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MULTIPLICATION IDEALS IN Γ-RINGS

ALI AKBAR ESTAJI, ALI SAGHAFI KHORASANI
AND SAMANEH BAGHDARI

Abstract. In this paper we introduce the notion of multiplication ideals in Γ-rings and we obtain some characterizations for multiplication ideals in Γ-rings.

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1. Introduction

We shall call an R-module M a multiplication module if every submodule of M is of the form IM, for some ideal I of R. Multiplication modules and ideals have been investigated in A. Barnard (1981), El-Bast and Smith (1988), P. F. Smith (1988) and others. For results on multiplication modules, the reader is referred to [1, 2, 5, 8, 12].

Nobusawa [9] developed the notion of a Γ-ring which is more general than a ring. After his research, Barnes studied Γ-rings in more details in [3]. But Barnes approached to Γ-rings in a different way than that of Nobusawa and he defined the concept of Γ-ring and related definitions. After these two papers were published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa, which are parallel to the results in the ring theory (for example [1, 4, 10, 12]). In this paper, we introduce the concepts of multiplication ideals in Γ-rings.

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∗Address correspondence to Ali Saghafi Khorasani; E-mail: saghafiali21@yahoo.com.

2. Preliminaries of Γ-rings

In the remainder of the paper we use some notation and results from the theory of Γ-rings. We present a few basic definitions here.

Let M and Γ be additive abelian groups. If we have a map from $M \times \Gamma \times M$ to M such that for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$

(1) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
(2) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring in the sense of Barnes [3]. Note that any ring R, can be regarded as an R-ring. A Γ-ring M is called commutative, if for any $x, y \in M$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. M is called a Γ-ring with unit, if there exist elements $1 \in M$ and $\gamma_0 \in \Gamma$ such that for any $m \in M$, $1\gamma_0 m = m = m\gamma_0 1$. Throughout this paper, M stands for a nonempty commutative Γ-ring with unit. If A and B are subsets of the Γ-ring M and $\Theta \subseteq \Gamma$, we denote by $A\Theta B$ the subset of M consisting of all finite sums of the form $\sum a_i\gamma_i b_i$ where $(a_i, \gamma_i, b_i) \in A \times \Theta \times B$. For singletone subsets we abbreviate this notation for example, $\{a\}\Theta B = a\Theta B$. An ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M = M\Gamma I \subseteq I$. We denote an ideal I in M by $I \trianglelefteq M$. An ideal $I \trianglelefteq M$ is called a proper ideal, if $I \subsetneqq M$. For each subset S of the Γ-ring M, the smallest ideal containing S is denoted by $< S >$ and is called the ideal generated by S. If S is finite, $< S >$ is called finitely generated.

A proper ideal P in the Γ-ring M is called a prime ideal, if for any ideals $A, B \trianglelefteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A proper ideal N in the Γ-ring M is called maximal ideal, if for any ideals J in M such that $N \subseteq J \subseteq M$, we have $N = J$ or $J = M$. It is easy to show that any maximal ideal is prime. We denote by $Max(M)$, the set of all maximal ideals in the Γ-ring M.

A subset S of the Γ-ring M is an m-system in M, if $S = \emptyset$ or if $a, b \in S$ implies that $< a > \Gamma < b > \cap S \neq \emptyset$. An ideal P in M is prime if and only if its complement P^c is an m-system, see [3]. The prime radical $P(A)$ of the ideal A in the Γ-ring M, is the set consisting of those elements r of M with the property that every m-system in M which contains r meet A (that is, has nonempty intersection with A). An ideal Q in the Γ-ring M is said to be semi-prime ideal if and only if it has the following property: if A is an ideal in M such that $A\Gamma A \subseteq Q$, then $A \subseteq Q$. It is clear that a prime ideal is semi-prime. More over the intersection of any set of semi-prime ideals is a semi-prime ideal, see [6]. It follows easy by induction that if Q is a semi-prime ideal, A is an ideal and $(A\Gamma)^n A \subseteq Q$ for an arbitrary positive integer n, then $A \subseteq Q$, see [6].

