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ON THE 2-ABSORBING IDEALS AND ZERO DIVISOR GRAPH OF EQUIVALENCE CLASSES OF ZERO DIVISORS

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Abstract. Let R be a commutative ring, I be a 2-absorbing ideal of R and let $I = Q_1 \cap \cdots \cap Q_n \;\; (n \geq 2)$ with $\sqrt{Q}_i = P_i$ for $i = 1, \cdots, n$, be a minimal primary decomposition of I. Let $\Gamma_E(R/I)$ denote the graph of equivalence classes of zero divisors of R/I. It is shown that $Q_1 \cap \cdots \cap Q_{n-1}, Q_1 \cap \cdots \cap Q_{n-2}, \cdots, Q_1, P_1, P_2 \cdots, P_n$ are all vertices of $\Gamma_E(R/I)$ and also the degrees of all vertices are determined.

Key Words: 2-absorbing ideal, zero divisor graph.
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1. Introduction

The concept of 2-absorbing ideals was introduced and investigated in [1]. A proper ideal I of a commutative ring R is called a 2-absorbing ideal if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. The reader is referred to [1, 3, 5] for more results and examples about 2-absorbing ideals. Let I be a 2-absorbing ideal of a commutative ring R and let x be an arbitrary element of R. The basic properties of the ideals $\mathrm{ann}_R(x + I)$ are studied in [3, 5]. It is shown that $\mathrm{ann}_R(x + I)$ is a prime or is a 2-absorbing ideal of R, and $\{\mathrm{ann}_R(x + I) \mid x \in R\}$ is a totally ordered set or is union of two totally ordered set.

The graph of equivalence classes of zero divisors of a ring R, which is constructed from classes of zero divisors determined by annihilator

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ideals, was introduced and investigated in [6, 8]. It will be denoted by $\Gamma_E(R)$. This graph has some advantages over zero divisor graph which introduced and studied in [2, 4]. In many cases zero divisor graph of equivalence classes of zero divisors in a commutative ring R is finite when the zero divisor graph is infinite. Another important aspect of zero divisor graph of equivalence classes of zero divisors is the connection to associated primes of R.

Let R be a commutative ring, I be a 2-absorbing decomposable ideal of R. Let $I = Q_1 \cap \cdots \cap Q_n$ with $\sqrt{Q_i} = P_i$ for $i = 1, \cdots, n$, be a minimal primary decomposition of I. In this article, we study the graph of equivalence classes of zero divisors of the ring R/I. For this reason, first in section 2 we study the associated prime ideals of I and then in section 3, we show that $Q_1 \cap \cdots \cap Q_{n-1}, Q_1 \cap \cdots \cap Q_{n-2}, \cdots, Q_1, P_1, P_2 \cdots, P_n$ are all vertices of the zero divisor graph of equivalence classes of zero divisors of R/I.

Throughout, R will denote a commutative ring with non-zero identity and I is an ideal of R. For notations and terminologies not given in this article, the reader is referred to [7].

2. Primary decomposition of 2-absorbing ideals

In this section we study 2-absorbing ideals which has primary decomposition. However, before going on to this study we should like to establish that 2-absorbing ideals with primary decomposition with at least two primary components do exist. Suppose that k is a field and $R = k[x, y]$ is the ring of polynomials over k in indeterminates x, y. Assume that $P = (x)$, $M = (x, y)$ and $I = (x^2, xy)$. It is easy to see that I is a 2-absorbing ideal of R, $I = P \cap M^2$ is a primary decomposition of I, $\sqrt{I} = P$ and $\mathrm{ass}(I) = \{M, P\}$.

In rest of this paper, we assume that I is a decomposable ideal of R, and $I = Q_1 \cap \cdots \cap Q_n$ with $\sqrt{Q_i} = P_i$ for $i = 1, \cdots, n$, is a minimal primary decomposition of I.

Theorem 2.1. *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P$ is a prime ideal of R. Then the following statements are true.*

(i) *$P = P_k$ for some k with $1 \leq k \leq n$.*
(ii) *$P_k = I :_R x$ for some $x \in R$.*
(iii) *There exists $x_i \in R$ such that $P_i = I :_R x_i$ for each $i = 1, \cdots, n$. Furthermore, either $P_i \subseteq P_j$ or $P_j \subseteq P_i$ for each $i, j = 1, \cdots, n$.*

Proof. (i) By assumption $P = \cap_{i=1}^{n} P_i$. Hence, $P = P_k$ for some k with $1 \leq k \leq n$ see Corollary 3.35 in [7].

