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SOFT CATEGORY THEORY - AN INTRODUCTION

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ABSTRACT. The soft category theory offers a way to study soft theories developed so far more generally. The main purpose of this paper is to introduce the basic notions of the theory of soft categories, to present some introductory results of the theory. Also we compare soft category with fuzzy category.

Key Words: Category, Soft set, Soft group, Soft ring, Soft category (balanced, normal, additive, with limits, abelian), Soft functor, Fuzzy category.
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1. Introduction

To deal with the complicated problems involving uncertainties in economics, engineering, environmental science, medical science and social science, methods of classical mathematics can not be successfully used. Alternatively, mathematical theories such as probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics were established by researchers to deal with uncertainties appearing in the above fields. These methods also have some inherent difficulties. To overcome these kinds of difficulties, Molodtsov [21] introduced the concept of soft sets. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applicable to many different fields. At present, works on soft set theory are progressing rapidly. Then Maji et al. [18] introduced several operations on soft sets. Aktaş and Çağman [3] defined soft groups and obtained the main properties of these groups. They also compared

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soft sets with fuzzy sets and rough sets. Besides, Jun [12] defined soft ideals on BCK/BCI-algebras. Feng et al. [10] defined soft semirings, soft ideals on soft semirings and idealistic soft semirings. Acar et al. [2] defined soft rings. Qiu-Mei Sun et al. [28] defined the concept of soft modules and studied their basic properties. We refer to [14, 9, 7, 5] for very recent works on soft algebraic structures.

In view of these development of soft algebraic structures and the fact that category theory unifies and simplifies many properties of mathematical systems, in this paper, we focus on introducing basic notions of soft category. Soft category is actually a parameterized family of subcategories of a category. Moreover, the concept of the soft functor is introduced. We show that soft sets, soft groups, soft rings are just special types of soft category and soft group homomorphism, soft ring homomorphism are special types of soft functor. We obtain some interesting properties of them. We also compare soft category with fuzzy category [26, 27].

2. Preliminaries

We assume that reader is familiar to the notations of category theory [16, 8, 1, 20, 6, 15, 22]. In this section we recall some basic definitions of soft set theory, soft group theory, soft ring theory and fuzzy category.

Definition 2.1. [21] Let U be an initial universe set, E a set of parameters, $\mathcal{P}(U)$ the power set of U, and $A \subseteq E$. A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. To illustrate this idea, let us consider the following example.

Example 2.2. Let us consider a soft set (F, E) which describes the attractiveness of houses that Mr.X is considering for purchase. Suppose that there are six houses in the universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ under consideration, and that $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of decision parameters. Let e_1 = expensive, e_2 = beautiful, e_3 = wooden, e_4 = cheap, and e_5 = in green surroundings. In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on.

Definition 2.3. [18] Let (F, A) be a soft set over U. Then (F, A) is called a *soft null set* if $F(x) = \emptyset$ for all $x \in A$.

Definition 2.4. [18] Let (F, A) and (G, B) be soft sets over a common universe U. Then (G, B) is called a *soft subset* of (F, A) if it satisfies the following:
(1) $B \subseteq A$;
(2) For all $x \in B$, $F(x)$ and $G(x)$ are identical approximations.

Definition 2.5. [18] Let (F, A) and (G, B) be two soft sets over U. Then they are said to be *soft equal* if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.6. [3] Let G be a group and A a set of parameters. Let (F, A) be a soft set over G. Then (F, A) is said to be a *soft group* over G if and only if $F(x)$ is a subgroup of G for all $x \in A$.

Example 2.7. [3] Suppose that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and that we define the set-valued function $F(x) = \{y \in G : y = x^n, n \in \mathbb{N}\}$. Then the soft group (F, A) is a parameterized family $\{F(x) : x \in A\}$ of subsets, which gives us a collection of subgroups of G given below:
$F(e) = \{e\}$, $F(12) = \{e, (12)\}$, $F(13) = \{e, (13)\}$, $F(23) = \{e, (23)\}$, $F(123) = F(132) = \{e, (123), (132)\}$.

Definition 2.8. [3] Let (F, A) and (H, B) be soft groups over a common universe G. Then (H, B) is called a *soft subgroup* of (F, A) if it satisfies the following:
(1) $B \subseteq A$;
(2) For all $x \in B$, $H(x)$ is a subgroup of $F(x)$.

Definition 2.9. [3] Let (F, A) and (H, B) be two soft groups over G and K respectively, and let $f : G \to K$ and $g : A \to B$ be two functions. Then we say that (f, g) is a *soft homomorphism*, and that (F, A) is soft homomorphic to (H, B) if the following conditions are satisfied:
(1) f is a group epimorphism from G to K,
(2) g is a surjection, and
(3) $f(F(x)) = H(g(x))$ for all $x \in A$.
In this definition, if f is an isomorphism from G to K and g is a one-to-one mapping from A onto B, then we say that (f, g) is a soft isomorphism and that (F, A) is soft isomorphic to (H, B).

