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CHARACTERIZATIONS OF FUZZY M-Γ-HEMIRINGS

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Abstract. In this paper, the concepts of M-fuzzy left h-ideals (right h-ideals, h-bi-ideals, h-quasi-ideals) in M-Γ-hemirings are introduced. Some new properties of these kinds of M-fuzzy h-ideals are also given. Finally, we show that h-hemiregular, h-intra-hemiregular and h-quasi-hemiregular M-Γ-hemirings can be described by M-fuzzy left(right) h-ideals, M-fuzzy h-bi-ideals and M-fuzzy h-quasi-ideals, respectively.

Key Words: M-Γ-hemiring, M-fuzzy (h-, h-bi-, h-quasi-) ideal, (h-hemiregular, h-intra-hemiregular, h-quasi-hemiregular) M-Γ-hemiring.
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1. Introduction

Semirings play an important role in studing matrices and determinants, have been recently found particular useage in solving problems in applied mathmatics and information sciences. We note that the ideals of semirings also play a crucial role in the structure theory. Although ideals in semirings are useful for ways, they do not in general coincide with the ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. In 2004, Jun [9] defined the fuzzy h-ideals of hemirings. Then, the h-hemiregular hemirings were described by Zhan by using the fuzzy h-ideals [27]. Furthermore, Yin introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings in [22]. After that, Ma [12] introduced the concepts of $(\in, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties.

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Recently, Ma [10] introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-)ideals of hemirings. In particular, some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings were investigated by these kinds of fuzzy h-ideals. The general properties of fuzzy h-ideals have been considered by Dudek, Jun, Ma, Zhan, and others. The reader is refereed to [2, 3, 7, 11, 14, 26, 21].

The concept of Γ-rings was first introduced in 1964 by Barnes [1], a concept more general than a ring. After the paper of Barnes, many researchers are engaged in studying some special Γ-rings. In 1992, applying the concept of fuzzy sets to the theory of Γ-ring, Jun and Lee gave the notion of fuzzy ideals in Γ-ring and some properties of fuzzy ideals of Γ-ring. After that, Hong and Jun defined the normalized fuzzy ideals and fuzzy maximal ideals in a Γ-ring and Jun further characterized the fuzzy prime ideals of a Γ-ring. In particular, Dutta-Chanda studied the fuzzy ideals of a Γ-ring and characterized the Γ-fields and Notherian Γ-ring by considering the fuzzy ideals via operator rings of Γ-ring. The concept of Γ-semirings was then introduced by Rao in [19] and some properties of such Γ-semirings have been studied, for example, see [5, 6, 20]. Recently, Ma and Zhan [13, 27] introduced the concept of h-hemiregular of a Γ-hemiring and gave a characterization of h-hemiregular Γ-hemirings in terms of fuzzy h-ideals. In 2007, Zhan and Davvaz[25] gave the fuzzy h-ideals with operators in hemirings and some properties are investigated. Then applying Zhan's idea, Pan[17] gave the concept of M-Γ-hemiring, and established a new fuzzy left h-ideal with operators.

The present paper is organized as follows. In Section 2, we recall some concepts and properties of M-Γ-hemirings and fuzzy sets. In Section 3, we introduce the concept of M-fuzzy h-ideals (h-bi-ideals, h-quasi-ideals) of M-Γ-hemirings and gave some related properties. In Section 4, we describe the characterizations of h-hemiregular (h-intra-hemiregular, h-quasi-hemiregular) M-Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals. Some conclusions and future work are presented in the last section of the paper.

2. Preliminaries

Let S and Γ be two commutative additive semigroups. Then S is said to be a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ (images are denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$;
(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$;
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a zero of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$, for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with a zero is said to be a Γ-hemiring[27].

Throughout this paper, S is a Γ-hemiring and we use the symbol 0_S to denote the zero element of S.

A left (resp., right) ideal of a Γ-hemiring S is a subset A of S which is closed under addition such that $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$), where $S\Gamma A = \{x\alpha y \mid x \in S, y \in A, \alpha \in \Gamma\}$. Naturally, a subset A of S is called an ideal of S if it is both a left and a right ideal of S. A subset A of S is called a bi-ideal if A is closed under addition such that $A\Gamma A \subseteq A$ and $A\Gamma S\Gamma A \subseteq A$. A subset A of S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.

A left ideal (right ideal, ideal) A of S is called a left h-ideal (right h-ideal, h-ideal) of S, respectively, if, for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z$ implies that $x \in A$.

The h-closure $\overline{A}$ of A in S is defined by $\overline{A} = \{x \in S \mid x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A, z \in S\}$.

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$, for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

A bi-ideal B of S is said to be an h-bi-ideal of S if $\overline{B\Gamma B} \subseteq B$, $\overline{B\Gamma S\Gamma B} \subseteq B$ and $x + a + z = b + z$ implies that $x \in A$, for all $x, z \in S, a, b \in A$.

A quasi-ideal A of S is called an h-quasi-ideal of S if $\overline{S\Gamma A} \cap \overline{A\Gamma S} \subseteq A$ and $x + a + z = b + z$ implies that $x \in A$, for all $x, z \in S, a, b \in A$.

Definition 2.1. [13, 27] (i) Let μ and ν be fuzzy subsets of S. Then the h-product of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigvee_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + a_1\gamma_1 b_1 + z = a_2\gamma_2 b_2 + z$.

(ii) Let μ and ν be fuzzy subsets of S. Then the h-intra-product of μ and ν is defined by

$$(\mu\widetilde{\Gamma}_h\nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i\gamma_i b_i+z=\sum\limits_{j=1}^{n} a'_j\gamma'_j b'_j+z} \min\{\mu(a_i),\mu(a'_j),\nu(b_i),\nu(b'_j)\}$$

$(\mu\widetilde{\Gamma}_h\nu)(x) = 0$ if x cannot be expressed as $x+\sum\limits_{i=1}^{m} a_i\gamma_i b_i+z = \sum\limits_{j=1}^{n} a'_j\gamma'_j b'_j+z$.

A fuzzy set is a function μ: $S \to [0,1]$. For any $A \subseteq S$, we denote the characteristic function of A by χ_A

$$\chi_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Proposition 2.2. [13, 27] *Let $A, B \subseteq S$. Then, the following statements holds:*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \chi_B$.
(2) $\chi_A \cap \chi_B = \chi_{A\cap B}$.
(3) $\chi_A\Gamma_h\chi_B = \chi_{\overline{A\Gamma B}}$.
(4) $\chi_A\widetilde{\Gamma}_h\chi_B = \chi_{\overline{A\Gamma B}}$.

3. M-fuzzy h-ideals in M-Γ-hemirings

In this section, we consider M-fuzzy h-bi-ideals of M-Γ-hemirings.

3.1. M-fuzzy left h-ideals.

Definition 3.1.1. An h-bi-ideal I of an M-Γ-hemiring S is called an M-h-bi-ideal of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 3.1.2. A fuzzy set μ over M-Γ-hemiring S is called an M-fuzzy h-bi-ideal over S if it satisfies the following conditions (F1a),(F1c),(F1d) and:

(F2a) $\mu(x\alpha y) \geq \min\{\mu(x),\mu(y)\}$ for all $x, y \in S, \alpha \in \Gamma$,
(F2b) $\mu(x\alpha y\beta z) \geq \min\{\mu(x),\mu(z)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Note that if μ is an M-fuzzy h-bi-ideal of S, then $\mu(0) \geq \mu(x), \forall x \in S$.

