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ROUGH SOFT SETS AND ROUGH SOFT GROUPS

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Abstract. In this paper we discuss some basic properties of rough soft sets and hence we introduce the notion of rough soft group. Basic properties of rough soft group are presented and supported by some illustrative examples.

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1. Introduction

Problems in economics, engineering, environmental science, social science, medical science and most of problems in every day life have various uncertainties. To solve these imprecise problems, methods in classical mathematics are not always adequate. Alternatively, some kind of theories such as probability theory, fuzzy set theory [12], rough set theory [10], soft set theory [9], vague set theory etc. are well known mathematical tools to deal with uncertainties. In 1982, Pawlak [10] introduced the concept of rough sets. In 1999, Molodtsov [9] proposed the soft set theory as a new mathematical tool for dealing with uncertainties which is free from the difficulties affecting existing methods.
Presently, works on soft set theory are progressing rapidly. Maji et al. [7] defined and studied several operations on soft sets. Aktas and Cagman [1] related soft set to fuzzy sets and rough sets, providing examples to clarify their differences and also defined soft groups and derived their basic properties. Feng et al. [5] initiated the study of soft semirings by using soft set theory. Ali et al. [2], A. Sezgin, A. O. Atagun [11]

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introduced some new operations on soft sets and illustrated their interconnections. In recent year, Feng et al. [4] introduced rough soft set, soft rough set, soft rough fuzzy set and examined related properties. D. Meng et al. [8] proposed soft fuzzy rough set and surveyed their related properties.
In this study we deal with the algebraic structure of group by applying rough soft set theory and examine their properties with suitable examples. This paper is organized as follows. In section 3, we verify some properties related to rough soft sets with respect to 'AND' operation and 'OR' operation. In section 4, we introduce the notion of rough soft group and focus on their algebraic properties with several illustrating examples. Proofs of certain results in the sequel are routine. However, we include them for the sake of completeness.

2. Preliminaries

Throughout this paper, (U, R) denotes a Pawlak approximation space and we write everywhere "Pawlak approximation space" with capital letter "P". This section contains some basic definitions and results which will be needed in the sequel. The symbol $\square$ marks the end of a proof.

Definition 2.1 ([4])**.** Let (U, R) be P and $\Omega = (F, A)$ be a soft set over U. The lower and upper approximations of Ω in (U, R) are denoted by $R_*(\Omega) = (F_*, A)$ and $R^*(\Omega) = (F^*, A)$, which are soft sets over U with the set valued mapping given by:

$$F_*(x) = R_*(\,F(x)\,) = \{\, y \in U : [y]_R \subseteq F(x)\,\},$$
$$F^*(x) = R^*(\,F(x)\,) = \{\, y \in U : [y]_R \cap F(x) \neq \phi\,\}$$

for all $x \in A$. The operators R_* and R^* are called the lower and upper rough approximation operators on soft sets. If $R_*(\Omega) = R^*(\Omega)$, the soft set Ω is said to be definable; otherwise Ω is called rough soft set.

Theorem 2.2 ([4])**.** *Suppose that (U, R) is P and $\Omega = (F, A)$ is a soft set over U. Then we have*
1. $R_*(\Omega) \subseteq \Omega \subseteq R^*(\Omega)$,
2. $R_*(\,R_*(\Omega)\,) = R_*(\Omega)$,
3. $R^*(\,R^*(\Omega)\,) = R^*(\Omega)$,
4. $R^*(\,R_*(\Omega)\,) = R_*(\Omega)$,
5. $R_*(\,R^*(\Omega)\,) = R^*(\Omega)$.

Definition 2.3 ([2])**.** Let (F, A) and (G, B) be two soft sets over a common universe U.

