left ideal of a Γ-semihyperring R, $A\Gamma R\Gamma A = A\Gamma(R\Gamma A) \subseteq A\Gamma A \subseteq R\Gamma A \subseteq A$. This gives a left ideal of a Γ-semihyperring R is a bi-ideal of R. On similar lines we can also prove the result for a right and a two sided ideal of a Γ-semihyperring R. □

Theorem 2.8. *Every quasi-ideal of a* Γ*-semihyperring* R *is a bi-ideal of* R*.*

Proof. Let Q be a quasi-ideal of a Γ-semihyperring R. Hence Q is a Γ-subsemihyperring of R. But as $R\Gamma Q$ is a left ideal of a Γ-semihyperring R then $Q\Gamma R\Gamma Q = Q\Gamma(R\Gamma Q) \subseteq R\Gamma Q$ and $Q\Gamma R\Gamma Q = (Q\Gamma R)\Gamma Q \subseteq Q\Gamma R$ as $Q\Gamma R$ is a right ideal of a Γ-semihyperring R. Thus we get, $Q\Gamma R\Gamma Q \subseteq R\Gamma Q \cap Q\Gamma R \subseteq Q$, since Q is a quasi-ideal of a Γ-semihyperring R. Therefore by definition we get, Q is a bi- ideal of a Γ-semihyperring R. □

Theorem 2.9. *Let* R *be a* Γ*-semihyperring. Then intersection of a right ideal and a left ideal of* R *is a bi-ideal of* R *provided that it is a non-empty.*

Proof. Let A be a right ideal and B be a left ideal of a Γ-semihyperring R. Then $A\cap B$ is a Γ-subsemihyperring of R. Also, $(A\cap B)\Gamma R\Gamma(A\cap B) \subseteq A\Gamma R\Gamma B \subseteq R\Gamma B \subseteq B$, since B is a left ideal of a Γ-semihyperring R. Similarly using A is a right ideal of a Γ-semihyperring R, we can show $(A\cap B)\Gamma R\Gamma(A\cap B) \subseteq A$. Hence we get, $(A\cap B)\Gamma R\Gamma(A\cap B) \subseteq A\cap B$. Therefore $A\cap B$ is a bi-ideal of a Γ-semihyperring R. □

Theorem 2.10. *Intersection of any family of bi-ideals of a* Γ*-semihyperring* R *is a bi-ideal of a* Γ*-semihyperring* R *provided that it is non-empty.*

Proof. Proof is obvious. □

Theorem 2.11. *Let* R *be a* Γ*-semihyperring. Then* $a\Gamma R\Gamma a$ *is a bi-ideal of a* Γ*-semihyperring* R*, for all* $a \in R$*.*

Proof. For any $a \in R$, $a\Gamma R\Gamma a$ is a Γ-subsemihyperring of a Γ-semihyperring R. Since $a\Gamma R$ is a right ideal of a Γ-semihyperring R. Therefore

$$\begin{aligned}(a\Gamma R\Gamma a)\Gamma R\Gamma(a\Gamma R\Gamma a) &= (a\Gamma R)\Gamma(a\Gamma R)\Gamma(a\Gamma R)\Gamma a\\ &\subseteq (a\Gamma R)\Gamma(a\Gamma R)\Gamma a\\ &\subseteq a\Gamma R\Gamma a.\end{aligned}$$

Hence we get, $(a\Gamma R\Gamma a)\Gamma R\Gamma(a\Gamma R\Gamma a) \subseteq a\Gamma R\Gamma a$. Thus we get, $a\Gamma R\Gamma a$ is a bi-ideal of R, for all $a \in R$. □

Theorem 2.12. *If B is a bi-ideal of a Γ-semihyperring R, then $a\Gamma B$ and $B\Gamma a$ are bi- ideals of a Γ-semihyperring R, for any $a \in R$.*

Proof. Let B be a bi-ideal of a Γ-semihyperring R. Then $a\Gamma B + a\Gamma B = a\Gamma(B+B) \subseteq a\Gamma B$ since B is a Γ-subsemihyperring of a Γ-semihyperring R and $(a\Gamma B)\Gamma(a\Gamma B) = a\Gamma(B\Gamma a\Gamma B) \subseteq a\Gamma(B\Gamma R\Gamma B) \subseteq a\Gamma B$ gives that $a\Gamma B$ is a Γ-subsemihyperring of R.
Now $(a\Gamma B)\Gamma R\Gamma(a\Gamma B) = a\Gamma B\Gamma(R\Gamma a)\Gamma B \subseteq a\Gamma(B\Gamma R\Gamma B) \subseteq a\Gamma B$ since B is a bi-ideal of R. it gives $a\Gamma B$ is a bi-ideal of R. On similar lines, we can prove $B\Gamma a$ is a bi-ideal of R. □