Example 3.1.3. Let $(S,+)$ and $(\Gamma,+)$ be two semigroups, where S and Γ are the sets of all non-negative integers and the operations are the usual additive operations. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = a\cdot\gamma\cdot b$, for all $a, b \in S$ and $\gamma \in \Gamma$, where "$\cdot$" is the usual multiplication. Then it can be easily verified that S, under the above multiplication and the structure Γ-mapping, is a Γ-hemiring, then let $M = \{1\}$, it is clear that S is an M-Γ-hemiring. Let $r, s \in [0, 1)$ be such that $r \leq s$. Define a fuzzy set μ over S by

$$\mu(x) = \begin{cases} s & \text{if } x \in < 2 >, \\ r & \textit{otherwise}, \end{cases}$$

Then μ is an M-fuzzy h-bi-ideal over S.

Lemma 3.1.4. A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy h-bi-ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an M-h-bi-ideal of S.

Proof. It is similar to the proof of Theorem 3.5 in [9].

Theorem 3.1.5. A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy h-bi-ideal of S if and only if the each nonempty level subset $U(\mu; t)$ of μ is a left M-h-bi-ideal of S.

Proof. Let μ be an M-fuzzy h-bi-ideal of S, and assume that $U(\mu; t) \neq \emptyset$ for $t \in [0, 1]$. Then by Lemma 3.2.4, $U(\mu; t)$ is a h-bi-ideal of S. For every $x \in U(\mu; t)$, $\alpha \in \Gamma$, and $m \in M$, we have

$$\mu(m\alpha x) \geqslant \mu(x) \geqslant t,$$

and so $m\alpha x \in U(\mu; t)$. Hence $U(\mu; t)$ is an M-h-bi-ideal of S. Conversely, suppose that $U(\mu; t) \neq \emptyset$ is an M-h-bi-ideal of S. Then μ is a fuzzy h-bi-ideal of S by Lemma 3.2.4. Now assume that there exist $y \in S$, $\beta \in \Gamma$ and $n \in M$ such that

$$\mu(n\beta y) < \mu(y).$$

Taking

$$t_0 = \frac{1}{2}(\mu(n\beta y) + \mu(y)),$$

we get $t_0 \in [0, 1]$ and

$$\mu(n\beta y) < t_0 < \mu(y)$$

This implies that $n\beta y \notin U(\mu; t_0)$ and $y \in U(\mu; t_0)$, this leads a contradiction. And therefor

$$\mu(n\beta y) \geq \mu(y),$$

for all $y \in S$, $\beta \in \Gamma$ and $n \in M$. This completes the proof.

3.2. M**-fuzzy** h**-quasi-ideals.**

Definition 3.2.1. An h-quasi-ideal I of an M-Γ-hemiring S is called a left M-h-quasi-ideal of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 3.2.2. A fuzzy set μ over M-Γ-hemiring S is called an M-fuzzy h-quasi-ideal over S if it satisfies the following conditions (F1a),(F1c), (F1d) and:

(F3a) $(\mu\Gamma_h^M \chi_M) \cap (\chi_M \Gamma_h^M \mu) \subseteq \mu$.

Note that if μ is an M-fuzzy h-quasi-ideal of S, then $\mu(0) \geq \mu(x), \forall x \in S$.

Example 3.2.3. Consider the M-Γ-hemiring S as given in Example 3.2.3. Define a fuzzy set μ over S by

$$\mu(x) = \begin{cases} 0.5 & \textit{if } x \in <2>, \\ 0.1 & \textit{otherwise}, \end{cases}$$

Then μ is an M-fuzzy h-quasi-ideal over S.

The following proposition is obvious.

Lemma 3.2.4. A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy h-quasi-ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an M-h-quasi-ideal of S.

Theorem 3.2.5. Let μ and ν be M-fuzzy right h-ideal and M-fuzzy left h-ideal of M-Γ-hemiring S, respectively. Then $\mu \cap \nu$ is an M fuzzy h-quasi-ideal of S, where $\mu \cap \nu$ is defined by

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad x \in S.$$

Proof. For $a, b \in S$,

$$\begin{aligned} (\mu \cap \nu)(a+b) &= \min\{\mu(a+b), \nu(a+b)\} \\ &\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\} \\ &= \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\} \\ &= \min\{(\mu \cap \nu)(a), (\mu \cap \nu)(b)\}. \end{aligned}$$

Now, let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$\begin{aligned} (\mu \cap \nu)(x) &= \min\{\mu(x), \nu(x)\} \\ &\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\} \\ &= \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\} \\ &= \min\{(\mu \cap \nu)(a), (\mu \cap \nu)(b)\}. \end{aligned}$$

On the other hand, we have

$$((\mu\cap\nu)\Gamma_h\chi_M)\cap(\chi_M\Gamma_h^M(\mu\cap\nu))\subseteq(\mu\Gamma_h^M\chi_M)\cap(\chi_M\Gamma_h^M\nu)\subseteq\mu\cap\nu.$$

Therefore $\mu\cap\nu$ is a fuzzy h-quasi-ideal of S. Let $m\in M$, $\alpha\in\Gamma$ we have

$$\begin{aligned}(\mu\cap\nu)(m\alpha x) &= \min\{\mu(m\alpha x),\nu(m\alpha x)\}\\ &\geq \min\{\mu(x),\nu(x)\}\\ &= (\mu\cap\nu)(x).\end{aligned}$$

Consequently, $\mu\cap\nu$ is an M fuzzy h-quasi-ideal of S.

Theorem 3.2.6. Any M-fuzzy h-quasi-ideal of an M-Γ-hemiring S is an M-fuzzy h-bi-ideal of S.

Proof. Let μ be any M-fuzzy h-quasi-ideal of S. It is sufficient to show that

$$\mu(x\alpha y\beta z)\geq\min\{\mu(x),\mu(z)\}$$

and

$$\mu(x\alpha y)\geq\min\{\mu(x),\mu(y)\}$$

for all $x,y,z\in S,\alpha,\beta\in\Gamma$.

In fact, we have

$$\begin{aligned}\mu(x\alpha y\beta z) &\geq ((\mu\Gamma_h^M\chi_M)\cap(\chi_M\Gamma_h^M\mu))(x\alpha y\beta z)\\ &= \min\{(\mu\Gamma_h^M\chi_M)(x\alpha y\beta z),(\chi_M\Gamma_h^M\mu)(x\alpha y\beta z)\}\\ &= \min\{\bigvee_{x\alpha y\beta z+m_1\gamma_1a_1+z=m_2\gamma_2a_2+z}\mu(m_1)\wedge\mu(m_2),\\ &\qquad \bigvee_{x\alpha y\beta z+m_1\gamma_1a_1+z=m_2\gamma_2a_2+z}\mu(a_1)\wedge\mu(a_2)\}\\ &\geq \min\{\mu(0)\wedge\mu(x),\mu(0)\wedge\mu(z)\}\\ &= \min\{\mu(x),\mu(z)\}.\end{aligned}$$

(since $x\alpha y\beta z+0\gamma_1a_1+0=x\alpha(y\beta z)+0$ and $x\alpha y\beta z+m_1\gamma_1 0+0=(x\alpha y)\beta z+0$)
Similarly, we can show that $\mu(x\alpha y)\geq\min\{\mu(x),\mu(y)\}$ for all $x,y,z\in S,\alpha,\beta\in\Gamma$. Hence, μ is an M-fuzzy h-bi-ideal of S.