Example 2.10. [3] Consider the groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}_m, \oplus)$. We define a homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_m$ such that $f(k) = \overline{k}$ for $k \in \mathbb{Z}$, and a mapping g from $\mathbb{Z}^+$ onto $\mathbb{Z}_m$ such that $g(k) = \overline{k}$ for $k \in \mathbb{Z}^+$. Let $F : \mathbb{Z}^+ \to P(\mathbb{Z})$ such that $F(x) = \{y \in \mathbb{Z} : y = 5kx, k \in \mathbb{Z}\}$ and

$H : \mathbb{Z}_m \to P(\mathbb{Z}_m)$ such that $H(\overline{u}) = \{\overline{y} \in \mathbb{Z}_m : y = uk, k \in 5\mathbb{Z}\}$. Then we obtain $F(x) = 5x\mathbb{Z}$ and $H(\overline{u}) = \{\overline{ku} : k \in 5\mathbb{Z}\}$. It is clear that $(F, \mathbb{Z}^+)$ and $(H, \mathbb{Z}_m)$ are soft groups over $\mathbb{Z}$ and $\mathbb{Z}_m$, respectively. Since $f(F(x)) = \{\overline{5xk} : k \in \mathbb{Z}\}$ and $H(g(x)) = \{\overline{xs} : s \in 5\mathbb{Z}\}$, we get $f(F(x)) = H(g(x))$. Hence (f, g) is a soft homomorphism, and $(F, \mathbb{Z}^+)$ is soft homomorphic to $(H, \mathbb{Z}_m)$.

Definition 2.11. [3] Let (F, A) be a soft group over G and (H, B) a soft subgroup of (F, A). Then we say that (H, B) is a *normal soft subgroup* of (F, A), if $H(x)$ is a normal subgroup of $F(x)$ for all $x \in B$.

Definition 2.12. [2] Let (F, A) be a soft set over a ring R. Then (F, A) is called a *soft ring* over R if $F(x)$ is a subring of R for all $x \in A$.

Example 2.13. [2] Let $R = A = \mathbb{Z}_6$. Consider the set-valued function $F : A \to P(R)$ given by $F(x) = \{y \in R : x.y = 0\}$. Here we see that $F(x)$ is a subring of R for all $x \in A$. Hence, (F, A) is a soft ring over R.

Definition 2.14. [2] Let (F, A) and (H, B) be soft rings over R. Then (H, B) is called a *soft subring* of (F, A) if it satisfies the following:
(1) $B \subseteq A$,
(2) $H(x)$ is a subring of $F(x)$, for all $x \in B$.

Definition 2.15. [2] Let (F, A) and (H, B) be soft rings over the rings R and R' respectively. Let $f : R \to R'$ and $g : A \to B$ be two mappings. The pair (f, g) is called a *soft ring homomorphism* if the following conditions are satisfied:
(1) f is a ring epimorphism,
(2) g is surjective, and
(3) $f(F(x)) = H(g(x))$ for all $x \in A$.
In this definition, if f is an isomorphism from R to R' and g is a one-to-one mapping from A onto B, then we say that (f, g) is a soft ring isomorphism and that (F, A) is soft isomorphic to (H, B).

Example 2.16. [2] Consider the rings $R = \mathbb{Z}$ and $R' = \{0\} \times \mathbb{Z}$. Let $A = 2\mathbb{Z}$ and $B = \{0\} \times 6\mathbb{Z}$. We see that (F, A) is a soft ring over R and (H, B) is a soft ring over R' if $F(x) = x18\mathbb{Z}$ and $G((0, y)) = \{0\} \times 6y\mathbb{Z}$. Then the function $f : R \to R'$ which is given by $f(x) = (0, x)$ is a ring isomorphism. Moreover, the function $g : A \to B$ which is defined by $g(y) = (0, 3y)$ is a surjective map. As we see, for all $x \in A$, we have $f(F(x)) = f(18x\mathbb{Z}) = \{0\} \times 18x\mathbb{Z} = H(\{0\} \times 3x\mathbb{Z}) = G(g(x))$.Consequently, (f, g) is a soft ring homomorphism.

Fuzzy category was defined in [26]. Here we recall that definition for the lattice $[0,1]$.

Definition 2.17. [27] Let $\mathcal{C}$ be a category and

$$Hom(\mathcal{C}) := \bigcup_{A,B\in Ob(\mathcal{C})} Hom(A,B).$$

Then a fuzzy category $\mathcal{FC}$ over the base category $\mathcal{C}$ is completely described by two mappings $\omega : Ob(\mathcal{C}) \to [0,1]$ and $\mu : Hom(\mathcal{C}) \to [0,1]$, satisfying the following properties:
(1) if $f : X \to Y$, then $\mu(f) \leq min\{\omega(x), \omega(Y)\}$;
(2) $\mu(g \circ f) \geq min\{\mu(g), \mu(f)\}$, whenever the composition is possible;
(3) if $i_X : X \to X$ is the identity map on X, then $\mu(i_X) = \omega(X)$.

We observe that, the "potential objects" of $\mathcal{FC}$ forms a fuzzy subclass $Ob(\mathcal{FC}) = \{(X, \omega(X)) : X \in Ob(\mathcal{C})\}$ of $Ob(\mathcal{C})$ and the "potential morphisms" of $\mathcal{FC}$ forms a fuzzy subclass $Hom(\mathcal{FC}) = \{(f, \mu(f)) : f \in Hom(\mathcal{C})\}$ of $Hom(\mathcal{C})$.

3. Soft category

In this section we introduce the notion of soft category along with some other basic definitions and study some of their properties.

Definition 3.1. Let $\mathcal{C}$ be a category, $\mathcal{P}(\mathcal{C})$ the set of all subcategories of $\mathcal{C}$ and A a set of parameters. Let $F : A \to \mathcal{P}(\mathcal{C})$ be a mapping. Then (F, A) is said to be a *soft category* over $\mathcal{C}$ if $F(x)$ is a subcategory of C [1], i.e. it is nothing but a parameterized family of subcategories of a category.

Example 3.2. Let *SET* be the category of all sets where the arrows are the set mappings and $A = \mathbb{N}$. Let $F(n)$ be the full subcategory [1] of the category *SET* consisting of all sets having cardinality n, for all $n \in \mathbb{N}$. Then (F, A) is a soft category over the category *SET*.