Theorem 2.1. *If Q is an ideal in the Γ-ring M, the following conditions are equivalent.*

(1) *Q is a semi-prime ideal.*
(2) *if $a \in M$ such that $< a > \Gamma < a > \subseteq Q$, then $a \in Q$.*

Proof. See Theorem 3.2 in [7]. □

Proposition 2.2. *If Q is an ideal in the Γ-ring M, then $P(Q)$ is the smallest semi-prime ideal in M which contains Q, i.e.*

$$P(Q) = \bigcap P$$

where P runs over all semi-prime ideals of M such that $Q \subseteq P$.

Proof. See Corollary 3.5 in [7]. □

The reader is referred to [6, 7, 8] for undefined terms and notations.

3. Multiplication ideals

In this section we give some important properties of multiplication ideals, starting with the following definition.

Definition 3.1. An ideal I in the Γ-ring M is called multiplication ideal, if for every ideal J contained in I, there exists ideal G in M such that $J = G\Gamma I$.

Let I and J be ideals in the Γ-ring M. $[I : J]$ is the set of all $m \in M$ such that $m\Gamma J \subseteq I$. $[I : J]$ is called the residual of I by J. The annihilator of I is denoted by $ann(I)$ and equals to $[0 : I]$. An ideal I in M is called faithful if $ann(I) = 0$. We say that I divides J, denoted by $I|J$, if there exists an ideal G in M such that $I\Gamma G = J$.

Proposition 3.2. *Let I be a multiplication ideal in the Γ-ring M and J be an arbitrary ideal in M. $I|J$ if and only if $J \subseteq I$.*

Proof. The proof is evident. □

Definition 3.3. Let M be a Γ-ring and N an ideal in M and $P \in Max(M)$. N is called P-cyclic if there exist $p \in P$ and $n \in N$ such that $(1-p)\gamma_0 N \subseteq M\Gamma n$ and also, it is clear that $(1-p)\gamma_0 N = (1-p)\Gamma N$. Define $T_P N$ as the set of all $n \in N$ such that $(1-p)\gamma_0 n = 0$ for some $p \in P$.

Lemma 3.4. *Let M be a Γ-ring and N an ideal in M and $P \in Max(M)$. Then $T_P N$ is an ideal in M.*

Proof. It is straightforward. $\square$

Proposition 3.5. *Let N be an ideal in the Γ-ring M. N is multiplication ideal if and only if for any ideal $P \in Max(M)$, either $N = T_P N$ or N is P-cyclic.*

Proof. Let N be a multiplication ideal and $P \in Max(M)$. First suppose that $N = P\Gamma N$. Since N is multiplication ideal, we conclude that for every $n \in N$, there exists an ideal A in M such that $< n >= A\Gamma N$. Hence $< n >= P\Gamma < n >$. So there exists $p \in P$ such that $(1-p)\gamma_0 n = 0$, it follows that $n \in T_P N$ and then $N = T_P N$.

Now suppose that $N \neq P\Gamma N$ and $x \in N \setminus P\Gamma N$. Then there exists an ideal B in M such that $< x >= B\Gamma N$ and $P + B = M$. Obviously, if we assume that $p \in P$, then $(1-p)\gamma_0 N \subseteq M\Gamma x$. Therefore N is P-cyclic.