Lemma 3.2.7. Let S be an M-Γ-hemiring and $A\subseteq S$. Then, the following statements holds:

(1) A is a left (resp. right) M-h-ideal of S if and only if χ_A is an M-fuzzy left (resp. right) h-ideal of S.

(2) A is an M-h-bi-ideal of S if and only if χ_A is an M-fuzzy h-bi-ideal of S.

(3) A is an M-h-quasi-ideal of S if and only if χ_A is an M-fuzzy h-quasi-ideal of S.

4. Characterizations of M-Γ-hemirings

In this section, we divide the results into three parts. In subsection 4.1, we describe the characterizations of h-hemiregular M-Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals. In subsection 4.2, we investigate the characterizations of h-intra-hemiregular M-Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals. Finally, we discuss the characterizations of h-quasi-hemiregular M-Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals in subsection 4.3.

4.1. h-hemiregular M-Γ-hemirings.

Definition 4.1.1. An M-Γ-hemiring S is said to be h-hemiregular if for each $x \in S$, there exist a, a', $z \in S$, α, α', β, $\beta' \in \Gamma$ and $m \in M$ such that $x + m\alpha a\beta x + z = m\alpha' a'\beta' x + z$.

Lemma 4.1.2. If A and B are a right M-h-ideal and a left M-h-ideal of M-Γ-hemiring S, respectively, then $\overline{M\Gamma(A\Gamma B)} \subseteq A \cap B$.

Proof. Let $x \in \overline{M\Gamma(A\Gamma B)}$. Then there exist $a, a' \in A$, $b, b' \in B, z \in S$, α, α', β, $\beta' \in \Gamma$ and $m, m' \in M$ such that $x + m\alpha(a\beta b) + z = m'\alpha'(a'\beta' b') + z$.
Since A is a right M-h-ideal of S, we have $m\alpha(a\beta b), m'\alpha'(a'\beta' b') \in A$, and in consequence, $x \in A$. Similarly, we also can prove that $x \in B$. Thus, $x \in A \cap B$, this shows that $\overline{M\Gamma(A\Gamma B)} \subseteq A \cap B$.

Lemma 4.1.3. An M-Γ-hemiring S is h-hemiregular if and only if for any right M-h-ideal A and any left M-h-ideal B, we have $\overline{M\Gamma(A\Gamma B)} = A \cap B$.

Proof. Assume that S be an h-hemiregular M-Γ-hemiring. For any $x \in A \cap B$, we have $x \in A$ and $x \in B$. Since S is h-hemiregular, there exist $a, a', z \in S$, α, α', β, $\beta' \in \Gamma$ and $m \in M$ such that $x + m\alpha a\beta x + z = m\alpha' a'\beta' x + z$. Since A is a right M-h-ideal of S, $m\alpha a \in A$, $a \in A$ and so $a\beta x \in A\Gamma B$, thus $m\alpha(a\beta x) \in M\Gamma(A\Gamma B)$, similarly, $m\alpha'(a'\beta' x) \in M\Gamma(A\Gamma B)$. These implies $x \in \overline{M\Gamma(A\Gamma B)}$, which means that $A \cap B \subseteq \overline{M\Gamma(A\Gamma B)}$. This, by Lemma 4.1.2, gives $\overline{M\Gamma(A\Gamma B)} = A \cap B$.

Lemma 4.1.4. Let S be an M-Γ-hemiring, A, B are a right M-h-ideal and a left M-h-ideal of S, respectively, $A, B \subseteq S$. Then we have $\chi_{\overline{M\Gamma(A\Gamma B)}} = \chi_A \Gamma_h^M \chi_B$.

Proof. Let $x \in S$. If $x \in \overline{M\Gamma(A\Gamma B)}$, then $\chi_{\overline{M\Gamma(A\Gamma B)}} = 1$ and $x + l\gamma(p\delta q)+z = l'\gamma'(p'\delta' q')+z$ for some $p, p' \in A, q, q' \in B, l, l' \in M, \gamma, \gamma', \delta$, $\delta' \in \Gamma$ and $z \in S$. Thus we have

$$\begin{aligned}
&(\chi_A\Gamma_h^M\chi_B)(x)\\
=&\bigvee_{x+m\alpha(a\beta b)+z=m'\alpha'(a'\beta'b')+z}(\min\{\chi_A(m\alpha a),\chi_A(m'\alpha'a'),\chi_B(m\alpha b),\chi_B(m'\alpha'b')\})\\
\geq&\bigvee_{x+l\gamma(p\delta q)+z=l'\gamma'(p'\delta'q')+z}(\min\{\chi_A(l\gamma p),\chi_A(l'\gamma'p'),\chi_B(l\gamma q),\chi_B(l'\gamma'q')\})\\
\geq&\bigvee_{x+l\gamma(p\delta q)+z=l'\gamma'(p'\delta'q')+z}(\min\{\chi_A(p),\chi_A(p'),\chi_B(q),\chi_B(q')\})=1
\end{aligned}$$

and so $(\chi_A\Gamma_h^M\chi_B)(x) = 1 = \chi_{\overline{M\Gamma(A\Gamma B)}}(x)$. If $x \notin \overline{M\Gamma(A\Gamma B)}$, then $\chi_{\overline{M\Gamma(A\Gamma B)}}(x) = 0$. If possible, let $(\chi_A\Gamma_M\chi_B)(x) \neq 0$. Then

$$\begin{aligned}
&(\chi_A\Gamma_{Mh}\chi_B)(x)\\
=&\bigvee_{x+m\alpha(a\beta b)+z=m'\alpha'(a'\beta'b')+z}(\min\{\chi_A(m\alpha a),\chi_A(m'\alpha'a'),\chi_B(m\alpha b),\chi_B(m'\alpha'b')\})\\
\neq&\ 0.
\end{aligned}$$

Hence there exist $p, p', q, q', z \in S, \gamma, \gamma', \delta, \delta' \in \Gamma$ and $l, l' \in M$ such that $x + l\gamma(p\delta q) + z = l'\gamma'(p'\delta'q') + z$, that is $x + (l\gamma p)\delta(l\gamma q) + z = (l'\gamma'p')\delta'(l'\gamma'q') + z$, so

$$\min\{\chi_A(l\gamma p),\chi_A(l'\gamma'p'),\chi_B(l\gamma q),\chi_B(l'\gamma'q')\}$$

$$\geq \min\{\chi_A(p),\chi_A(p'),\chi_B(q),\chi_B(q')\} \neq 0,$$

we can get $\chi_A(p) = \chi_A(p') = \chi_B(q) = \chi_B(q') = 1$, hence $p, p' \in A, q, q' \in B$, and so $l\gamma p, l'\gamma'p' \in M\Gamma A$ and $l\gamma q, l'\gamma'q' \in M\Gamma B$, thus $x \in \overline{(M\Gamma A)\Gamma(M\Gamma B)} = \overline{M\Gamma(A\Gamma B)}$, which contradicts $\chi_{\overline{M\Gamma(A\Gamma B)}}(x) = 0$. Then we have $(\chi_A\Gamma_h^M\chi_B)(x) = 0 = \chi_{\overline{M\Gamma(A\Gamma B)}}(x)$.