Example 3.3. Let *GRP* be the category of all groups where the arrows are the group homomorphisms. Let $A = \{cyclic, finite, commutative, free\}$. Then (F, A) is a soft category over *GRP*, where $F(x)$ is the subcategory of all groups with the property $x \in A$.
This soft category identifies cyclic groups or finite groups etc in the category *GRP*.

One may say that, this conception can also be represented by soft set theory. But here in soft category theory not only we have considered the groups with certain properties, but also the morphisms between them. This is the advantage of this theory to the soft set theory, particularly, while studying algebraic systems.
In a similar manner, we can construct examples with rings, modules, semirings, semigroups, vector spaces etc.

Now we are going to "softify" some basic definitions known in the category theory and study some results similar to category theory.

Definition 3.4. Let (F, A) and (H, B) be two soft categories over $\mathcal{C}$. Then we say that, (H, B) is a *soft subcategory* of (F, A) if the following conditions are satisfied:
(1) $B \subseteq A$,
(2) $H(x)$ is a subcategory of $F(x)$, for all $x \in B$.

Example 3.5. Let (F, A) be the soft category of Example 3.3 and (H, B) be another soft category over GRP where $B = \{cyclic\}$ and $H(cyclic)$ be the subcategory of all finite cyclic groups. Then clearly (H, B) is a soft subcategory of (F, A).

The following theorem follows easily from the definition.

Theorem 3.6. *If (H, B) is a soft subcategory of (F, A) and (G, D) is a soft subcategory of (H, B), then (G, D) is a soft subcategory of (F, A).*

Definition 3.7. Two soft categories (F, A) and (H, B) over same category $\mathcal{C}$ is said to be *soft equal* if (H, B) is a soft subcategory of (F, A) and (F, A) is a soft subcategory of (H, B).

Theorem 3.8. *Let (F, A) over $\mathcal{C}$ be a soft category and $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ a functor* [1] *from the category $\mathcal{C}$ to the category $\mathcal{D}$. Then $\mathcal{G}(F, A) = (\mathcal{G}F, A)$, defined by $\mathcal{G}F(x) = \mathcal{G}(F(x))$ for all $x \in A$, is a soft category over $\mathcal{D}$.*

Proof. As $F(x)$ is a subcategory of $\mathcal{C}$ and $\mathcal{G}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, $\mathcal{G}(F(x))$ is a subcategory of $\mathcal{D}$. Hence the result follows. □

Definition 3.9. Let (H, B) be a soft subcategory of (F, A) over $\mathcal{C}$. Then
(1) (H, B) is said to be a *full soft subcategory* of (F, A) if $H(x)$ is a full subcategory [1] of $F(x)$, for all $x \in B$.

(2) (H, B) is said to be a *lluf soft subcategory* of (F, A) if $H(x)$ is a lluf subcategory [1] of $F(x)$, for all $x \in B$.

Example 3.10. Let (F, A) is the soft category of Example 3.2. Let us consider (G, B) and (H, C), where $B = C = \{1, 2, 3, 4\}$ and $G(n)$ denotes the full subcategory of the category SET consisting of all subsets of the set of natural numbers of cardinality n; and $H(n)$ denotes the lluf subcategory of of SET consisting of all sets of cardinality n, but only with the bijective mappings. Then both of them are soft categories over SET. Moreover we see that, (G, B) is a full soft subcategory of (F, A) and (H, C) is a lluf soft subcategory of (F, A).

Definition 3.11. Let (F, A) be a soft category over $\mathcal{C}$ and $\mathcal{C}^{op}$ the dual category of $\mathcal{C}$ (see [1]). Then $(F, A)^{op} = (F^{op}, A)$ is said to be the *dual soft category* of (F, A) if $F^{op}(x)$ corresponds to the dual subcategory of $F(x)$, i.e. $F^{op}(x) = (F(x))^{op}$, for all $x \in A$. Clearly $(F, A)^{op}$ is a soft category over $\mathcal{C}^{op}$.

Then we easily obtain the following result.

Theorem 3.12. $((F, A)^{op})^{op} = (F, A)$.

Definition 3.13. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *balanced soft category* over $\mathcal{C}$ if $F(x)$, as a category, is a balanced category (see [20]), for all $x \in A$.
We observe that, if $\mathcal{C}$ is balanced category, then (F, A) is balanced soft category.

Example 3.14. Let us consider the following category $\mathcal{C}$ with three objects, which are sets:
$Ob(\mathcal{C}) = \{I, J, K\}$, where $I = \{1\}$, $J = \{2\}$, $K = \{3\}$; and
$Hom[I, I] = \{i_I\}$, $Hom[J, J] = \{i_J\}$, $Hom[K, K] = \{i_K\}$,(here i denotes the identity map on the set); and
$Hom[I, J] = \{f_{12}\}$, $Hom[J, I] = \emptyset$, $Hom[I, K] = \{f_{13}\}$, $Hom[K, I] = \emptyset$, $Hom[J, K] = \{f_{23}\}$, $Hom[K, J] = \{f_{32}\}$, where f_{ij} denotes the mapping $f_{ij}(i) = j$. Then, clearly, $\mathcal{C}$ is a category. Moreover we see that, $\mathcal{C}$ is not balanced, as f_{12}, though being a monomorphism and epimorphism, has no inverse in $\mathcal{C}$, and hence not an isomorphism.
Now consider a soft category, (F, A) over $\mathcal{C}$, where $A = \{x, y\}$ and $F(x)$ is the full subcategory consisting objects J and K only and $F(y)$ is the full subcategory with the object I only. Then, clearly, $F(x)$ and $F(y)$ are balanced categories. Hence (F, A) is a balanced soft category over $\mathcal{C}$.