(ii) First note that if $Q_k = P_k$, then $I :_R x = Q_k :_R x = Q_k$ for each $x \in \cap_{i=1, i\neq k}^{n} Q_i \setminus Q_k$ in view of [7, Lemma 4.14(iii)]. Now, suppose that $Q_k \subset P_k$. First of all, we show that $\cap_{i=1,i\neq k}^{n} Q_i \cap P_k \nsubseteq Q_k$. Assume that $a \in \cap_{i=1,i\neq k}^{n} Q_i \setminus Q_k$. If $a \in P_k$, then we have an element of the desired form. We therefore assume henceforth in this proof $a \notin P_k$. By assumption there exists $b \in P_k \setminus Q_k$. Now define $c = ab$ and note that $c \in \cap_{i=1,i\neq k}^{n} Q_i \cap P_k$ but $c \notin Q_k$. Suppose that $x \in \cap_{i=1,i\neq k}^{n} Q_i \cap P_k \setminus Q_k$. Thus $I :_R x = Q_k :_R x$ is a prime ideal of R containing $P = P_k$ by [3, Theorem 2.5]. On the other hand, in view of [7, Lemma 4.14(ii)] $Q_k :_R x$ is a P_k-primary ideal of R so that $P_k = I :_R x = Q_k :_R x$.

(iii) Assume that $1 \leq i, j \leq n$ and $i \neq k$ and $j \neq k$. There exists $x_i \in \cap_{s=1,s\neq i}^{n} Q_s \setminus Q_i$ by definition of primary decomposition. Thus $x_i \in P_k$ and with a similar argument to that of (ii) one can see that $I :_R x_i = Q_i :_R x_i = P_i$. In addition it is easy to see that $P_j = I :_R x_j$ for some $x_j \in R$. Now, we have $P_i \subseteq P_j$ or $P_j \subseteq P_i$ in view of [3, Theorem 2.5]. There is nothing to prove for $i = k$ or $j = k$, by (i) and (ii). □

Corollary 2.2. *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P$ is a prime ideal of R. Then $\operatorname{ass}(I) = \{ P_1, \cdots, P_n\}$ is a totally ordered set.*

Proof. This is immediate from Theorem 2.1. □

Remark 2.3. Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P$ is a prime ideal of R. In the rest of this paper, we suppose that $P_1, \cdots, P_n$ have been numbered (renumbered if necessary) such that $P = P_1$ and $P_1 \subset P_2 \subset \cdots \subset P_n$.

Theorem 2.4. *Let the situation be as in Remark 2.3. Then*

(i) $\cap_{i=j+1}^{n} Q_i \nsubseteq P_j$ *for each j with $1 \leq j \leq n-1$.*

(ii) *There exists $a_j \in R$ such that $\cap_{i=1}^{j} Q_i = I :_R a_j$ for each j with $1 \leq j \leq n-1$.*

(iii) *For each $x \in R$ either $I :_R x = P_j$ or $I :_R x = \cap_{i=1}^{j} Q_i$ for some j with $1 \leq j \leq n$.*

(iv) *There exist $a_1, \cdots, a_n$ and $x_1, \cdots, x_n$ in R such that*

$$\begin{aligned} I :_R a_n = \cap_{i=1}^{n} Q_i &\subset I :_R a_{n-1} = \cap_{i=1}^{n-1} Q_i \subset \cdots \subset I :_R a_1 = Q_1 \\ &\subseteq P_1 = I :_R x_1 \subset P_2 = I :_R x_2 \subset \cdots \subset P_n = I :_R x_n. \end{aligned}$$

Proof. (i) It is obvious by arrangements that we made in Remark 2.3.