In any case, we have $(\chi_A\Gamma_h^M\chi_B)(x) = \chi_{\overline{M\Gamma(A\Gamma B)}}(x)$. This completes the proof.

Theorem 4.1.5. An M-Γ-hemiring S is h-hemiregular if and only if $\mu \cap \nu = \mu\Gamma_h^M\nu$, for any fuzzy right M-h-ideal μ and fuzzy left M-h-ideal ν.

Proof. Let S be an h-hemiregular M-Γ-hemiring, μ and ν be M-fuzzy right h-ideal and M-fuzzy left h-ideal of S, respectively, then, $\mu\Gamma_h^M\nu \subseteq \mu\Gamma_h^M\chi_M \subseteq \mu$ and $\mu\Gamma_h^M\nu \subseteq \chi_M\Gamma_h^M\nu \subseteq \nu$. So, $\mu\Gamma_h^M\nu \subseteq \mu \cap \nu$. For any $x \in S$, there exist $a, a', z \in S$, α, α', β, $\beta' \in \Gamma$ and $m \in M$ such that

$x + m\alpha a\beta x + z = m\alpha' a'\beta' x + z$ since S is h-hemiregular. Then

$$\begin{aligned}(\mu\Gamma_h^M\nu)(x) &= \bigvee_{x+m\alpha a\beta x+z=m\alpha' a'\beta' x+z} (\min\{\mu(m\alpha a), \mu(m\alpha' a'), \nu(x)\}) \\ &\geq \min\{\mu(m\alpha x), \mu(m\alpha' x), \nu(x)\} \\ &\geq \min\{\mu(x), \nu(x)\} \\ &= (\mu\cap\nu)(x).\end{aligned}$$

That is, $\mu\cap\nu \subseteq \mu\Gamma_h^M\nu$, therefor $\mu\cap\nu = \mu\Gamma_h^M\nu$.

Conversely, let A and B be any right M-h-ideal and left M-h-ideal of S, respectively. Then by Lemma 3.3.7, the characteristic functions χ_A and χ_B of A and B are an M-fuzzy right h-ideal and M-fuzzy left h-ideal of S, respectively. Now by Lemma 4.1.4, we have

$$\chi_{\overline{M\Gamma(A\Gamma B)}} = \chi_A\Gamma_h^M\chi_B = \chi_A\cap\chi_B = \chi_{A\cap B}.$$

It follows from Proposition 2.2, we have $\overline{M\Gamma(A\Gamma B)} = A\cap B$. Hence, S is h-hemiregular by Lemma 4.1.3.

Lemma 4.1.6. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is h-hemiregular.

(2) $B = \overline{M\Gamma S\Gamma B}$, for every M-h-bi-ideal B of S.

(3) $Q = \overline{M\Gamma S\Gamma Q}$, for every M-h-quasi-ideal Q of S.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let B be any M-h-bi-ideal of S and x any element of B. Then there exist $a, a', z \in S$, α, α', β, $\beta' \in \Gamma$ and $m \in M$ such that $x + m\alpha a\beta x + z = m'\alpha' a'\beta' x + z$. It is easy to see that $m\alpha a\beta x, m'\alpha' a'\beta' x \in M\Gamma S\Gamma B$, and so $x \in \overline{M\Gamma S\Gamma B}$. Hence $B \subseteq \overline{M\Gamma S\Gamma B}$. On the other hand, since B is an M-h-bi-ideal of S, we have $M\Gamma S\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$, so $\overline{M\Gamma S\Gamma B} \subseteq \overline{B} = B$. Therefor $B = \overline{M\Gamma S\Gamma B}$.

(2) $\Rightarrow$ (3) This part is obvious.

(3) $\Rightarrow$ (1) Let A, B are any right M-h-ideal and any left M-h-ideal of S, respectively. Then we have $\overline{M\Gamma(A\Gamma B)} \subseteq \overline{A\Gamma B} \subseteq \overline{A\cap B} \subseteq \overline{A}\cap\overline{B} = A\cap B$, and thus $A\cap B$ is an M-h-quasi-ideal of S. We have $A\cap B = \overline{M\Gamma S\Gamma(A\cap B)} \subseteq \overline{M\Gamma S\Gamma(A\cap B)}\cap\overline{M\Gamma S\Gamma(A\cap B)} \subseteq \overline{M\Gamma S\Gamma A}\cap\overline{M\Gamma S\Gamma B} \subseteq \overline{M\Gamma A}\cap\overline{M\Gamma B} \subseteq \overline{(M\Gamma A)\Gamma(M\Gamma B)} = \overline{M\Gamma(A\Gamma B)}$. So, $\overline{M\Gamma(A\Gamma B)} = A\cap B$. Hence, S is an h-hemiregular by Lemma 4.1.3.

Theorem 4.1.7. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is h-hemiregular.

(2) $\mu=\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu$ for every M-fuzzy h-bi-ideal μ of S.
(3) $\mu=\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu$, for every M-fuzzy h-quasi-ideal μ of S.

Proof. (1) $\Rightarrow$ (2) Let μ be an M-fuzzy h-bi-ideal of S. Then for any $x\in S$, there exist $a,a',z\in S$, α, α', β, $\beta'\in\Gamma$ and $m\in M$ such that $x+m\alpha a\beta x+z=m\alpha' a'\beta' x+z$ since S is h-hemiregular. Then we have

$$\begin{aligned}(\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu)(x) &= \bigvee_{x+m_1\gamma_1b_1+z=m_2\gamma_2b_2+z}(\min\{\chi_M(m_i),(\chi_S\Gamma_h^M\mu)(b_i)\})\\ &\geq \min\{\mu(x),(\chi_S\Gamma_h^M\mu)(m\alpha x),(\chi_S\Gamma_h^M\mu)(m'\alpha' x)\}\\ &= \min\{\mu(x),\bigvee_{m\alpha x+a_1\gamma_1b_1+z=a_2\gamma_2b_2+z}\min(\mu(b_i)),\\ &\quad \bigvee_{m'\alpha' x+a_1\gamma_1b_1+z=a_2\gamma_2b_2+z}\min(\mu(b_i))\}\\ &\geq \min\{\mu(x),\mu(m\alpha a\beta x),\mu(m\alpha' a'\beta' x)\}\\ &\geq \mu(x).\end{aligned}$$

This implies that $\mu\subseteq\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu$. Since μ is an M-fuzzy h-bi-ideal of S, $\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu\subseteq\mu\Gamma_h^M\chi_S\Gamma_h^M\mu\subseteq\mu$. This shows that $\mu=\chi_M\Gamma_h^M\chi_S\Gamma_h^M\mu$.

(2) $\Rightarrow$ (3) This part is obvious.