Definition 3.15. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with limits* over $\mathcal{C}$ if $F(x)$, as a category, has limits for all $x \in A$.
Here also we observe, if $\mathcal{C}$ is a category with limits, then (F, A) is a soft category with limits.
Similarly we can define *soft category with colimits*. We observe that, if (F, A) is soft category with limits, then $(F, A)^{op}$ is a soft category with colimits.

As equalizers [20], pullbacks [20], pushouts [20], intersections [20], unions [20], images [20], inverse images [20], products [20], coproducts [20], kernels [20], cokernels [20] are nothing but special types of limits or colimits, so soft category with equalizers, soft category with pullbacks etc are now defined to us.

Example 3.16. Let us consider the following category $\mathcal{C}$ with three objects, which are sets:
$Ob(\mathcal{C}) = \{I, J, K\}$, where $I = \{1\}$, $J = \{1, 2\}$, $K = \{2, 3\}$; and
$Hom[I, I] = \{i_I\}$, $Hom[J, J] = \{i_J\}$, $Hom[K, K] = \{i_K\}$, (here i denotes the identity map on the set); and
$Hom[I, J] = \{e\}$, $Hom[J, I] = \emptyset$, $Hom[J, K] = \{f, g, h\}$, $Hom[K, J] = \emptyset$, $Hom[I, K] = \{f \circ e, g \circ e, h \circ e\}$, $Hom[K, I] = \emptyset$, where $e(1) = 1$, $f(1) = 2$, $f(2) = 3$, $g(1) = 2$, $g(2) = 2$, $h(1) = 3$, $h(2) = 3$. Then, clearly, $\mathcal{C}$ is a category. Moreover we see that, the diagram of $\mathcal{C}$ consisting of objects J and K and arrows g and h, has no limit.
Now consider a soft category, (F, A) over $\mathcal{C}$, where $A = \{x, y\}$ and $F(x)$ is the lluf subcategory obtained by removing the arrows h and $h \circ e$ and $F(y)$ is the full subcategory with the object I only. Clearly, $F(x)$ and $F(y)$ both are categories with limits. Hence (F, A) is a soft category with limits over $\mathcal{C}$.

Definition 3.17. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with initial objects* over $\mathcal{C}$ if $F(x)$, as a category, has initial object [1], for all $x \in A$.
We observe that, if $\mathcal{C}$ is category with initial object, then (F, A) may not be a soft category with initial objects.

Definition 3.18. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with terminal objects* over $\mathcal{C}$ if $F(x)$, as a category, has terminal object [1], for all $x \in A$.
We observe that, if $\mathcal{C}$ is category with terminal object, then (F, A) may not be a soft category with terminal objects.

Example 3.19. Let us consider the following category $\mathcal{C}$ with three objects, which are sets:
$Ob(\mathcal{C}) = \{I, J, K\}$, where $I = \{1, 2\}$, $J = \{1, 2\}$, $K = \{2, 3\}$; and
$Hom[I, I] = \{i_I, e_I\}$, $Hom[J, J] = \{i_J, e_J\}$, $Hom[K, K] = \{i_K, e_K\}$, (here i denotes the identity map on the set); and
$Hom[I, J] = \{f_1, f_2\}$, $Hom[J, I] = \{f_1^{-1}, f_2^{-1}\}$, $Hom[J, K] = \{g_1, g_2\}$, $Hom[K, J] = \{g_1^{-1}, g_2^{-1}\}$, $Hom[I, K] = \{h_1, h_2\}$, $Hom[K, I] = \{h_1^{-1}, h_2^{-1}\}$, where f_i, g_i, h_i are the possible bijections between the sets and e_i are the bijections sending one element to the other element. Then, clearly, $\mathcal{C}$ is a category. Moreover we see that, this category has no initial or terminal objects.
Now consider a soft category (F, A) over $\mathcal{C}$, where $A = \{x, y\}$. Let $F(x)$ is the subcategory of C consisting of objects I and J and arrows i_I, i_J and f_1. Let $F(y)$ is the subcategory of C consisting of objects K and I and arrows i_I, i_K and h_1. Clearly, $F(x)$ and $F(y)$ are categories with initial and terminal objects. Hence (F, A) is a soft category with initial and terminal objects over $\mathcal{C}$.

We know from category theory that, every initial (resp., terminal) object becomes terminal (resp., initial) in its dual category. Hence we get the following result.

Theorem 3.20. *If (F, A) is a soft category with initial (resp., terminal) objects over $\mathcal{C}$, then $(F, A)^{op}$ is a soft category with terminal (resp., initial) objects over $\mathcal{C}$.*

Definition 3.21. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with zero objects* over $\mathcal{C}$ if $F(x)$, as a category, has zero object [20], for all $x \in A$.

Definition 3.22. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *normal soft category* over $\mathcal{C}$ if $F(x)$, as a category, is a normal category [20], for all $x \in A$.
Similarly, we can also define conormal soft category. We observe that, if (F, A) is normal soft category, then $(F, A)^{op}$ is a conormal soft category.