(1) The extended intersection of (F, A) and (G, B), denoted by $(F, A) \sqcap_{\mathcal{E}} (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$\begin{aligned} H(e) &= F(e), && \text{if } e \in A \setminus B, \\ &= G(e), && \text{if } e \in B \setminus A, \\ &= F(e) \cap G(e), && \text{if } e \in A \cap B. \end{aligned}$$

(2) The restricted intersection of (F, A) and (G, B), denoted by $(F, A) \Cap (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

(3) The extended union of (F, A) and (G, B), denoted by $(F, A) \widetilde{\cup} (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$\begin{aligned} H(e) &= F(e), && \text{if } e \in A \setminus B, \\ &= G(e), && \text{if } e \in B \setminus A, \\ &= F(e) \cup G(e), && \text{if } e \in A \cap B. \end{aligned}$$

(4) The restricted union of (F, A) and (G, B), denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cup G(e)$ for all $e \in C$.

Note that restricted intersection was also known as bi-intersection in Feng et al. (2008), and extended union was at first introduced and called union by Maji et al. (2003).

Theorem 2.4 ([4]). *Suppose that (U, R) is P and $\Omega = (F, A)$, $\Psi = (G, B)$ are soft sets over U. Then we have*

1. $R_*(\Omega \Cap \Psi) = R_*(\Omega) \Cap R_*(\Psi)$,
2. $R_*(\Omega \sqcap_{\mathcal{E}} \Psi) = R_*(\Omega) \sqcap_{\mathcal{E}} R_*(\Psi)$,
3. $R^*(\Omega \Cap \Psi) \subseteq R^*(\Omega) \Cap R^*(\Psi)$,
4. $R^*(\Omega \sqcap_{\mathcal{E}} \Psi) \subseteq R^*(\Omega) \sqcap_{\mathcal{E}} R^*(\Psi)$,
5. $R_*(\Omega \cup_{\mathcal{R}} \Psi) \supseteq R_*(\Omega) \cup_{\mathcal{R}} R_*(\Psi)$,
6. $R_*(\Omega \widetilde{\cup} \Psi) \supseteq R_*(\Omega) \widetilde{\cup} R_*(\Psi)$,
7. $R^*(\Omega \cup_{\mathcal{R}} \Psi) = R^*(\Omega) \cup_{\mathcal{R}} R^*(\Psi)$,
8. $R^*(\Omega \widetilde{\cup} \Psi) = R^*(\Omega) \widetilde{\cup} R^*(\Psi)$,
9. $\Omega \subseteq \Psi \Rightarrow R_*(\Omega) \subseteq R_*(\Psi)$, $R^*(\Omega) \subseteq R^*(\Psi)$.

Definition 2.5 ([7]). For a soft set (F, A), the set $Supp(F, A) = \{x \in A : F(x) \neq \phi\}$ is called the support of the soft set (F, A). Thus the null soft set is a soft set with an empty support, and we say that soft set (F, A) is non-null if $Supp(F, A) \neq \phi$.

3. On operations of soft sets in pawlak approximation space

Here we state two operations on soft sets, introduced by Maji et al. [7] and then we prove a few theorems.

Definition 3.1. If (F, A), and (G, B), are two soft sets over a common universe U, then $(F, A)\, AND\, (G, B)$, denoted by $(F, A) \wedge (G, B)$, is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$, $\forall\, (x, y) \in A \times B$.

Definition 3.2. If (F, A), and (G, B), are two soft sets over a common universe U, then $(F, A)\, OR\, (G, B)$, denoted by $(F, A) \vee (G, B)$, is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cup G(y)$, $\forall\, (x, y) \in A \times B$.

Theorem 3.3. *Suppose that (U, R) is P and $\Omega = (F, A)$, $\Psi = (G, B)$ are soft sets over U. Then we have*
(1) $R^*(\Omega \wedge \Psi) \subseteq R^*(\Omega) \wedge R^*(\Psi)$,
(2) $R_*(\Omega \wedge \Psi) = R_*(\Omega) \wedge R_*(\Psi)$.