(3) $\Rightarrow$ (1) Let Q be any M-h-quasi-ideal of S, then χ_Q is an M-fuzzy h-quasi-ideal of S by Lemma 3.3.7, thus, $\chi_Q=\chi_M\Gamma_h^M\chi_S\Gamma_h^M\chi_Q=\chi_{\overline{M\Gamma S\Gamma Q}}$.

This implies that $Q=\overline{M\Gamma S\Gamma Q}$. So, S is h-hemiregular by Lemma 4.1.6.

Theorem 4.1.8. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is h-hemiregular.
(2) $\mu\cap\nu=\chi_M\Gamma_h^M\nu\Gamma_h^M\mu$ for every M-fuzzy h-bi-ideal μ and every M-fuzzy h-ideal ν of S.
(3) $\mu\cap\nu=\chi_M\Gamma_h^M\nu\Gamma_h^M\mu$, for every M-fuzzy h-quasi-ideal μ and every M-fuzzy h-ideal ν of S.

Proof. (1) $\Rightarrow$ (2) Let μ and ν be an M-fuzzy h-bi-ideal and M-fuzzy h-ideal of S. Then for any $x\in S$, there exist $a,a',z\in S$, α, α', β, $\beta'\in\Gamma$

and $m \in M$ such that $x + m\alpha a\beta x + z = m\alpha' a'\beta' x + z$ since S is h-hemiregular. Then we have

$$\begin{array}{rcl}(\chi_M \Gamma_h^M \nu \Gamma_h^M \mu)(x) & = & \bigvee\limits_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\chi_M(m_i), (\nu\Gamma_h^M\mu)(b_i)\}) \\ & \geq & \min\{\mu(x), (\nu\Gamma_h^M\mu)(m\alpha x), (\nu\Gamma_h^M\mu)(m'\alpha' x)\} \\ & = & \min\{\mu(x), \bigvee\limits_{m\alpha x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min(\nu(a_i), \mu(b_i)), \\ & & \bigvee\limits_{m'\alpha' x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min(\nu(a_i), \mu(b_i))\} \\ & \geq & \min\{\min\{\mu(x), \nu(m\alpha x\beta a), \nu(m\alpha x\beta' a')\}, \\ & & \min\{\mu(x), \nu(m\alpha x\beta' a'), \nu(m'\alpha' x\beta' a')\}\} \\ & \geq & \min\{\mu(x), \nu(x)\} \\ & = & (\mu \cap \nu)(x).\end{array}$$

This implies that $\mu \cap \nu \subseteq \chi_M \Gamma_h^M \nu \Gamma_h^M \mu$.

On the other hand, we have $\chi_M \Gamma_h^M \nu \Gamma_h^M \mu \subseteq \mu \Gamma_h^M \nu \Gamma_h^M \mu \subseteq \mu$ and $\chi_M \Gamma_h^M \nu \Gamma_h^M \mu \subseteq \chi_M \Gamma_h^M \nu \Gamma_h^M \chi_M \subseteq \nu$, and so $\chi_M \Gamma_h^M \nu \Gamma_h^M \mu \subseteq \mu \cap \nu$. Thus $\mu \cap \nu = \chi_M \Gamma_h^M \nu \Gamma_h^M \mu$.

(2) $\Rightarrow$ (3) This part is obvious.

(3) $\Rightarrow$ (1) Let μ be any M-fuzzy h-quasi-ideal of S, then we have $\mu = \mu \cap \chi_S = \chi_M \Gamma_h^M \chi_S \Gamma_h^M \mu$, therefor S is h-hemiregular by Theorem 4.1.7.

4.2. **h-intra-hemiregular M-Γ-hemirings.** In this section, we describe the characterizations of h-intra-hemiregular M-Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals.

Definition 4.2.1. An M-Γ-hemiring S is said to be h-intra-hemiregular if, for each $x \in S$, there exist $d_i, d'_j, z \in S$, $\alpha_i, \beta_i, \gamma_i, \alpha'_j, \beta'_j, \gamma'_j \in \Gamma$ and $m_i, m'_j \in M$ such that $x + \sum\limits_{i=1}^{m} m_i\alpha_i x\beta_i x\gamma_i d_i + z = \sum\limits_{j=1}^{n} m'_j\alpha'_j x\beta'_j x\gamma'_j d'_j + z$.

Equivalent definitions:

(1) $x \in \overline{M\Gamma x\Gamma x\Gamma S}, \forall x \in S$,

(2) $A \subseteq \overline{M\Gamma S\Gamma S\Gamma S}, \forall A \subseteq S$.

Example 4.2.2. Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then S is an M-Γ-hemiring that is both h-hemiregular and h-intra-hemiregular, where $\Gamma = M = S$.

Lemma 4.2.3. Let S be an M-Γ-hemiring. Then the following statements are equivalent:

(1) S is h-intra-hemiregular;

(2) $L \cap R \subseteq \overline{M\Gamma(L\Gamma R)}$, for every left h-ideal L and every right h-ideal R of S.

Proof. The proof is similar to Lemma 4.1.3.

Theorem 4.2.4. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is h-intra-hemiregular.

(2) $\mu \cap \nu \subseteq \mu\widetilde{\Gamma}_h\nu$ for every M-fuzzy left h-ideal μ and every M-fuzzy right h-ideal ν of S.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let μ and ν be any M-fuzzy left h-ideal and any M-fuzzy right h-ideal of S, respectively. Now let $x \in S$, there exist $d_i, d'_j, z \in S$, $\alpha_i, \beta_i, \gamma_i, \alpha'_j, \beta'_j, \gamma'_j \in \Gamma$ and $m_i, m'_j \in M$ such that $x + \sum\limits_{i=1}^{m} m_i\alpha_i x\beta_i x\gamma_i d_i + z = \sum\limits_{j=1}^{n} m'_j\alpha'_j x\beta'_j x\gamma'_j d'_j + z$. That is, $x + \sum\limits_{i=1}^{m}(m_i\alpha_i x)\beta_i(x\gamma_i d_i) + z = \sum\limits_{j=1}^{n}(m'_j\alpha'_j x)\beta'_j(x\gamma'_j d'_j) + z$.

Then we have

$$\begin{aligned}(\mu\widetilde{\Gamma}_h^M\nu)(x) &= \bigvee_{x+\sum\limits_{i=1}^{m}(m_i\gamma_i b_i)+z=\sum\limits_{j=1}^{n}(m'_j\gamma'_j b'_j)+z} (\min\{\mu(m_i), \mu(m'_j), \nu(b_i), \nu(b'_j)\}) \\ &\geq \min\{\mu(m_i\alpha_i x), \mu(m'_j\alpha'_j x), \nu(x\gamma_i d_i), \nu(x\gamma'_j d'_j)\} \\ &\geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \cap \nu)(x).\end{aligned}$$

This implies that $\mu \cap \nu \subseteq \mu\widetilde{\Gamma}_h^M\nu$.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let P and R be any left M-h-ideal and right M-h-ideal of S, respectively. Then by Lemma 3.3.7, the characteristic functions χ_P and χ_R of P and R are an M-fuzzy left h-ideal and M-fuzzy right h-ideal of S, respectively. Now by Lemma 4.1.4, we have $\chi_{\overline{M\Gamma(P\Gamma R)}} = \chi_P\widetilde{\Gamma}_h^M\chi_R \supseteq \chi_P \cap \chi_R = \chi_{P\cap R}$. It follows from Proposition 2.2, we have $\overline{M\Gamma(P\Gamma R)} \supseteq P \cap R$. Hence, S is h-hemiregular by Lemma 4.2.3.