Example 3.23. Let us consider the following category $\mathcal{C}$ with three objects, which are subgroups of $\mathbb{Z}_4$:
$Ob(\mathcal{C}) = \{I, J, K\}$, where $I = \{\bar{0}\}$, $J = \{\bar{0}, \bar{2}\}$, $K = \mathbb{Z}_4$; and
$Hom[I, I] = \{i_I\}$, $Hom[J, J] = \{i_J\}$, $Hom[K, K] = \{i_K\}$,(here i denotes the identity morphism on the group); and
$Hom[I, J] = \{f\}$, $Hom[J, I] = \{e\}$, $Hom[J, K] = \{g\}$, $Hom[K, J] = \emptyset$,

$Hom[I, K] = \{h\}$, $Hom[K, I] = \emptyset$, where f, g and h are inclusion morphisms and e is the null morphism. Then, clearly, $\mathcal{C}$ is a category. But it is not normal, as for the monomorphism g there is no arrow such that g is kernel of that arrow.
Now consider a soft category, (F, A) over $\mathcal{C}$, where $A = \{x, y\}$ and $F(x)$ is the full subcategory consisting objects I and J only and $F(y)$ is the full subcategory with the object K only. Then, clearly, $F(x)$ and $F(y)$ are normal categories. Hence (F, A) is a normal soft category over $\mathcal{C}$. Moreover we see that this is also an example of soft category with zero objects.

Theorem 3.24. *If (F, A) is a normal soft category over $\mathcal{C}$, then (F, A) is a balanced soft category.*

Proof. We know from category theory that, every normal category is a balanced category (see [20]). Now $F(x)$, being a normal category, is a balanced category, for all $x \in A$. Hence the proof. □

Definition 3.25. Let (F, A) be a normal and conormal soft category with kernels and cokernels over $\mathcal{C}$. Then (F, A) is said to be an *exact soft category* over $\mathcal{C}$ if $F(x)$, as a category, is an exact category [20], for all $x \in A$.

Example 3.26. Let us consider the following category $\mathcal{C}$ with four objects, which are groups.
$Ob(\mathcal{C}) = \{I, J, K, L\}$, where $I = \mathbb{Z}_4$, $J = \mathbb{Z}_2$, $K = \{\bar{0}\}$ = the subgroup of $\mathbb{Z}_2$ and $L = \mathbb{Z}_3$; and
$Hom[I, I] = \{i_I\}$, $Hom[J, J] = \{i_J\}$, $Hom[K, K] = \{i_K\}$, $Hom[L, L] = \{i_L\}$,(here i denotes the identity morphism on the group); and
$Hom[I, J] = \{f\}$, $Hom[J, I] = \emptyset$, $Hom[J, K] = \emptyset$, $Hom[K, J] = \{h\}$, $Hom[I, K] = \{g\}$, $Hom[K, I] = \emptyset$, $Hom[I, L] = \{e\}$, $Hom[L, I] = \emptyset$, where f, g and e are null morphisms and h is the inclusion morphism. Then, clearly, $\mathcal{C}$ is a category. But it is not exact, as the morphism e does not split into composition of epimorphism and monomorphism.
Now consider a soft category, (F, A) over $\mathcal{C}$, where $A = \{x, y\}$ and $F(x)$ is the full subcategory consisting objects I, J and K only and $F(y)$ is the full subcategory with the object L only. Then, clearly, $F(x)$ and $F(y)$ both are exact categories. Hence (F, A) is an exact soft category over $\mathcal{C}$.

Theorem 3.27. *A normal and conormal soft category with cokernels and equalizers is an exact soft category.*

Proof. Let (F, A) is a normal and conormal soft category with cokernels and equalizers over C. Then for all $x \in A$, $F(x)$ is a normal and conormal category with cokernels and equalizers. Hence $F(x)$ is exact (see [20]). Thus we get the desired result. □

Theorem 3.28. *If (F, A) is an exact soft category over $\mathcal{C}$, then $(F, A)^{op}$ is an exact soft category over $\mathcal{C}^{op}$.*

Proof. As we know, dual of an exact category is again exact (see [20]), the result follows. □

Theorem 3.29. *If (F, A) is an exact soft category over $\mathcal{C}$, then it has finite unions.*

Proof. To prove this it is sufficient to state that, an exact category has finite unions [20]. □

We know in category theory that, a category has finite intersection and finite product if and only if it has equalizers and finite products. Moreover, as a terminal object is the product of empty family and pullbacks are nothing but a kind of finite product, the above two equivalent statements together imply that the category has pullbacks and terminal object. Hence the following proposition follows.

Theorem 3.30. *Let (F, A) be a soft category over $\mathcal{C}$. Then the following statements are equivalent.*
(1) (F, A) has finite intersections and finite products.
(2) (F, A) has equalizers and finite products.
The above statements imply
(3) (F, A) has pullbacks and terminal objects.

Definition 3.31. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be an *additive (resp., semiadditive) soft category* over $\mathcal{C}$ if $F(x)$, as a category, is an additive (resp., semiadditive) category (see [20]), for all $x \in A$.
We observe that, a soft category over an additive category may not be an additive soft category.

Theorem 3.32. *An additive soft category has kernels if and only if it has equalizers.*

Proof. As an additive category has kernels if and only if it has equalizers (see [20]), the result follows. □

Theorem 3.33. *An exact soft category with biproducts of the form $A \oplus B$ is additive soft category.*

Proof. Let (F, A) be an exact soft category with biproducts of the above form over $\mathcal{C}$. Then for all $x \in A$, $F(x)$ is exact category with biproducts of the above form. So, $F(x)$ is additive category (see [20]). Hence (F, A) is an additive soft category. □

Definition 3.34. An exact additive soft category with finite products is said to be an *abelian soft category.*

Theorem 3.35. *Let (F, A) be a soft category over $\mathcal{C}$. Then the following statements are equivalent.*
(1) (F, A) is an abelian soft category.
(2) (F, A) is a normal and conormal soft category with kernels, cokernels, finite products, finite coproducts.
(3) (F, A) is a normal and conormal soft category with pushouts and pullbacks.

Proof. We skip the proof, as it is easily derivable from the corresponding result of abelian category in the category theory (see [20]). □

Definition 3.36. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with exponentials* over $\mathcal{C}$ if $F(x)$, as a category, has exponentials [6], for all $x \in A$.