Theorem 2.13. *If B is a bi- ideal of a Γ-semihyperring R and R_1, R_2 be a non-empty subsets of R. Then $R_1\Gamma B\Gamma R_2$ is a bi-ideal of a Γ-semihyperring R.*

Proof. Since B is a bi-ideal of a Γ-semihyperring R,
$(R_1\Gamma B\Gamma R_2) + (R_1\Gamma B\Gamma R_2) \subseteq R_1\Gamma(B + B)\Gamma R_2 \subseteq R_1\Gamma B\Gamma R_2$ and

$$\begin{aligned}(R_1\Gamma B\Gamma R_2)\Gamma(R_1\Gamma B\Gamma R_2) &= R_1\Gamma B\Gamma(R_2\Gamma R_1)\Gamma B\Gamma R_2\\ &\subseteq R_1\Gamma(B\Gamma R\Gamma B)\Gamma R_2\\ &\subseteq R_1\Gamma B\Gamma R_2.\end{aligned}$$

It gives that $R_1\Gamma B\Gamma R_2$ is a Γ-subsemihyperring of R. Now,

$$\begin{aligned}(R_1\Gamma B\Gamma R_2)\Gamma R\Gamma(R_1\Gamma B\Gamma R_2) &= (R_1\Gamma B)\Gamma(R_2\Gamma R\Gamma R_1)\Gamma(B\Gamma R_2)\\ &\subseteq R_1\Gamma(B\Gamma R\Gamma B)\Gamma R_2\\ &\subseteq R_1\Gamma B\Gamma R_2.\end{aligned}$$

Therefore we get, $R_1\Gamma B\Gamma R_2$ is a bi-ideal of R. □

Corollary 2.14. *Let A be any non-empty subset of a Γ-semihyperring R. Then $A\Gamma R\Gamma A$ is a bi-ideal of a Γ-semihyperring R.*

Corollary 2.15. *Let x be any element of a Γ-semihyperring R. Then $x\Gamma R\Gamma x$ is a bi-ideal of a Γ-semihyperring R.*

Corollary 2.16. *Let x and y be any element of a Γ-semihyperring R. Then $x\Gamma R\Gamma y$ is a bi-ideal of a Γ-semihyperring R.*

Theorem 2.17. *If B is a bi-ideal of a Γ-semihyperring R. Then $B \cap S$ is a bi-ideal of S, where S is a Γ-subsemihyperring of R.*

Proof. Let B be a bi-ideal of a Γ-semihyperring R. Assume S is a Γ-subsemihyperring of R. Then clearly $B \cap S$ is a Γ-subsemihyperring of R and as $B \cap S \subseteq S$, gives $B \cap S$ is a Γ-subsemihyperring of S. Also, $(B \cap S)\Gamma S\Gamma(B \cap S) \subseteq B\Gamma S\Gamma B \subseteq B\Gamma R\Gamma B \subseteq B$, as B is a bi-ideal of a Γ-subsemihyperring of R. Now, $(B \cap S)\Gamma S\Gamma(B \cap S) \subseteq S\Gamma S\Gamma S \subseteq S\Gamma S \subseteq S$, since S is a Γ -subsemihyperring of R. Therefore we get, $(B \cap S)\Gamma S\Gamma(B \cap S) \subseteq B \cap S$. Hence $B \cap S$ is a bi-ideal of S. □

Theorem 2.18. *If B is a bi-ideal and S is a Γ-subsemihyperring of a Γ-semihyperring R. Then $B\Gamma S$ and $S\Gamma B$ are bi-ideals of R.*

Proof. Let B be a bi-ideal and S be a Γ-subsemihyperring of R. Then $B\Gamma S + B\Gamma S = B\Gamma(S + S) \subseteq B\Gamma S$, since S is a Γ-subsemihyperring of R and $(B\Gamma S)\Gamma(B\Gamma S) = (B\Gamma S\Gamma B)\Gamma S \subseteq (B\Gamma R\Gamma B)\Gamma S \subseteq B\Gamma S$, as B is a bi-ideal of R. It gives $B\Gamma S$ is a Γ-subsemihyperring of R. Now, $(B\Gamma S)\Gamma R\Gamma(B\Gamma S) = B\Gamma(S\Gamma R)\Gamma B\Gamma S \subseteq (B\Gamma R\Gamma B)\Gamma S \subseteq B\Gamma S$, since B is a bi-ideal of R. This shows that $B\Gamma S$ is a bi-ideal of R. Similarly we can show $S\Gamma B$ is a bi-ideal of R. □