Conversely, suppose that J is an ideal in M and $J \subseteq N$. Define I as the set of all $m \in M$, where $m\gamma_0 n \in J$ for any $n \in N$. Clearly I is an ideal in M and $I\Gamma N \subseteq J$. Let $y \in J$. Define K as the set of all $m \in M$, where $m\gamma_0 y \in I\Gamma N$. We claim $K = M$. Assume that $K \subsetneq M$. Then, by Zorn's Lemma, there exists $Q \in Max(M)$ such that $K \subseteq Q \subset M$. By hypothesis $N = T_Q N$ or N is Q-cyclic. If $N = T_Q N$, then there exists $s \in Q$ such that $(1-s)\gamma_0 y = 0$. Hence $(1-s) \in K \subseteq Q$, it follows that $1 \in Q$, a contradiction. If N is Q-cyclic then there exist $t \in Q$ and $z \in N$ such that $(1-t)\gamma_0 N \subseteq M\Gamma z =< z >$. Define L as the set of all $m \in M$ such that $m\gamma_0 z \in (1-t)\gamma_0 J$. Clearly L is an ideal in M and $L\gamma_0 z \subseteq (1-t)\gamma_0 J$. Since $J \subseteq N$, we conclude that $(1-t)\gamma_0 J \subseteq< z >$. Hence $(1-t)\gamma_0 J \subseteq L\gamma_0 z$. So $(1-t)\gamma_0 J = L\gamma_0 z$, it follows that $(1-t)\gamma_0 L\gamma_0 N \subseteq (1-t)\gamma_0 J \subseteq J$ and $(1-t)\gamma_0 L \subseteq I$. Therefore $(1-t)\gamma_0(1-t)\gamma_0 J \subseteq I\Gamma M$. Hence $(1-t)\gamma_0(1-t) \in K \subseteq Q$. Thus $(1-t) \in Q$, it follows that $1 \in Q$, a contradiction. Hence $K = M$ and $y \in I\Gamma N$. Thus N is a multiplication ideal. $\square$

Proposition 3.6. *Let N be a faithful ideal in the Γ-ring M. N is multiplication ideal if and only if*

(1) *For any nonempty collection $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals in M,*

$$\bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma N) = (\bigcap_{\lambda \in \Lambda} I_\lambda)\Gamma N$$

(2) *For any ideal K in M which $K \subseteq N$ and any ideal A in M with $K \subset A\Gamma N$, there exists ideal B in M such that $B \subset A$ and $K \subseteq B\Gamma N$.*

Proof. Suppose (1) and (2) hold. Let K be an ideal in M contained in N and

$$\mathcal{S} = \{I : \ I \text{ is an ideal of } M \text{ and } K \subseteq I\Gamma N\}.$$

Clearly $M \in \mathcal{S}$. Since the statement (1) is correct, by Zorn's Lemma, $\mathcal{S}$ has a minimal member, A say. Since $K \subseteq A\Gamma N$ and A is minimal element of $\mathcal{S}$, we can then conclude from (2) that $K = A\Gamma N$. It follows that N is a multiplication ideal.

Conversely, suppose that N is a multiplication ideal in M. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a nonempty collection of ideals in M and $I = (\bigcap_{\lambda\in\Lambda} I_\lambda)$. Clearly $I\Gamma N \subseteq \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma N)$. Let $x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma N) \subseteq N$ and we put $L = \{m \in M : m\gamma_0 x \in I\Gamma N\}$. We claim $L = M$. Assume that $L \subsetneqq M$. By Zorn's Lemma, there exists $P \in Max(M)$ such that $L \subseteq P$. It is clear that $x \notin T_P N$. Hence $T_P N \neq N$ and by Proposition 3.5, N is P-cyclic. Hence there exist $n \in N$ and $p \in P$ such that $(1-p)\gamma_0 N \subseteq M\Gamma n =< n >$. Thus $(1-p)\gamma_0 x \in \bigcap_{\lambda\in\Lambda}(I_\lambda\gamma_0 n)$ and so for any $\lambda \in \Lambda$, $(1-p)\gamma_0 x \in I_\lambda\gamma_0 n$. It is clear that $(1-p)\gamma_0(1-p) \in L \subseteq P$, in view of the fact that N is faithful. Hence $1 \in P$, a contradiction. Therefore $L = M$, it follows that $x = 1\gamma_0 x \in I\Gamma N$ and (1) holds. Now suppose K is an ideal in M with $K \subseteq N$ and A is an ideal in M with $K \subset A\Gamma N$. Since N is multiplication ideal, there exists an ideal C in M such that $K = C\Gamma N$. Let $B = A \cap C$. Clearly, $B \subset A$ and by the statement (1), $K \subseteq B\Gamma N$. This proves the statement (2). □

Let P be a proper ideal in the Γ-ring M. It is clear that the following conditions are equivallent.