Theorem 4.2.5. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is both h-hemiregular and h-intra-hemiregular.

(2) $B = \overline{M\Gamma B}$, for every M-h-bi-ideal B of S.

(3) $Q = \overline{M\Gamma Q}$, for every M-h-quasi-ideal Q of S.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let B be any M-h-bi-ideal of S and x any element of B. Then $\overline{M\Gamma B} \subseteq \overline{B\Gamma B} \subseteq \overline{B} \subseteq B$. Since S is both h-hemiregular and h-intra-hemiregular, there exist some elements $a_1, a_2, p_i, p'_j, q_j, q'_j, z \in S$, $\alpha_t, \beta_t, \gamma_t, \delta_t \in \Gamma$ and $m_i, m_j \in M$ such that

$$\begin{aligned}
x &+ \sum_{j=1}^{n}(m_j\alpha_1 a_2\beta_1 q_j\gamma_1 x)\delta_1(m_j\alpha_1 q'_j\beta_2 a_1\gamma_2 x) \\
&+ \sum_{j=1}^{n}(m_j\alpha_2 a_1\beta_3 q_j\gamma_3 x)\delta_2(m_j\alpha_2 q'_j\beta_4 a_2\gamma_4 x) \\
&+ \sum_{i=1}^{m}(m_i\alpha_3 a_1\beta_5 p_i\gamma_5 x)\delta_3(m_i\alpha_3 p'_i\beta_6 a_1\gamma_6 x) \\
&+ \sum_{i=1}^{m}(m_i\alpha_4 a_2\beta_7 p_i\gamma_7 x)\delta_4(m_i\alpha_4 p'_i\beta_8 a_2\gamma_8 x) + z \\
&= \sum_{i=1}^{m}(m'_i\alpha'_1 a_2\beta'_1 p_i\gamma'_1 x)\delta'_1(m'_i\alpha'_1 p'_i\beta'_2 a_1\gamma'_2 x) \\
&+ \sum_{i=1}^{m}(m'_i\alpha'_2 a_1\beta'_3 p_i\gamma'_3 x)\delta'_2(m'_i\alpha'_2 p'_i\beta'_4 a_2\gamma'_4 x) \\
&+ \sum_{j=1}^{n}(m'_j\alpha'_3 a_1\beta'_5 q_j\gamma'_5 x)\delta'_3(m'_j\alpha'_3 q'_j\beta'_6 a_1\gamma'_6 x) \\
&+ \sum_{j=1}^{n}(m'_j\alpha'_4 a_2\beta'_7 q_j\gamma'_7 x)\delta'_4(m'_j\alpha'_4 q'_j\beta'_8 a_2\gamma'_8 x) + z.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{j=1}^{n}(m_j\alpha_1 a_2\beta_1 q_j\gamma_1 x)\delta_1(m_j\alpha_1 q'_j\beta_2 a_1\gamma_2 x) \\
&\quad = \sum_{j=1}^{n} m_j\alpha_1[(a_2\beta_1 q_j\gamma_1 x)\delta_1(q'_j\beta_2 a_1\gamma_2 x)] \in M\Gamma B.
\end{aligned}$$

The case for others can be similarly verified.

Thus we have $x \in \overline{M\Gamma B}$ and so $B \subseteq \overline{M\Gamma B}$. Therefor $B = \overline{M\Gamma B}$.

(2) $\Rightarrow$ (3) This part is straightforward.

(3) $\Rightarrow$ (1) Let L, R are any left M-h-ideal and any right M-h-ideal of S, respectively. Then it is clear that $L \cap R$ is an M-h-quasi-ideal of S. By the assumption, we have $L \cap R = \overline{M\Gamma(L \cap R)} \subseteq \overline{M\Gamma(L \cap R)} \cap \overline{M\Gamma(L \cap R)} \subseteq \overline{M\Gamma L} \cap \overline{M\Gamma P} \subseteq \overline{(M\Gamma L)\Gamma(M\Gamma R)} = \overline{M\Gamma(L\Gamma R)}$. So, $L \cap R \subseteq \overline{M\Gamma(L\Gamma R)}$. Hence, S is an h-hemiregular by Lemma 4.2.3.

Theorem 4.2.6. Let S be an M-Γ-hemiring. Then the following conditions are equivalent:

(1) S is both h-hemiregular and h-intra-hemiregular.

(2) $\mu = \chi_M \widetilde{\Gamma}_h \mu$, for every M-fuzzy h-bi-ideal μ of S.

(3) $\mu = \chi_M \widetilde{\Gamma}_h \mu$, for every M-fuzzy h-quasi-ideal μ of S.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let μ be any M-fuzzy h-bi-ideal of S and x any element of S. Then $\chi_M \widetilde{\Gamma}_h \mu \subseteq \mu \widetilde{\Gamma}_h \mu \subseteq \mu$. Since S is both h-hemiregular and h-intra-hemiregular, there exist some elements $a_1, a_2, p_i, p'_j, q_j, q'_j, z \in S$, $\alpha_t, \beta_t, \gamma_t, \delta_t \in \Gamma$ and $m_i, m_j \in M$ such that

$$\begin{aligned}
x + \sum_{j=1}^{n} (m_j\alpha_1 a_2\beta_1 q_j\gamma_1 x)\delta_1(m_j\alpha_1 q'_j\beta_2 a_1\gamma_2 x) \\
+ \sum_{j=1}^{n} (m_j\alpha_2 a_1\beta_3 q_j\gamma_3 x)\delta_2(m_j\alpha_2 q'_j\beta_4 a_2\gamma_4 x) \\
+ \sum_{i=1}^{m} (m_i\alpha_3 a_1\beta_5 p_i\gamma_5 x)\delta_3(m_i\alpha_3 p'_i\beta_6 a_1\gamma_6 x) \\
+ \sum_{i=1}^{m} (m_i\alpha_4 a_2\beta_7 p_i\gamma_7 x)\delta_4(m_i\alpha_4 p'_i\beta_8 a_2\gamma_8 x) + z \\
= \sum_{i=1}^{m} (m'_i\alpha'_1 a_2\beta'_1 p_i\gamma'_1 x)\delta'_1(m'_i\alpha'_1 p'_i\beta'_2 a_1\gamma'_2 x) \\
+ \sum_{i=1}^{m} (m'_i\alpha'_2 a_1\beta'_3 p_i\gamma'_3 x)\delta'_2(m'_i\alpha'_2 p'_i\beta'_4 a_2\gamma'_4 x) \\
+ \sum_{j=1}^{n} (m'_j\alpha'_3 a_1\beta'_5 q_j\gamma'_5 x)\delta'_3(m'_j\alpha'_3 q'_j\beta'_6 a_1\gamma'_6 x)
\end{aligned}$$

$$+\sum_{j=1}^{n}(m'_j\alpha'_4 a_2\beta'_7 q_j\gamma'_7 x)\delta'_4(m'_j\alpha'_4 q'_j\beta'_8 a_2\gamma'_8 x)+z.$$

Then we have

$$\begin{aligned}(\chi_M\widetilde{\Gamma}_h^M\mu)(x) &= \bigvee_{x+\sum_{i=1}^{m}(m_i\gamma_i b_i)+z=\sum_{j=1}^{n}(m'_j\gamma'_j b'_j)+z}(\min\{\mu(b_i),\mu(b'_j)\})\\ &\geq \min\{\mu(m_j\alpha_1 q'_j\beta_2 a_1\gamma_2 x),\mu(m_j\alpha_2 q'_j\beta_4 a_2\gamma_4 x),\\ &\quad \mu(m'_j\alpha'_3 q'_j\beta'_6 a_1\gamma'_6 x),\mu(m'_j\alpha'_4 q'_j\beta'_8 a_2\gamma'_8 x)\}\\ &\geq \mu(x).\end{aligned}$$

This implies that $\mu\subseteq\chi_M\widetilde{\Gamma}_h^M\mu$. Therefor $\mu=\chi_M\widetilde{\Gamma}_h^M\mu$.