Corollary 2.19. *Let A and B are bi-ideals of a Γ-semihyperring R. Then $A\Gamma B$ and $B\Gamma A$ are bi-ideals of R.*

Corollary 2.20. *Let L be a right ideal and M be a left ideal of a Γ-semihyperring R. Then $L\Gamma M$ and $M\Gamma L$ are bi-ideals of R.*

Corollary 2.21. *Let P and Q are quasi-ideals of a Γ-semihyperring R. Then $P\Gamma Q$ and $Q\Gamma P$ are a bi-ideals of R.*

Theorem 2.22. *If B is a bi-ideal of a Γ-semihyperring R and E is a bi-ideal of B. Then E is a bi-ideal of R if E is an Γ-idempotent subset of R.*

Proof. As E is a Γ-subsemihyperring of B, E is a Γ-subsemihyperring of R. Now as E is an Γ-idempotent subset of R,

$$\begin{aligned} E\Gamma R\Gamma E &= (E\Gamma E)\Gamma R\Gamma(E\Gamma E) \\ &= E\Gamma(E\Gamma R\Gamma E)\Gamma E \\ &\subseteq E\Gamma(B\Gamma R\Gamma B)\Gamma E \qquad \text{(since } E \subseteq B) \\ &\subseteq E\Gamma B\Gamma E \qquad \text{(since } E \text{ is a bi-ideal of } B) \\ &\subseteq E \end{aligned}$$

This shows that E is a bi-ideal of R. □

Theorem 2.23. *Let I is an ideal of a Γ-semihyperring R. Then each quasi-ideal of I is a bi-ideal of R.*

Proof. Let I be an ideal of a Γ-semihyperring R and Q be a quasi-ideal of I. Then $Q\Gamma R\Gamma Q \subseteq (I\Gamma R)\Gamma Q \subseteq I\Gamma Q$ and $Q\Gamma R\Gamma Q \subseteq Q\Gamma(R\Gamma I) \subseteq Q\Gamma I$, since I is an ideal of R. Therefore $Q\Gamma R\Gamma Q \subseteq (I\Gamma Q) \cap (Q\Gamma I) \subseteq Q$ as Q is a quasi-ideal of I. Thus we get, Q is a bi-ideal of R. □

Theorem 2.24. *For a non-empty subset B of a Γ-semihyperring R, following statements are equivalent.*
1. B is a bi-ideal of R.
2. B is a left ideal of some right ideal of R.
3. B is a right ideal of some left ideal of R.

Proof. (1) $\Rightarrow$ (2)
Assume that B is a bi-ideal of a Γ-semihyperring R. Therefore $B\Gamma R\Gamma B \subseteq B$. We know $B\Gamma R$ is a right ideal of R. Shows that B is a left ideal of a right ideal $B\Gamma R$ of R.
(2) $\Rightarrow$ (3)
Assume that B is a left ideal of a right ideal A of R. Hence we have $A\Gamma B \subseteq B$ and $A\Gamma R \subseteq A$. Therefore $B\Gamma R\Gamma B \subseteq A\Gamma R\Gamma B \subseteq A\Gamma B \subseteq B$, gives that B is a right ideal of a left ideal $R\Gamma B$ of R.
(3) $\Rightarrow$ (1)
Assume that B is a right ideal of some left ideal C of R. Hence we have $B\Gamma C \subseteq B$ and $R\Gamma C \subseteq C$. Therefore $B\Gamma R\Gamma B \subseteq B\Gamma R\Gamma C \subseteq B\Gamma C \subseteq B$. Thus we get, B is a bi-ideal of R. □

Theorem 2.25. *A Γ-subsemihyperring B of a Γ-semihyperring R is a bi-ideal of R if and only if there exists a left ideal P and right ideal Q of R such that $Q\Gamma P \subseteq B \subseteq P \cap Q$.*

3. Bi-Simple Γ-semihyperrings

In this section we introduced the notion of a bi-simple Γ-semihyperring analogous to the concept of a bi-simple Γ-semigroup and a 0-bi-simple Γ-semigroup defined by Iampan in [2] and we proved some properties and result in this respect.