(1) P is semi-prime.
(2) For any $a \in M$, if $a\gamma_0 a \in P$ then $a \in P$.
(3) For any $a \in M$ and $n \in \mathbb{N}$, if $(a\gamma_0)^n a \in P$ then $a \in P$.

Proposition 3.7. *Let C be an ideal in Γ-ring M and A be the set of all $x \in M$ such that $(x\gamma_0)^n x \in C$ for some $n \in \mathbb{N} \cup \{0\}$, where $(x\gamma_0)^0 x = x$. Then $A = P(C)$.*

Proof. Suppose that $x \in A$. So $(x\gamma_0)^n x \in C$ for some $n \in \mathbb{N} \cup \{0\}$. Let P be a semi-prime ideal in M containing C. So $x \in P$. It follows from Proposition 2.2 that $x \in P(C)$. Thus $A \subseteq P(C)$. Now suppose $x \notin A$. Let Σ be the set of all ideals I in M such that $C \subseteq I$ and

$(x\gamma_0)^n x \notin I$ for any $n \in \mathbb{N} \cup \{0\}$. By Zorn's Lemma, Σ has maximal element P. Suppose that $z, y \notin P$. Then there exists $m \in \mathbb{N} \cup \{0\}$ such that $(x\gamma_0)^m x \in P+ < z\gamma_0 y >$. Hence $P+ < z\gamma_0 y > \notin \Sigma$ and so $z\gamma_0 y \notin P$. Now if $z = y$, by the above argument $z \notin P$ implies that $z\gamma_0 z \notin P$. So P is semi-prime and $x \notin P$. Hence, by Proposition 2.2, $x \notin P(C)$. Thus $x \notin A$ implies that $x \notin P(C)$, whence $P(C) \subseteq A$. □

Proposition 3.8. *Let J be a faithful multiplication ideal in the Γ-ring M and A, B be two ideals in M. Then, $A\Gamma J \subseteq B\Gamma J$ if and only if either $A \subseteq B$ or $J = [B : A]\Gamma J$.*

Proof. Let $A \nsubseteq B$. Note that $[B : A] = \bigcap_{a \in X} [B :< a >]$ where X is the set of all elements $a \in A$ with $a \notin B$. By Proposition 3.6,

$$[B : A]\Gamma J = \bigcap_{a \in X} ([B :< a >]\Gamma J)$$

If for every $a \in X$, $J = [B :< a >]\Gamma J$, then $J = [B : A]\Gamma J$, which finishes the proof. Let $a \in X$ and $C = [B :< a >]$. It is clear that $C \neq M$. Let Ω denote the collection of all semi-prime ideals P in M containing C. Suppose that there exists $P \in \Omega$ such that $J \neq P\Gamma J$ and $x \in J \setminus P\Gamma J$. Since J is a multiplication ideal in the Γ-ring M, we conclude that there the exists an ideal D in M such that $< x >= J\Gamma D$ and $D \nsubseteq P$. Thus $c\Gamma J \subseteq < x >$ for some $c \in D \setminus P$. Now we have $c\Gamma a\Gamma J \subseteq B\Gamma < x >$. It is easily to show that for any $\gamma \in \Gamma$, there exist $\gamma_1 \in \Gamma$ and $b \in B$ such that $(c\gamma a - 1\gamma_1 b)\gamma_0 x = 0$, it follows that $(c\gamma a - 1\gamma_1 b)\Gamma c\Gamma J = 0$. Hence $c\gamma c \in [B :< a >] = C$. Since P is a semi-prime ideal containing C, we conclude that $c \in P$, a contradiction. Therefore for every $P \in \Omega$, $J = P\Gamma J$ and, by Propositions 2.2 and 3.6, $J = P(C)\Gamma J$. Let $j \in J$. It is easily to show that $< j >= P(C)\Gamma < j >$. Then there exists $s \in P(C)$ such that for every $n \in \mathbb{N}$, $j = (s\gamma_0)^n j$. By Proposition 3.7, there exists $t \in \mathbb{N} \cup \{0\}$ such that $(s\gamma_0)^t s \in C$, it follows that $j = (s\gamma_0)^t s\gamma_0 j \in C\Gamma J$, i.e., $J \subseteq C\Gamma J$. Hence $C\Gamma J = J$. The converse is evident. □