Definition 3.37. A soft category with terminal objects and products is said to be a *Cartesian soft category* or *CSC*. Moreover, if it has exponentials, then it is defined to be *Cartesian closed soft category* or *CCSC*.

Example 3.38. Consider the category *SET*. Let $A = \mathbb{N}$. Also let $F(n) = \{A \in Ob(SET) : A \text{ has cardinality } n\} \cup \{1\}$, for all $n \in N$. Hence (F, A) is a soft category over *SET*. This soft category is a CCSC.

Definition 3.39. Let (F, A) be a soft category over $\mathcal{C}$. Then (F, A) is said to be a *soft category with subobject classifiers* over $\mathcal{C}$ if $F(x)$, as a category, has a subobject classifier [6], for all $x \in A$.

Definition 3.40. A soft category with terminal objects, subobject classifiers, pullbacks and exponentials is defined to be a *soft topos.*

The following result follows from category theory (see [6]) as we shown previously.

Theorem 3.41. *A soft topos is always a CCSC.*

4. Soft functor

In this section soft functor will be defined with some other basic definitions. Also we present some results parallel to category theory.

Definition 4.1. Let (F, A) over $\mathcal{C}$ and (H, B) over $\mathcal{D}$ be two soft categories. Also suppose that $g : A \to B$ is a set mapping and $\mathcal{K} : \mathcal{C} \to \mathcal{D}$ is a functor [20]. Then $(\mathcal{K}, g)$ is said to be a *soft functor* from (F, A) to (H, B) if:
(1) $\mathcal{K}$ is full, i.e. image of $\mathcal{C}$ under $\mathcal{K}$ is all of $\mathcal{D}$,
(2) g is a surjection from A onto B, and
(3) $\mathcal{K}(F(x)) = H(g(x))$ for all $x \in A$.

Example 4.2. Let us consider two categories $\mathcal{C}$ and $\mathcal{D}$ defined as follows:
$Ob(\mathcal{C}) = \{I, J, K, L\}$, $Ob(\mathcal{D}) = \{I', J', K'\}$; and
$Hom[I, I] = \{i_I\}$, $Hom[J, J] = \{i_J\}$, $Hom[K, K] = \{i_K\}$, $Hom[L, L] = \{i_L\}$, $Hom[I', I'] = \{i_{I'}\}$, $Hom[J', J'] = \{i_{J'}\}$, $Hom[K', K'] = \{i_{K'}\}$, (here i denotes the identity morphism on the group); and
$Hom[I, J] = \{f\}$, $Hom[J, I] = \emptyset$, $Hom[J, K] = \{g\}$, $Hom[K, J] = \emptyset$, $Hom[I, K] = \{h\} = \{g \circ f\}$, $Hom[K, I] = \emptyset$, $Hom[I, L] = \{e\}$, $Hom[L, I] = \emptyset$; and
$Hom[I', J'] = \{f'\}$, $Hom[J', I'] = \emptyset$, $Hom[J', K'] = \{g'\}$, $Hom[K', J'] = \emptyset$, $Hom[I', K'] = \{h'\} = \{g' \circ f'\}$, $Hom[K', I'] = \emptyset$. Clearly, $\mathcal{C}$ and $\mathcal{D}$ are categories.
Now consider the functor $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{H}(I) = \mathcal{H}(L) = I'$, $\mathcal{H}(J) = J'$, $\mathcal{H}(K) = K'$; $\mathcal{H}(i_I) = \mathcal{H}(i_L) = \mathcal{H}(e) = i_{I'}$, $\mathcal{H}(i_J) = i_{J'}$, $\mathcal{H}(i_K) = K'$, $\mathcal{H}(f) = f'$, $\mathcal{H}(g) = g'$, $\mathcal{H}(h) = h'$.
Now consider two soft categories (F, A) over $\mathcal{C}$ and (G, B) over $\mathcal{D}$, where $A = \{x, y\}$, $B = \{z\}$. Let $F(x)$ is the full subcategory of $\mathcal{C}$ consisting the objects I, K and L and $F(y)$ is the full subcategory of $\mathcal{C}$ consisting the objects I and K and $G(z)$ is the full subcategory of $\mathcal{D}$ consisting the objects I' and K'. Let $m : A \to B$ where $m(x) = m(y) = z$. Then $\mathcal{H}(F(x)) = G(m(x))$ and $\mathcal{H}(F(y)) = G(m(y))$. Hence $(\mathcal{H}, m)$ is a soft functor from (F, A) to (G, B).

Definition 4.3. Let (F, A) over $\mathcal{C}$ and (H, B) over $\mathcal{D}$ be two soft categories and $(\mathcal{K}, g)$ a soft functor from (F, A) to (H, B). Then we say that $(\mathcal{K}, g)$ is a *soft functor with a certain property* P, if the functor $\mathcal{K}$ has property P when it is restricted to the subcategory $F(x)$ of $\mathcal{C}$ for all $x \in A$.
We observe that when a functor $\mathcal{K}$ has a property P, then the soft functor $(\mathcal{K}, g)$ also has that property.

Now, terms like additive soft functor, limit preserving soft functor, faithful soft functor etc are now defined to us.

Now we assert some results similar to category theory.

We know from category theory that, a functor from a category with products to an arbitrary category preserves limits if and only if it preserves products and finite intersections if and only if it preserves product and equalizers (see [20]). Hence the following result is clear from the Definition 4.3.