Definition 3.1. A Γ-semihyperring R without zero is a bi-simple Γ-semihyperring if R has no proper bi-ideal of R.

Definition 3.2. A Γ-semihyperring R with zero is said to be a 0-bi-simple Γ-semihyperring if R has no proper non zero bi-ideal.

Definition 3.3. A bi-ideal of a Γ-semihyperring R generated by a non-empty subset K of R is the smallest bi-ideal of R containing K. It is denoted by $(K)_b$.

Definition 3.4. A bi-ideal of a Γ-semihyperring R generated by an element $k \in R$ is the smallest bi-ideal of R containing k. It is denoted by $(k)_b$.

Example 3.5. [10] Let $R = \{a, b, c, d\}$ then R is a commutative semihypergroup with following hyperoperations

$+$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,b,c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{d\}$

Then R be a Γ-semihyperring, where Γ-is any commutative group with operation $x\alpha y \to x \cdot y$ for $x, y \in R$ and $\alpha \in \Gamma$.

Here R does not have any proper bi-ideal. Therefore R is a bi-simple Γ-semihyperring.

Theorem 3.6. *Let R be a Γ-semihyperring without Zero. Then R is a bi-simple Γ-semihyperring if and only if $a\Gamma R\Gamma a = R$, for all $a \in R$.*

Proof. Let R be a bi-simple Γ-semihyperring. Then by theorem 2.11 $a\Gamma R\Gamma a$, for all $a \in R$ is a bi-ideal of R. Thus we get, $a\Gamma R\Gamma a = R$, for all $a \in R$, since R be a bi-simple Γ-semihyperring. Conversely, assume $a\Gamma R\Gamma a = R$, for all $a \in R$ and let B be a bi-ideal of R. Therefore for any $b \in B$, $b\Gamma R\Gamma b = R$ by assumption. Then $R = b\Gamma R\Gamma b \subseteq B\Gamma R\Gamma B \subseteq B$ since $b \in B$ and B is a bi-ideal of R. Hence $R \subseteq B$ and $B \subseteq R$ gives $B = R$. Therefore R is a bi-simple Γ-semihyperring. □

Theorem 3.7. *Let R be a Γ-semihyperring without Zero. Then R is a bi-simple Γ-semihyperring if and only if $(k)_b = R$, for all $k \in R$.*

Proof. Let R be a bi-simple Γ-semihyperring. As for any $k \in R$, $(k)_b$ is bi-ideal of a Γ-semihyperring R generated by k then by definition of bi-simple Γ-semihyperring imply $(k)_b = R$. Conversely, let B be a bi-ideal of a Γ-semihyperring R. Then for any $k \in B$, we have $(k)_b = R$ by assumption $R = (k)_b \subseteq B$. Therefore $R = B$, as B is subset of R. Hence gives R is a bi-simple Γ-semihyperring. □

Theorem 3.8. *Let R be a Γ-semihyperring without Zero, B be a bi-ideal and S be a Γ-subsemihyperring of R. Then $S \subseteq B$ if S is bi-simple and $S \cap B$ is non-empty.*

Proof. Let S be a bi-simple Γ-subsemihyperring of R with $S \cap B$ is non-empty. Then for $a \in S \cap B$, we have $a\Gamma S\Gamma a$ is a bi-ideal of S by theorem 2.11. Then S is a bi-simple imply $a\Gamma S\Gamma a = S$ by theorem 3.6. Hence $S = a\Gamma S\Gamma a \subseteq B\Gamma S\Gamma B \subseteq B$ as B is a bi-ideal. Thus we get $S \subseteq B$. □

Theorem 3.9. *Let R be a 0-bi-simple Γ-semihyperring. If R has an identity element e then $a\Gamma S\Gamma a = R$, for any $a \in R \setminus \{0\}$.*

Proof. Let R be a 0-bi-simple Γ-semihyperring with e is an identity element of R. For any $a \in R \setminus \{0\}$, $a\Gamma R\Gamma a$ is a bi-ideal of R by theorem 2.11. But $a \in a\alpha e\beta a \subseteq a\Gamma R\Gamma a$ as e is an identity element of R. Thus $a\Gamma R\Gamma a$ contains a non zero element a. Therefore $a\Gamma R\Gamma a = R$ since R 0-bi-simple Γ-semihyperring. □

Theorem 3.10. *Let R be a 0-bi-simple Γ-semihyperring, then $(k)_b = R$, for any $k \in R \setminus \{0\}$.*