(2) $\Rightarrow$ (3) This part is straightforward.

(3) $\Rightarrow$ (1) Let Q be any M-h-quasi-ideal of S, then χ_Q is an M-fuzzy h-quasi-ideal of S by Lemma 3.3.7, thus, $\chi_Q=\chi_M\widetilde{\Gamma}_h^M\chi_Q=\chi_{\overline{M\Gamma Q}}$. Then $Q=\overline{M\Gamma Q}$. Hence, S is both h-hemiregular and h-intra-hemiregular by Theorem 4.2.5.

4.3. **h-quasi-hemiregular M-Γ-hemirings.** In this section, we investigate the characterizations of h-quasi-hemiregular Γ-hemirings in terms of these kinds of generalized M-fuzzy h-ideals.

Definition 4.3.1. (1) A subset A of S is called Γ-idempotent if $A=\overline{A\Gamma A}$

(2) An M-Γ-hemiring S is called left (resp., right) h-quasi-hemiregular if every left (resp., right) M-h-ideal is Γ-idempotent, and is called h-quasi-hemiregular if every left M-h-ideal and every right M-h-ideal are all Γ-idempotent.

The following Lemma is obvious.

Lemma 4.3.2. An M-Γ-hemiring S is left h-quasi-hemiregular if and only if one of the following statements holds:

(1) There exist $c_i,c'_j,z\in S$, $\alpha_i,\beta_i,\gamma_i,\alpha'_j,\beta'_j,\gamma'_j\in\Gamma$ and $m_i,m'_j\in M$ such that

$$x+\sum_{i=1}^{m}m_i\alpha_i x\beta_i c_i\gamma_i x+z=\sum_{j=1}^{n}m'_j\alpha'_j x\beta'_j c'_j\gamma'_j x+z$$

for all $x\in S$;

(2) $x\in\overline{M\Gamma x\Gamma S\Gamma x}$, for all $x\in S$;

(3) $A\subseteq\overline{M\Gamma A\Gamma S\Gamma A}$, for all $A\in S$;

(4) $I \cap L = \overline{I\Gamma L}$, for every M-h-ideal I and every left M-h-ideal L of S.

Definition 4.3.3. A fuzzy set μ over S is called M-fuzzy idempotent if $\mu = \mu\widetilde{\Gamma}_h\mu$.

Theorem 4.3.4. An M-Γ-hemiring S is left (resp., right) h-quasi-hemiregular if and only if every M-fuzzy left (resp., right) h-ideal over S is M-fuzzy idempotent.

Proof. Let S be a left h-quasi-hemiregular M-Γ-hemiring, μ any M-fuzzy left h-ideal over S. Now let x be any element of S. Since S is h-quasi-hemiregular, then, by Lemma 4.3.2, there exist $c_i, c'_j, z \in S$, $\alpha_i, \beta_i, \gamma_i, \alpha'_j, \beta'_j, \gamma'_j \in \Gamma$ and $m_i, m'_j \in M$ such that $x + \sum\limits_{i=1}^{m} m_i\alpha_i x\beta_i c_i\gamma_i x + z = \sum\limits_{j=1}^{n} m'_j\alpha'_j x\beta'_j c'_j\gamma'_j x + z$.

Then we have

$$\begin{aligned}
&(\mu\widetilde{\Gamma}_h^M\mu)(x)\\
&= \bigvee_{x+\sum\limits_{i=1}^{m} m_i\gamma_i b_i + z = \sum\limits_{j=1}^{n} m'_j\gamma'_j b'_j + z} (\min\{\mu(m_i), \mu(m'_j), \mu(b_i), \mu(b'_j)\})\\
&\geq \min\{\mu(m_i\alpha_i x), \mu(m'_j\alpha'_j x), \mu(c_i\gamma_i x), \mu(c'_j\gamma'_j x)\}\\
&\geq \min\{\mu(x)\}.
\end{aligned}$$

It follows that $\mu \subseteq \mu\widetilde{\Gamma}_h^M\mu$. Since μ is an M-fuzzy left h-ideal over S, we have $\mu\widetilde{\Gamma}_h^M\mu \subseteq \mu$. Whence, $\mu = \mu\widetilde{\Gamma}_h^M\mu$. This implies that μ is an M-fuzzy left h-ideal over S which is M-fuzzy idempotent.

Conversely, let L be any left M-h-ideal of S. Then χ_L is an M-fuzzy left h-ideal over S. Now, by the assumption, we have $\chi_L = \chi_L\widetilde{\Gamma}_h^M\chi_L = \chi_{\overline{L\Gamma L}}$, it follows from Proposition 2.2 that $L = \overline{L\Gamma L}$. Therefore S is left h-quasi-hemiregular by Definition 4.3.1. The case for the M-fuzzy right h-ideals can be similarly proved.

Theorem 4.3.5. Let S be an M-Γ-hemiring. Then the following are equivalent:

(1) S is h-quasi-hemiregular;

(2) $\mu \cap \nu = \mu\widetilde{\Gamma}_h^M\nu$, for any M-fuzzy h-ideal μ and any M-fuzzy left h-ideal ν of S;

(3) $\mu \cap \nu \subseteq \mu \widetilde{\Gamma}_h^M \nu$, for any M-fuzzy h-ideal μ and any M-fuzzy h-bi-ideal ν of S;

(4) $\mu \cap \nu \subseteq \mu \widetilde{\Gamma}_h^M \nu$, for any M-fuzzy h-ideal μ and any M-fuzzy h-quasi-ideal ν of S.