Theorem 4.4. *Let (F, A) over $\mathcal{C}$ and (H, B) over $\mathcal{D}$ be two soft categories and $(\mathcal{K}, g)$ a soft functor from (F, A) to (H, B). Also let (F, A) be with products. Then the following statements are equivalent.*
(1) $(\mathcal{K}, g)$ is limit preserving.
(2) $(\mathcal{K}, g)$ preserves products and finite intersections.
(3) $(\mathcal{K}, g)$ preserves products and equalizers.

Theorem 4.5. *Let (F, A) over $\mathcal{C}$ and (H, B) over $\mathcal{D}$ be two soft categories and $(\mathcal{K}, g)$ a soft functor from (F, A) to (H, B). Also let (F, A) be a normal soft category with products. Then $(\mathcal{K}, g)$ preserves limits if and only if it preserves products and kernels.*

Proof. By definition, for all $x \in A$, $F(x)$ is a normal category and $\mathcal{K}$,when restricted on $F(x)$, preserves limits. Then by category theory, it preserves products and kernels on $F(x)$ (see [20]). Other part of the proposition can be deduced similarly. □

We skip the proofs of the followings as they can be derived, similarly, using the corresponding results of category theory [20].

Theorem 4.6. *Let (F, A) over $\mathcal{C}$ and (H, B) over $\mathcal{D}$ be two additive soft categories and $(\mathcal{K}, g)$ a soft functor from (F, A) to (H, B). Also let (F, A) be a category with finite products. Then $(\mathcal{K}, g)$ preserves limits if and only if it preserves products and kernels.*

Theorem 4.7. *If (F, A) is an abelian soft category over $\mathcal{C}$ and (H, B) is an exact and additive soft category over $\mathcal{D}$, then a soft functor $(\mathcal{K}, g)$ from (F, A) to (H, B) preserves limits of finite diagrams if and only if it is kernel preserving.*

5. Soft category as a generalization

In this section, in the following remarks we show that soft set, soft group and soft ring are just special cases of soft category. We also show that soft homomorphisms of soft groups and soft rings are nothing but a type of soft functor. Again we compare soft category with fuzzy category.

Remark 5.1 (Soft set). Let (F, D) over U be a soft set. Then the set U can be considered as a category $\mathcal{C}$ by defining $Ob(\mathcal{C}) = U$, $Hom(A, A) = I_A$ and otherwise $Hom(A, B) = \emptyset$, for all $A, B \in Ob(\mathcal{C})$. As (F, D) is a soft set over U, $F(x)$ is a subset of U, for each $x \in D$. Now if we consider $F(x)$ with its identity arrows, it becomes a subcategory of the category C. Hence we find that (F, D) can be considered as a soft category over $\mathcal{C}$.

Remark 5.2 (Soft group). Let (F, D) be a soft group over G. Then the group G can be considered as a category $\mathcal{C}$ with one object, say A, where the arrows are the elements of the group and the composition of arrows are nothing but the binary operation of the group (see [20]). As (F, D) is a soft group over G, $F(x)$ is a subgroup of G, for each $x \in D$. Hence $F(x)$ can be considered as a subcategory of $\mathcal{C}$, as it, being a subgroup, is closed under binary composition and has the identity element. Clearly (F, D) can be considered as a soft category over $\mathcal{C}$.

Remark 5.3 (Soft ring). Let (F, D) be a soft ring over R with identity. Then the ring R can be considered as an additive category $\mathcal{C}$ with one object, say A, where the arrows are the elements of the ring and the composition of arrows are nothing but the multiplication operation of the ring and the addition of arrows are the addition of elements of the ring (see [20]). As (F, D) is a soft ring over R, $F(x)$ is a subring of R, for each $x \in D$. Hence $F(x)$ can be considered as an additive subcategory of $\mathcal{C}$, as it, being a subring, is closed under addition and multiplication and has the multiplicative identity. Clearly (F, D) can be considered as a soft category over $\mathcal{C}$.

Remark 5.4 (Soft homomorphism of soft groups). Let (f, g) be a soft homomorphism from soft group (F, A) over G to (H, B) over K. Then
(1) f is a homomorphism from G onto K,
(2) g is a mapping from A onto B, and
(3) $f(F(x)) = H(g(x))$ for all $x \in A$.
Now, considering a group as a category as in the Remark 5.2 and f being

a group homomorphism, f can be considered as a functor from category G to category K. So by (1),(2),(3), (f, g) become a soft functor from soft category (F, A) over G to (H, B) over K.

Remark 5.5 (Soft homomorphism of soft rings). Let (f, g) be a soft homomorphism from soft ring (F, A) over R to (H, B) over R', where both rings rings have multiplicative identity and f a ring homomorphism sending 1_R to $1_{R'}$. Then
(1) f is a ring epimorphism,
(2) g is surjective, and
(3) $f(F(x)) = H(g(x))$ for all $x \in A$.
Now, considering a ring as an additive category as in the Remark 5.3 and f being a ring homomorphism sending 1_R to $1_{R'}$, f can be considered as an additive soft functor from additive soft category (F, A) over R to (H, B) over R'.