(ii) Let $1 \leq j \leq n-1$ and $a_j \in \cap_{i=j+1}^{n} Q_i \setminus P_j$. Then $a_j \notin P_i$ for all $i = 1, \cdots, j$. Thus Lemma 4.14(ii) in [7] shows that

$$I :_R a_j = \cap_{i=1}^{n} Q_i :_R a_j = \cap_{i=1}^{j} (Q_i :_R a_j) = \cap_{i=1}^{j} Q_i.$$

(iii) Let $x \in P_1$. Then in view of [3, Theorem 2.5], $I :_R x$ is a prime ideal of R, furthermore $I :_R x$ is an associated prime ideal of I so that there is $1 \leq j \leq n$ such that $I :_R x = P_j$. Now, suppose that $x \in R \setminus P_1$ and suppose that x lies in all $P_{k+1}, \cdots, P_n$ but in none of $P_1, \cdots, P_k$. If $k = n$, then $I :_R x = \cap_{i=1}^{n} (Q_i :_R x) = \cap_{i=1}^{n} Q_i = I$, see Lemma 4.14 in [7]. We therefore assume henceforth in this proof that $k < n$. In this case, there exists $t \in \mathbb{N}$ such that $x^t \in \cap_{i=k+1}^{n} Q_i$. Thus $I :_R x^t = \cap_{i=1}^{n} (Q_i :_R x^t) = \cap_{i=1}^{k} (Q_i :_R x^t) = \cap_{i=1}^{k} Q_i$. To complete the proof, it is enough to show that $I :_R x = I :_R x^t$. It is obvious that $I :_R x \subseteq I :_R x^t$. Assume that $a \in I :_R x^t$. Thus $ax^t \in I$ which implies that $ax \in I$ or $x^2 \in I$ since I is a 2-absorbing ideal. If $ax \in I$ we are done. Otherwise, $x^2 \in I$ which shows that $x \in P_1$ and this is a contradiction.

(iv) It is obvious by (ii) and Theorem 2.1(iii). □

Theorem 2.5. *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P \cap P'$, where P, P' are the only distinct prime ideals of R that are minimal over I. Then the following statements are true:*

(i) *$P = P_k$ and $P' = P_s$ for some k, s with $1 \leq k, s \leq n$ and $k \neq s$.*
(ii) *$Q_k = P_k$ and $Q_k = I :_R a$ for each $a \in \cap_{i=1, i \neq k}^{n} Q_i \setminus Q_k$.*
(iii) *$Q_s = P_s$ and $Q_s = I :_R a$ for each $a \in \cap_{i=1, i \neq s}^{n} Q_i \setminus Q_s$.*
(iv) *If $I \neq \sqrt{I}$, then any primary decomposition of I has at least three components.*
(v) *There exists $x_i \in R$ such that $P_i = I :_R x_i$ for each $i = 1, \cdots, n$. Furthermore, either $P_i \subseteq P_j$ or $P_j \subseteq P_i$ for each $i, j = 1, \cdots, n$ with $i \neq k, s$ and $j \neq k, s$.*

Proof. (i) By assumptions $P \cap P' = \cap_{i=1}^{n} P_i$ thus there is $1 \leq k \leq n$ such that $P = P_k$ since P is a minimal prime ideal of I. By a same argument there is $1 \leq s \leq n$ with $k \neq s$ such that $P' = P_s$.

(ii) Assume that $a \in \cap_{i=1, i \neq k}^{n} Q_i \setminus Q_k$. Then $I :_R a = Q_k :_R a$. If $a \in P_k$, then $a \in \sqrt{I}$ and so by [3, Theorem 2.6], $I :_R a$ is a prime ideal of R containing $P = P_k$ and $P' = P_s$. On the other hand, $Q_k :_R a$ is a P_k-primary ideal of R so that $P_k = I :_R a = Q_k :_R a$ thus $P' = P_s \subseteq P = P_k$ which is a contradiction. Hence, $a \notin P_k$ and so $a \notin \sqrt{I}$. Now, in view of [5, Theorem 2.1(iii)] and [7, Lemma 4.14(iii)], it follows that $P_k = I :_R a = Q_k :_R a = Q_k$.

(iii) The proof is similar to that of (ii).

(iv) Assume that I has a primary decomposition with only two components. Then by (ii) and (iii) it follows that $I = \sqrt{I}$ which is a contradiction.