Let M be a Γ-ring and let $Mat_{n\times n}(M)$ be the set of all $n \times n$ matrices over M.

Definition 3.9. Let M be a Γ-ring and $A = (a_{ij}) \in Mat_{n\times n}(M)$. If σ is a permutation on $\{1, 2, \ldots, n\}$, let $sign(\sigma) = 1$ if σ is an even permutation, and $sign(\sigma) = -1$ if σ is an odd permutation. The determinant is defined by

$$det_\Gamma(A) = \sum_{\text{all } \sigma} sign(\sigma) a_{1,\sigma(1)}\gamma_0 a_{2,\sigma(2)}\gamma_0 \cdots \gamma_0 a_{n,\sigma(n)}.$$

Let $M_{i,j}$ be the determinant of the $(n-1)\times(n-1)$ matrix obtained by removing row i and column j from A. Let $C_{i,j} = (-1)^{i+j}M_{i,j}$. $M_{i,j}$ and $C_{i,j}$ are called the (i,j) minor and cofactor of A.

Proposition 3.10. *For any* $1 \leq i \leq n$, $det_\Gamma(A) = a_{i1}\gamma_0 C_{i,1} + a_{i2}\gamma_0 C_{i,2} + \cdots + a_{in}\gamma_0 C_{i,n}$. *For any* $1 \leq j \leq n$, $det_\Gamma(A) = a_{1j}\gamma_0 C_{1,j} + a_{2j}\gamma_0 C_{2,j} + \cdots + a_{nj}\gamma_0 C_{n,j}$.

Let M be a Γ-ring and $\{a_i | i \in \mathbb{N}_n\} \subseteq M$. It is clear that

$$< a_1, \ldots, a_n >= \{\sum_{i=1}^{n} m_i\gamma_0 a_i | \forall i \in \mathbb{N}_n (m_i \in M\}.$$

Also, if I is an ideal of the Γ-ring M and $J =< a_1, \ldots, a_n >$, then

$$I\Gamma J = \{x_1\gamma_0 a_1 + \ldots + x_n\gamma_0 a_n | x_i \in I, \text{ for all } 1 \leq i \leq n\}.$$

Proposition 3.11. *Let M be a Γ-ring, I an ideal in M, J an ideal generated by n elements, and x an element of M satisfying $x\Gamma J \subseteq I\Gamma J$. Then there exists $y \in I$ such that $((x\gamma_0)^{n-1}x + y)\gamma_0 J = 0$.*

Proof. If $J =< a_1, \ldots, a_n >$, then there exist $y_{i1}, \ldots, y_{in} \in I$ such that

$$x\gamma_0 a_i = \sum_{j \in \mathbb{N}_n} y_{ij}\gamma_0 a_j.$$

Now we put

$$B = \begin{bmatrix} x - y_{11} & -y_{12} & \cdots & -y_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -y_{n1} & -y_{n2} & \cdots & x - y_{nn} \end{bmatrix}.$$

It is clear that there exists $y \in I$ such that $det(B) = ((x\gamma_0)^{n-1}x) + y)$ and also, for every $1 \leq i \leq n$, $(detB)\gamma_0 a_i = 0$. Therefore $((x\gamma_0)^{n-1}x + y)\gamma_0 J = 0$. □

We denote by S_Γ, the set of all finitely generated faithful multiplication ideals in the Γ-ring M.