Proof. Proof is obvious as $k \in (k)_b$, for any $k \in R \setminus \{0\}$. □

Theorem 3.11. *Let R be a Γ-semihyperring with Zero, B is a bi-ideal and S be a Γ-subsemihyperring of R such that $e \in S$, where e is an identity element of R. If S is 0-bi-simple with $S \setminus \{0\} \cap B \neq \phi$ is a non-empty then $S \subseteq B$.*

Proof. Let S be a 0-bi-simple Γ-subsemihyperring R with $S\setminus\{0\}\cap B \neq \phi$. For any $a(\neq 0) \in S \setminus \{0\} \cap B$, $a\Gamma S\Gamma a$ is a bi-ideal of S by theorem 2.11. As S be a 0-bi-simple implies $a\Gamma S\Gamma a = S$ by theorem 3.9. Therefore $S = a\Gamma S\Gamma a \subseteq B\Gamma S\Gamma B \subseteq B$, since B is a bi-ideal. Therefore we get $S \subseteq B$. □

4. Minimal Bi-ideals of Γ-semihyperrings

Definition 4.1. A Γ-semihyperring R without zero. A bi-ideal B of R is said to be a minimal bi-ideal of R if B does not contain any other proper bi-ideal of R.

Definition 4.2. A Γ-semihyperring R with zero. A bi-ideal $B \neq 0$ of R is said to be a 0-minimal bi-ideal of R if B does not contain any other proper non zero bi-ideal of R.

Example 4.3. [9] Let $R = \{a, b, c, d\}, \Gamma = \mathbb{Z}_2$ and $\alpha = \bar{0}, \beta = \bar{1}$. Then R is Γ-semihyperring with the following hyperoperations

$+$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
c	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$
d	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$

β	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$

For any $x, y \in R$ we define $x\alpha y = \{a, b\}$.

Here $\{a, b\}$ is a proper bi-ideal of Γ-semihyperring R which is a minimal.

Theorem 4.4. *Let R be a Γ-semihyperring without zero. If P is a minimal right ideal and Q is a minimal left ideal of R, then $P\Gamma Q$ is a minimal bi-ideal of R.*

Proof. Let P be a minimal right ideal and Q be a minimal left ideal of R. Take $B = P\Gamma Q$ which is a bi-ideal of R by corollary 2.20 . Assume A be a bi-ideal of R such that $A \subseteq B$. As $R\Gamma A$ is a left ideal of R and $R\Gamma A \subseteq R\Gamma B \subseteq R\Gamma P\Gamma Q \subseteq Q$ since Q is left ideal of R. But as Q is minimal left ideal of R, therefore we get, $R\Gamma A = Q$. Similarly, we can show $A\Gamma R = P$. Therefore $B = P\Gamma Q = A\Gamma R\Gamma R\Gamma A \subseteq A\Gamma R\Gamma A \subseteq A$ as A is bi-ideal of R. Hence $B \subseteq A$ and $A \subseteq B$ implies that $B = A$, . This shows that $B = P\Gamma Q$ is a minimal bi-ideal of R. □

Theorem 4.5. *Let R be a Γ-semihyperring without zero.Then any given proper bi-ideal of R is a minimal if and only if the intersection of any two distinct proper bi-ideals is empty.*

Proof. Let any given proper bi-ideal of R is a minimal. Let P and Q be any two distinct proper bi-ideals of R. Suppose $P \cap Q \neq \phi$. Therefore $P \cap Q$ is a bi-ideal of R. So, we get $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$ but any proper bi-ideals are minimal so P and Q gives that $P \cap Q = P = Q$. Contradicts the fact P and Q are distinct proper bi-ideals of R. Therefore our supposition $P \cap Q \neq \phi$ is a wrong. Conversely, if the intersection of any two distinct proper bi-ideals is empty then no any proper bi-ideal of R is contained in any other proper bi-ideal gives converse. □

Theorem 4.6. *Let B be a bi-ideal of a Γ-semihyperring R with zero. If B itself is a 0-bi-simple Γ-semihyperring, then B is a 0-minimal bi-ideal of R.*

Proof. Let B be a bi-ideal of a Γ-semihyperring R and B itself is a 0-bi-simple Γ-semihyperring. Let$A \neq 0$ be a bi-ideal of R such that $A \subseteq B$. Then we have $A\Gamma B\Gamma A \subseteq A\Gamma R\Gamma A \subseteq A$, as A is bi-ideal of S. But then A becomes a bi-ideal of 0-bi-simple Γ-semihyperring B imply $A = B$. Therefore B is a 0-minimal bi-ideal of R. □

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