(v) Assume that $1 \leq i, j \leq n$ are such that $i \neq k, s$ and $j \neq k, s$ and assume that $a \in \cap_{t=1, t\neq i}^{n} Q_t \setminus Q_i$. Thus $I :_R a = Q_i :_R a$ also $i \neq k, s$ implies that $a \in P_k \cap P_s$ which means that $I :_R a = Q_i :_R a$ is a prime ideal of R containing P_k and P_s, see [3, Theorem 2.6]. On the other hand, $Q_i :_R a$ is a P_i-primary ideal of R so that $P_i = I :_R a = Q_i :_R a$. With a similar argument one can show that $I :_R a = Q_j :_R a = P_j$ for $a \in \cap_{t=1, t\neq j}^{n} Q_t \setminus Q_j$. Hence, using [3, Theorem 2.6] again shows that $P_i \subseteq P_j$ or $P_j \subseteq P_i$ that is claimed. □

Corollary 2.6. *Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P \cap P'$, where P, P' are the only distinct prime ideals of R that are minimal over I. Then $\mathrm{ass}(I)$ is union of two totally ordered sets such as $\mathrm{ass}(I) = \{P_k\} \cup \{P_1, \cdots, P_{k-1}, P_{k+1}, \cdots, P_n\}$ or $\mathrm{ass}(I) = \{P_s\} \cup \{P_1, \cdots, P_{s-1}, P_{s+1}, \cdots, P_n\}$.*

Proof. This is immediate from Theorem 2.5. □

Remark 2.7. Let I be a 2-absorbing ideal of R such that $\sqrt{I} = P \cap P'$, where P, P' are the only distinct prime ideals of R that are minimal over I. In the rest of this paper, we suppose that $P_1, \cdots, P_n$ have been numbered (renumbered if necessary) such that $P = P_1$, $P' = P_2$ and $P_1 \subset P_3 \subset \cdots \subset P_n$, $P_2 \subset P_3 \subset \cdots \subset P_n$.

Theorem 2.8. *Let the situation be as in Remark 2.7. Then the following statements are true.*

(i) *$\cap_{i=j+1}^{n} Q_i \not\subseteq P_j$ for each j with $1 \leq j \leq n-1$ also $\cap_{i=3}^{n} Q_i \not\subseteq P_1 \cup P_2$.*

(ii) *There exists $a_j \in R$ such that $\cap_{i=1}^{j} Q_i = I :_R a_j$ for each j with $1 \leq j \leq n-1$.*

(iii) *For each $x \in R$ either $I :_R x = P_j$ or $I :_R x = \cap_{i=1}^{j} Q_i$ for some j with $1 \leq j \leq n$.*

(iv) *There exist $a_2, \cdots, a_n$ and $x_1, \cdots, x_n$ in R such that*

$$\begin{aligned} I :_R a_n = \cap_{i=1}^{n} Q_i &\subset I :_R a_{n-1} = \cap_{i=1}^{n-1} Q_i \subset \cdots \subset I :_R a_2 = Q_1 \cap Q_2 \\ &\subset P_1 = I :_R x_1 \subset P_3 = I :_R x_3 \subset \cdots \subset P_n = I :_R x_n. \end{aligned}$$

Proof. (i) It is obvious by arrangements that we made in Remark 2.7.

(ii) For $j = 1$ it is immediate by Theorem 2.5 (ii). Assume that $a_2 \in \cap_{i=3}^{n} Q_i \setminus P_1 \cup P_2$. Thus $I :_R a_2 = \cap_{i=1}^{n} Q_i :_R a_2 = \cap_{i=1}^{2}(Q_i :_R a_j) = \cap_{i=1}^{2} Q_i$. Now, assume that $a_j \in \cap_{i=j+1}^{n} Q_i \setminus P_j$ and $3 \leq j \leq n-1$. In this case $a_j \notin P_i$ for all $i = 1, \cdots, j$. Thus Lemma 4.14(ii) in [7] shows that

$$I :_R a_j = \cap_{i=1}^{n} Q_i :_R a_j = \cap_{i=1}^{j}(Q_i :_R a_j) = \cap_{i=1}^{j} Q_i.$$

(iii) If $x \in P_1 \cap P_2$, then $I :_R x$ is a prime ideal of R by Theorem 2.6 in [3] also $I :_R x$ is an associated prime ideal of I so that $I :_R x = P_j$ for some $1 \leq j \leq n$. Now, suppose that $x \in P_1 \setminus P_2$. Then Theorem 2.1(iii) in [5] shows that $I :_R x = P_2$ also $I :_R x = P_1$ for $x \in P_2 \setminus P_1$. By a similar argument to that of Theorem 2.4(iii) we reach the desired conclusion for each $x \notin P_1 \cup P_2$.