Proposition 3.12. *Let I be an ideal of the Γ-ring M. If $I\Gamma J = J$ for some $J \in S_\Gamma$, then there exists $i \in I$ such that $(1 - i)\gamma_0 J = 0$.*

Proof. We know that $1\Gamma J = J$. Now for $x = 1$ in Proposition 3.11, there exists $n \in \mathbb{N}$ such that $((1\gamma_0)^n 1 + y)\gamma_0 J = 0$ and by setting $i = -y$ the proof will be completed. □

Corollary 3.13. *Let A, B be two ideals of the Γ-ring M and $J \in S_\Gamma$. Then $A \subseteq B$ if and only if $A\Gamma J \subseteq B\Gamma J$.*

Proof. Assume that $A\Gamma J \subseteq B\Gamma J$, then by Proposition 3.8, $A \subseteq B$ or $J = [B : A]\Gamma J$. Suppose that $J = [B : A]\Gamma J$. By Proposition 3.12, there exists $r \in [B : A]$ such that $(1 - r)\gamma_0 J = 0$. Since $J \in S_\Gamma$, we conclude that $r = 1$ and so $A = 1\Gamma A \subseteq B$. The converse is evident. □

Lemma 3.14. *Let I be a multiplication ideal of the Γ-ring M and $I \subseteq J$. Then*

$J = I\Gamma[J : I]$.

Proof. Since I is a multiplication ideal of M, then $J = I\Gamma G$ for some ideal G of M, and $G \subseteq [J : I]$. Therefore $J \subseteq I\Gamma[J : I]$. On the other hand we can see easily that $I\Gamma[J : I] \subseteq J$. So $J = I\Gamma[J : I]$. □

Definition 3.15. Let M be a Γ-ring. A left M_Γ-module is an additive abelian group A together with a mapping $\cdot : M \times \Gamma \times A \longrightarrow A$ (the image of (m, γ, a) is denoted by $m\gamma a$), such that for all $a, a_1, a_2 \in A$, $\gamma, \gamma_1, \gamma_2 \in \Gamma$, and $m, m_1, m_2 \in M$ the following hold:

(1) $m\gamma(a_1 + a_2) = m\gamma a_1 + m\gamma a_2$ and $(m_1 + m_2)\gamma a = m_1\gamma a + m_2\gamma a$,
(2) $m_1\gamma_1(m_2\gamma_2 a) = (m_1\gamma_1 m_2)\gamma_2 a$,
(3) $1\gamma_0 a = a$.

A right M_Γ-module is defined in a similar way.

Definition 3.16. If A is a left M_Γ-module and $\mathcal{S}$ is the set of all M_Γ-submodules B of A such that $B \neq A$, then $\mathcal{S}$ is partially ordered by set-theoretic inclusion. B is a maximal M_Γ-submodule if and only if B is a maximal element in the partially ordered set $\mathcal{S}$.

Proposition 3.17. *If A is a non-zero finitely generated left M_Γ-module, then the following statements hold.*

(1) *If K is a proper M_Γ-submodule of A, then there exists a maximal M_Γ-submodule of A which contains K.*
(2) *A has a maximal M_Γ-submodule.*

Proof. (1) Let $A = \langle a_1, \ldots, a_n |$ and

$$\mathcal{S} = \{L : K \subseteq L \text{ and } L \text{ is a proper } M_\Gamma\text{-submodule of } A\}.$$