Proof. (1) $\Rightarrow$ (3): Let μ and ν be any M-fuzzy h-ideal and any M-fuzzy h-bi-ideal over S, respectively. Now let x be any element of S, since S is left h-quasi-hemiregular, then by Lemma 4.3.2, we have $x \in \overline{M\Gamma x\Gamma S\Gamma x} \subseteq \overline{M\Gamma \overline{M\Gamma x\Gamma S\Gamma x}\Gamma S\Gamma x} \subseteq \overline{M\Gamma x\Gamma S\Gamma x\Gamma S\Gamma x}$, and so there exist $d_i, d_j', e_i, e_j', z \in S$, $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta_i, \alpha_j', \beta_j', \gamma_j', \delta_j', \eta_j' \in \Gamma$ and $m_i, m_j' \in M$ such that

$$x + \sum_{i=1}^{m} m_i\alpha_i x\beta_i d_i\gamma_i x\delta_i e_i\eta_i x + z = \sum_{j=1}^{n} m_j'\alpha_j' x\beta_j' d_j'\gamma_j' x\delta_j' e_j'\eta_j' x + z.$$

Then we have

$$\begin{aligned}
&(\mu\widetilde{\Gamma}_h^M\nu)(x)\\
&= \bigvee_{x+\sum_{i=1}^{m} m_i\gamma_i b_i + z = \sum_{j=1}^{n} m_j'\gamma_j' b_j' + z} (\min\{\mu(a_i), \mu(a_j'), \nu(b_i), \nu(b_j')\})\\
&\geq \min\{\mu(m_i\alpha_i x\beta_i d_i), \mu(m_j'\alpha_j' x\beta_j' d_j'), \nu(x\delta_i e_i\eta_i x), \nu(x\delta_j' e_j'\eta_j' x)\}\\
&\geq \min\{\mu(x), \nu(x)\}\\
&= \mu(x) \cap \nu(x).
\end{aligned}$$

It follows that $\mu \cap \nu \subseteq \mu\widetilde{\Gamma}_h^M\nu$, thus (3) holds.

It is clear that (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2).

(2) $\Rightarrow$ (1): Let I and L be any M-h-ideal and any left M-h-ideal of S, respectively. Then χ_I and χ_L are an M-fuzzy h-ideal and an M-fuzzy left h-ideal over S, respectively. Now, we have

$$\chi_(I \cap L) = \chi_I \cap \chi_L = \chi_I \widetilde{\Gamma}_h^M \chi_L = \chi_{\overline{I\Gamma L}}.$$

It follows from Proposition 2.2 that $I \cap L = \overline{I\Gamma L}$. Therefore S is h-quasi-hemiregular by Lemma 4.3.2.

Now, we can describe an important characterization of h-quasi-hemiregular M-Γ-hemirings.

Theorem 4.3.6. An M-Γ-hemiring S is h-quasi-hemiregular if and only if, $\mu = (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu) \cap (\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M)$, for every M-fuzzy h-quasi-ideal over S.

Proof. Let S be an h-quasi-hemiregular M-Γ-hemiring, and μ any M-fuzzy h-quasi-ideal over S. We know that $\chi_M\widetilde{\Gamma}_h^M\mu$ and $\mu\widetilde{\Gamma}_h^M\chi_M$ are an M-fuzzy left h-ideal and an M-fuzzy right h-ideal over S, respectively, and so both $\chi_M\widetilde{\Gamma}_h^M\mu$ and $\mu\widetilde{\Gamma}_h^M\chi_M$ are M-fuzzy idempotent by Theorem 4.3.4. Hence, we have

$(\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu)\cap(\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M) = (\chi_M\widetilde{\Gamma}_h^M\mu)\cap(\mu\widetilde{\Gamma}_h^M\chi_M)$ $\subseteq \mu$.

Now for $x \in S$. Since S is left h-quasi-hemiregular, then there exist $d_i, d'_j, z \in S$, $\alpha_i, \beta_i, \gamma_i, \alpha'_j, \beta'_j, \gamma'_j \in \Gamma$ and $m_i, m'_j \in M$ such that $x + \sum_{i=1}^{m} m_i\alpha_i x\beta_i d_i\gamma_i x + z = \sum_{j=1}^{n} m'_j\alpha'_j x\beta'_j d'_j\gamma'_j x + z$.

Thus, we have

$$
\begin{aligned}
&(\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu)\\
&= \bigvee_{x+\sum_{i=1}^{m} m_i\delta_i b_i+z=\sum_{j=1}^{n} m'_j\delta'_j b'_j+z} \min\{(\chi_M\widetilde{\Gamma}_h^M\mu)(m_i), (\chi_M\widetilde{\Gamma}_h^M\mu)(m'_j),\\
&(\chi_M\widetilde{\Gamma}_h^M\mu)(b_i), (\chi_M\widetilde{\Gamma}_h^M\mu)(b'_j)\}\\
&\geq \min\{(\chi_M\widetilde{\Gamma}_h^M\mu)(m_i\alpha_i x), (\chi_M\widetilde{\Gamma}_h^M\mu)(m'_j\alpha'_j x),\\
&(\chi_M\widetilde{\Gamma}_h^M\mu)(d_i\gamma_i x), (\chi_M\widetilde{\Gamma}_h^M\mu)(d'_j\gamma'_j x)\}\\
&\geq \{\mu(x)\}.
\end{aligned}
$$

which implies $\mu \subseteq (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu)$. Similarly, we can prove

$$\mu \subseteq (\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M)$$

and so,

$$\mu \subseteq (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu) \cap (\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M),$$

that is,

$$\mu = (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu) \cap (\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M).$$

Conversely, assume that the given condition holds. Let μ be any M-fuzzy h-quasi-ideal over S. Now, by the assumption, we have

$$
\begin{aligned}
\mu &= (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu) \cap (\mu\widetilde{\Gamma}_h^M\chi_M)\Gamma(\mu\widetilde{\Gamma}_h^M\chi_M)\\
&\subseteq (\chi_M\widetilde{\Gamma}_h^M\mu)\Gamma(\chi_M\widetilde{\Gamma}_h^M\mu) \subseteq (\mu\widetilde{\Gamma}_h^M\mu)\Gamma(\mu\widetilde{\Gamma}_h^M\mu) \subseteq (\chi_M\widetilde{\Gamma}_h^M\mu) \subseteq \mu.
\end{aligned}
$$

which implies, $\mu = \mu\widetilde{\Gamma}_h^M\mu$. It follows from Theorem 4.3.4 that S is left h-quasi-hemiregular. Similarly, we can prove S is right h-quasi-hemiregular. Therefore, S is h-quasi-hemiregular.

Applying Definitions 4.2.1, 4.3.1 and Lemma 4.3.2, we have the following result:

Lemma 4.3.7. A Γ-hemiring S is both left h-quasi-hemiregular and h-intra-hemiregular if and only if, for any $x \in S$, there exist $d_i, d'_j, z \in S$, $\alpha_i, \beta_i, \gamma_i, \delta_i, \alpha'_j, \beta'_j, \gamma'_j, \delta'_j \in \Gamma$ and $m_i, m'_j \in M$ such that

$$x + \sum_{i=1}^{m} m_i\alpha_i x\beta_i x\gamma_i d_i\delta_i x + z = \sum_{j=1}^{n} m'_j\alpha'_j x\beta'_j x\gamma'_j d'_j\delta'_j x + z.$$

Similar to Theorem 4.3.5, we have the following statements:

Theorem 4.3.8. Let S be an M-Γ-hemiring. Then the following are equivalent:

(1) S is both left h-quasi-hemiregular and h-intra-hemiregular;

(2) $\mu \cap \nu = \mu\widetilde{\Gamma}_h^M\nu$ for any M-fuzzy left h-ideal μ and any M-fuzzy h-bi-ideal ν of S;

(3) $\mu \cap \nu = \mu\widetilde{\Gamma}_h^M\nu$ for any M-fuzzy left h-ideal μ and any M-fuzzy h-quasi-ideal ν of S.

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