(iv) It is obvious by (ii). □

3. Zero divisor graph of equivalence classes of zero divisors

Recall that R is a commutative ring. The following are some basic facts about zero divisor graph of equivalence classes of zero divisors in a commutative ring R. Let $Z^*(R)$ denote the zero divisors of R and $Z(R) = Z^*(R) \cup \{0\}$. For $x, y \in Z^*(R)$ we say that $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$. As noted in [8], $\sim$ is an equivalence relation. Furthermore, if $x_1 \sim x_2$ and $x_1 y = 0$, then $y \in \mathrm{ann}(x_1) = \mathrm{ann}(x_2)$ and hence, $x_2 y = 0$. It follows that multiplication is well defined on the equivalence classes of $\sim$; that is if $[x]$ denotes the class of x, then the product $[x][y] = [xy]$ makes sense. Note that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$; the other equivalence classes form a partition of $Z^*(R)$.

Definition 3.1. The graph of equivalence classes of zero divisors of a ring R, denoted $\Gamma_E(R)$, is the graph associated to R whose vertices are the classes of elements in $Z^*(R)$, and with each pair of distinct classes $[x], [y]$ joined by an edge if and only if $[x][y] = [0]$.

Lemma 3.2. [8, Lemma 1.2] *Let R be a commutative Noetherian ring. Then any two distinct elements of* $\mathrm{Ass}(R)$ *are connected by an edge. Furthermore, every vertex $[v]$ of $\Gamma_E(R)$ is either an associated prime or adjacent to an associated prime maximal in* $\{\mathrm{ann}(x) \ : \ 0 \neq x \in R\}$.

The degree of a vertex v in a graph, denoted $\deg v$, is the number of edges incident to v. By a graph we mean that a simple graph in the sense that there are no loops or double edges.

Proposition 3.3. [8, Proposition 3.4] *Let R be a commutative Noetherian ring. Let $x_1, \cdots, x_r$ be elements of R, with $r \geq 2$, and suppose* $\mathrm{ann}(x_1) \subset \cdots \subset \mathrm{ann}(x_r)$ *is a chain in* $\mathrm{Ass}(R)$. *If* $3 \leq |\Gamma_E(R)| < \infty$, *then* $\deg[x_1] < \cdots < \deg[x_r]$.

Recall that I is a decomposable ideal of R, and $I = Q_1 \cap \cdots \cap Q_n$ with $\sqrt{Q_i} = P_i$ for $i = 1, \cdots, n$ is a minimal primary decomposition of I.

Corollary 3.4. *If I is a 2-absorbing ideal of R, then the vertices set of $\Gamma_E(R/I)$ has at most $2n-1$ elements. Moreover, $Q_1 \cap \cdots \cap Q_{n-1}, Q_1 \cap \cdots \cap Q_{n-2}, \cdots, Q_1, P_1, \cdots, P_n$ are all vertices of $\Gamma_E(R/I)$.*

Proof. It is clear by Theorems 2.4 and 2.8. $\square$

We will slightly abuse terminology and refer to $[a_i + I]$ as $Q_1 \cap \cdots \cap Q_i$ for all i, with $1 \leq i \leq n-1$ and refer to $[x_i + I]$ as P_i for all i, with $1 \leq i \leq n$, where a_i and x_i are elements of R such as chosen in the proofs of Theorems 2.1,2.4, 2.5 and 2.8.

Theorem 3.5. *Let the situations be as in Remark 2.3, $Q_1 \neq P_1$ and $n \geq 2$. Then the following statements are true.*

(i) $[a_i + I][x_1 + I] \neq [0]$ *for all* $i = 1, \cdots, n-1$, *so that* $\deg[x_1 + I] = n-1$.

(ii) $[a_{n-1} + I][x_i + I] = [0]$ *if and only if* $i = n$ *so that* $\deg[x_n + I] = 2n-2$.

(iii) $\deg[x_i + I] = n+i-2$ *for all* $i = 1, \cdots, n$, *and* $\deg[a_i + I] = n-i$ *for all* $i = 1, \cdots, n-1$.