$\mathcal{S}$ is partially ordered by inclusion and note that $\mathcal{S} \neq \emptyset$, since $K \in \mathcal{S}$. If $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, then $L = \bigcup_{\lambda\in\Lambda} L_\lambda$ is a M_Γ-submodule of A. We show that $L \neq A$. If $L = A$, then for every $1 \leq i \leq n$, there exists $\lambda_i \in \Lambda$ such that $a_i \in L_{\lambda_i}$. Since $\{L_\lambda\}_{\lambda\in\Lambda}$ is a chain in $\mathcal{S}$, we conclude that there exists $1 \leq j \leq n$ such that $a_1, \ldots, a_n \in L_{\lambda_j}$. Therefore $A = L_{\lambda_j} \in \mathcal{S}$ which contradicts the fact that $A \notin \mathcal{S}$. It follows easily that L is an upper bound for $\{L_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{S}$. By Zorn's Lemma, there exists a proper M_Γ-submodule B of A that is maximal in $\mathcal{S}$. It is a clear that B a maximal M_Γ-submodule of A containing K.

(2) By part (1), it suffices to we put $K = (0)$. □

Proposition 3.18. *Let J be a finitely generated ideal of the Γ-ring M contained in multiplication ideal I. If $A = ann(J)$, then $\frac{I}{A\Gamma I}$ is finitely generated.*

Proof. Suppse that $B = A + \sum_{x\in I}[< x >: I]$. If $B \neq M$ then, by Proposition 3.17, there exists a maximal ideal P of the Γ-ring M such that $B \subseteq P$. By Lemma 3.14, $< x >= [< x >: I]\Gamma I \subseteq P\Gamma I$ for any $x \in I$, it follows that $I \subseteq P\Gamma I$. Since $P\Gamma I \subseteq I$, we conclude that $I = P\Gamma I$. By hypothesis, there exists $m_1, \ldots, m_k \in J$ such that $J =< m_1, \ldots, m_k >$. Since I is a multiplication ideal, we can then conclude from Lemma 3.14 that for each $1 \leq i \leq k$, $< m_i >= [< m_i >: I]\Gamma I = [< m_i >: I]\Gamma P\Gamma I =< m_i > \Gamma P$. Therefore, there exists $p_i \in P$ such that $(1 - p_i)\gamma_0 m_i = 0$, for each $1 \leq i \in\leq k$. If we put $p = 1 - (1 - p_1)\gamma_0 \ldots \gamma_0(1 - p_k)$, then $p \in P$ and $(1 - p)\Gamma J = 0$. Hence $(1 - p) \in Ann(J) \subseteq B \subseteq P$, it follows that $1 \in P$, a contradiction. Thus $B = M$ and there exists $x_1, x_2, \ldots x_n \in I$ such that $1 \in [< x_1 >: I] + \cdots + [< x_n >: I] + A$. Therefore $I =< x_1 > + \cdots + < x_n > +A\Gamma I$. On the other hand, $\frac{I}{A\Gamma I} =< x_1 + A\Gamma I, \ldots, x_n + A\Gamma I >$, then $\frac{I}{A\Gamma I}$ is finitely generated. □

Proposition 3.19. *Let I be a multiplication ideal of the Γ-ring M. I is finitely generated if and only if $ann(I) = ann(J)$ for some finitely generated ideal J contained in I.*

Proof. Suppose that $ann(I) = ann(J)$ for some finitely generated ideal J contained in I. By Proposition 3.18, $\frac{I}{ann(J)\Gamma I}$ is finitely generated. On the other hand $\frac{I}{ann(J)\Gamma I} = \frac{I}{ann(I)\Gamma I} \cong I$. Hence I is a finitely generated ideal of M. For the converse it's enough to we put $J = I$. □

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Ali Akbar Estaji
Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O.Box 397, Sabzevar, Iran
Email: `aa_estaji@yahoo.com`

Ali Saghafi Khorasani
Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O.Box 397, Sabzevar, Iran
Email: `saghafiali21@yahoo.com`

Samaneh Baghdari
Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O.Box 397, Sabzevar, Iran
Email: `m.baghdari@yahoo.com`