Remark 5.6 (Fuzzy category). Let $\mathcal{FC}$ be a fuzzy category as defined in Definition 2.17. For each $\alpha \in [0, 1]$, we define $G(\alpha) = (Ob(G(\alpha))$, $Hom(G(\alpha))$, where

$$Ob(G(\alpha)) := \{X \in Ob(\mathcal{C}) : \omega(X) \geq \alpha\} \text{ and}$$
$$Hom(G(\alpha)) := \{f \in Hom(\mathcal{C}) : \mu(f) \geq \alpha\}.$$

Then each $G(\alpha)$ is a subcategory of the base category $\mathcal{C}$, since

$$\begin{aligned} X \in Ob(G(\alpha)) &\Rightarrow \omega(X) \geq \alpha \\ &\Rightarrow \mu(i_X) = \omega(X) \geq \alpha \\ &\Rightarrow i_X \in Hom(G(\alpha)) \quad \text{and} \end{aligned}$$

$$\begin{aligned} f, g \in Hom(G(\alpha)) &\Rightarrow \mu(f) \geq \alpha \text{and} \mu(g) \geq \alpha \\ &\Rightarrow \mu(g \circ f) \geq min\{\mu(g), \mu(f)\} \geq min\{\alpha, \alpha\} = \alpha \\ &\Rightarrow g \circ f \in Hom(G(\alpha)). \end{aligned}$$

Hence for any fuzzy category $\mathcal{FC}$, $(G, [0, 1])$ becomes a soft category over the category $\mathcal{C}$.

6. Conclusion

In this paper we have introduced the notion of soft category with a motivation to unify and simplify all the soft algebraic structures. Then we have defined different types of soft categories. We also have deduced, some elementary results in soft category theory. After that we have given the notion of soft functor and stated some propositions about soft functor. Some other soft notions are shown as soft category or soft functor. Now one may ask that: "What are the benefits of having soft category? Do we really need it?" The answers will certainly be positive. There are several benefits of having soft category theory. In comparison to the soft set theory, soft category is capable to parameterize not only the objects but also the morphisms, which is far more rich than the soft set theory. Again, as the soft category is a generalized version of fuzzy category, we may supplement fuzzy category theory by soft category theory. Also, as the days are passing by, more and more soft algebraic structures are being introduced. And we think that there should exist such a structure like soft category to observe all the soft structures from a generalized point of view. In the light of this paper one can try to "softify" more results of category theory and it may be possible to deduce some results of category theory itself from these results. Also, one can study properties of algebraic structures of soft category.

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References

[1] S. Abramsky and N. Tzevelekos, *Introduction to Categories and Categorical Logic*, Lec. Notes in Phy., **813** (2011), 3-94.

[2] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings*, Comput. Math. Appl., **59(11)** (2010), 3458–3463.

[3] H. Aktas and N. Cagman, *Soft sets and soft groups*, Inform. Sci., **177** (2007), 2726–2735.

[4] M. I. Ali, F. Feng, X. Liu and W. K. Min, *On some new operations in soft set theory*, Comput. Math. Appl., **57(9)** (2009), 1547–1553.

[5] M. I. Ali, M. Shabir and M. Naz, *Algebraic structures of soft sets associated with new operations*, Comput. Math. Appl., **61(9)** (2011), 3458–3463.

[6] A. Asperti and G. Longo, *Categories types and structures* USA : MIT Press (1991).

[7] A. O. Atagn and A. Sezgin, *Soft substructures of rings, fields and modules*, Comput. Math. Appl., **61(3)** (2011), 592–601.

[8] S. Awodey, *Category Theory*, New York : Oxford University Press Inc. (2010).

[9] Y. Celik, C. Ekiz, and S. Yamak, *A new view on soft rings*, Hacettepe Journal of Mathematics and Statistics, **40(2)** (2011), 273–286.

[10] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Comput. Math. Appl., **56** (2008), 2621–2628.

[11] K. Gong, Z. Xiao and X. Zhang, *The bijective soft set with its operations*, Comput. Math. Appl., **60(8)** (2010), 2270–2278.
[12] Y. B. Jun, *Soft BCK/BCI-algebra*, Comput. Math. Appl., **56** (2008), 1408–1413.
[13] Y. B. Jun, and C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci., **178** (2008), 2466–2475.
[14] O. Kazanci, S. Yilmaz and S. Yamak, *Soft set and soft BCH-algebras*, Hacettepe Journal of Mathematics and Statistics, **39(2)** (2010), 205–217.
[15] G. M. Kelly, *Basic concepts of basic category theory*, Cambridge : Cambridge University Press (1982).
[16] S. Mac Lane, *Categories for working mathematician*, New York : Springer-Verlag Inc. (1998).
[17] P. K. Maji, A. R. Roy and R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl., **44** (2002), 1077–1083.
[18] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. Appl., **45** (2003), 555–562.
[19] P. Majumdar and S. K. Samanta, *On soft mappings*, Comput. Math. Appl., **60(9)** (2010), 2666–2672.
[20] B. Mitchell, *Theory of categories*, New York : Academic press Inc. (1965).
[21] D. Molodtsov, *Soft set theory-first result*, Comput. Math. Appl., **37** (1999), 19–31.
[22] B. C. Pierce, *Basic category theory for computer scientists*, USA : MIT Press (1991).
[23] A. Sezgin, A. O. Atagn, and E. Aygn, *A note on soft near-rings and idealistic soft near-rings*, Filomat, **25(2)** (2011), 53–68.
[24] A. Sezgin and A. O. Atagn, *On operations of soft sets*, Comput. Math. Appl., **61(5)** (2011), 1457–1467.
[25] A. Sezgin and A. O. Atagn, *Soft groups and normalistic soft groups*, Comput. Math. Appl., **62** (2011), 685–698.
[26] A. P. Sostak, *Towards the concept of a fuzzy category*, Mathematics (Russian), Latv. Univ. Zinat. Rakstí, **562** (1991), 85–94.
[27] A. P. Sostak, *Fuzzy functions and an extension of the category L-TOP of Chang-Goguen L-Topological spaces*, Proceedings of the Ninth Prague Topological Symposium, Prague, Czech Republic (2001).
[28] Q. M. Sun, Z. L. Zhang and J. Liu, *Soft sets and soft modules*, Lecture Notes in Comput. Sci., **5009** (2008), 403–409.

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