Proof. (i) It is enough to show that $[a_1 + I][x_1 + I] \neq 0$ by Theorem 2.4(iv). In view of Theorems 2.1 and 2.4 there are $x_1 \in \cap_{i=2}^n Q_i \setminus Q_1$ and $a_1 \in \cap_{i=2}^n Q_i \setminus P_1$ such that $Q_1 = \mathrm{ann}(a_1 + I)$ and $P_1 = \mathrm{ann}(x_1 + I)$. If $a_1 x_1 + I = 0$, then $a_1 x_1 \in I$ and so $a_1 x_1 \in Q_1$ which is a contradiction since neither $x_1 \in Q_1$ nor $a_1 \in P_1$. Thus $a_1 x_1 + I \neq 0$ therefore $[a_i + I][x_1 + I] \neq [0]$ for all $i = 1, \cdots, n-1$. The last assertion follows by Lemma 3.2.

(ii) ($\Rightarrow$) The vertex $[a_{n-1} + I]$ is adjacent to $[x_n + I]$ in view of Lemma 3.2 so that $[a_{n-1} + I][x_n + I] = [0]$.

($\Leftarrow$) Assume that $[a_{n-1} + I][x_i + I] = [0]$ for some i with $1 \leq i \leq n$. Then $a_j x_i + I = 0$ and also $x_j x_i + I = 0$ for all $j \geq 1$, see Theorem 2.4(iv). Thus $\deg[x_i + I] \geq 2n-2$. Now, Proposition 3.3 implies that $i = n$.

(iii) We have $\deg[x_1 + I] = n - 1$ and $\deg[x_n + I] = 2n - 2$ by (i) and (ii). Thus Proposition 3.3 shows that $\deg[x_i + I] = n + i - 2$. Using (i) again the last assertion follows. □

Corollary 3.6. *If $Q_1 = P_1$, then any primary decomposition of I has at most two components, so that $n = 2$.*

Proof. We have $\deg[x_1 + I] = n - 1$ and $\deg[x_n + I] = 2n - 3$ in view of Theorem 3.5(i) and Lemma 3.2. On the other hand, if $n \geq 3$, then Proposition 3.3 shows that $\deg[x_n + I] \geq 2n - 2$ which is a contradiction. So that $n \leq 2$ and any primary decomposition of I has at most two components. □

Theorem 3.7. *Let the situations be as in Remark 2.7. Then the following statements are true.*

(i) $[a_i + I][x_1 + I] \neq [0]$ *and* $[a_i + I][x_2 + I] \neq [0]$ *for each* $i = 2, \cdots, n - 1$, *so that* $\deg[x_1 + I] = \deg[x_2 + I] = n - 1$

(ii) $[a_{n-1} + I][x_i + I] = [0]$ *if and only if* $i = n$. *So that* $\deg[x_n + I] = 2n - 3$.

(iii) $\deg[x_i + I] = n + i - 3$ *for each* $i = 3, \cdots, n$, *so that* $\deg[a_i + I] = n - i$ *for all* $i = 2, \cdots, n - 1$.

Proof. (i) It is enough to show that $[a_2 + I][x_1 + I] \neq 0$, see Theorem 2.8(iv). In view of Theorems 2.5 and 2.8 there are $x_1 \in \cap_{i=2}^{n} Q_i \setminus Q_1$ and $a_2 \in \cap_{i=2}^{n} Q_i \setminus P_1 \cup P_2$ such that $Q_1 \cap Q_2 = \mathrm{ann}(a_2 + I)$ and $P_1 = \mathrm{ann}(x_1 + I)$. If $a_2 x_1 + I = 0$, then $a_2 x_1 \in I$ and so $a_2 x_1 \in Q_1$ which is a contradiction since neither $x_1 \in Q_1$ nor $a_2 \in P_1$. Thus $a_2 x_1 + I \neq 0$ therefore $[a_i + I][x_1 + I] \neq [0]$ for all $i = 2, \cdots, n - 1$. By a similar argument one can show that $[a_i + I][x_2 + I] \neq [0]$ for all $i = 2, \cdots, n - 1$. The last assertion follows by Lemma 3.2.

(ii) It is similar to that of (ii) in Theorem 3.3

(iii) We have $\deg[x_1 + I] = \deg[x_2 + I] = n - 1$ and $\deg[x_n + I] = 2n - 3$ by (i) and (ii). Now, Proposition 3.3 implies that $\deg[x_i + I] = n + i - 3$, for all $i = 3, \cdots, n$. Using (i) again the last assertion follows. □

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