Proof. (1) Let $\Omega \wedge \Psi = (F, A) \wedge (G, B) = (H, A \times B)$. Then $H(x, y) = F(x) \cap G(y)$, $\forall\, (x, y) \in A \times B$ and $R^*(\Omega \wedge \Psi) = (H^*, A \times B)$.
Again, let $R^*(\Omega) \wedge R^*(\Psi) = (\widetilde{H}, A \times B)$. Then $\widetilde{H}(x, y) = F^*(x) \cap G^*(y), \forall\, (x, y) \in A \times B$. Now $\forall\, (x, y) \in A \times B$,
$H^*(x, y) = R^*(\, H(x, y)\,)$, by Definition 2.1
$= R^*(\, F(x) \cap G(y)\,)$
$\subseteq R^*(\, F(x)) \cap R^*(G(y)\,)$, by the property of upper rough approximation
$= F^*(x) \cap G^*(y) = \widetilde{H}(x, y)$.
So, $R^*(\Omega \wedge \Psi) \subseteq R^*(\Omega) \wedge R^*(\Psi)$.
(2) Since in proof of (1), $\Omega \wedge \Psi = (H, A \times B)$, then $R_*(\Omega \wedge \Psi) = (H_*, A \times B)$.
Let $R_*(\Omega) \wedge R_*(\Psi) = (\underline{H}, A \times B)$. Then $\underline{H}(x, y) = F_*(x) \cap G_*(y)$, $\forall\, (x, y) \in A \times B$.
Now for all $(x, y) \in A \times B$, we have
$H_*(x, y) = R_*(\, H(x, y)\,)$, by Definition 2.1
$= R_*(\, F(x \cap G(y)\,)$
$= R_*(F(x)) \cap R_*(G(y))$, by the property of lower rough approximation
$= F_*(x) \cap G_*(y) = \underline{H}(x, y)$.
So, $R_*(\Omega \wedge \Psi) = R_*(\Omega) \wedge R_*(\Psi)$. □

Theorem 3.4. *Suppose that (U, R) is P and $\Omega = (F, A)$, $\Psi = (G, B)$ are soft sets over U. Then we have*

(1) $R^*(\Omega \vee \Psi) = R^*(\Omega) \vee R^*(\Psi)$,
(2) $R_*(\Omega \vee \Psi) \supseteq R_*(\Omega) \vee R_*(\Psi)$.

Proof. (1) Let $\Omega \vee \Psi = (F, A) \vee (G, B) = (H, A \times B)$. Then $H(x, y) = F(x) \cup G(y)$, $\forall\, (x, y) \in A \times B$ and $R^*(\Omega \vee \Psi) = (H^*, A \times B)$.
Again, let $R^*(\Omega) \vee R^*(\Psi) = (\widetilde{H}, A \times B)$. Then $\widetilde{H}(x, y) = F^*(x) \cup G^*(y)$, $\forall\, (x, y) \in A \times B$. Now $\forall\, (x, y) \in A \times B$,
$H^*(x, y) = R^*(\, H(x, y)\,)$, by Definition 2.1
$= R^*(\, F(x) \cup G(y)\,)$
$= R^*(\, F(x)) \cup R^*(G(y)\,)$, by the property of upper rough approximation
$= F^*(x) \cup G^*(y) = \widetilde{H}(x, y)$.
So, $R^*(\Omega \vee \Psi) = R^*(\Omega) \vee R^*(\Psi)$.
(2) Since in proof of (1), $\Omega \vee \Psi = (H, A \times B)$, then $R_*(\Omega \vee \Psi) = (H_*, A \times B)$.
Let $R_*(\Omega) \vee R_*(\Psi) = (\underline{H}, A \times B)$. Then $\underline{H}(x, y) = F_*(x) \cup G_*(y)$, $\forall\, (x, y) \in A \times B$.
Now for all $(x, y) \in A \times B$, we have
$H_*(x, y) = R_*(\, H(x, y)\,)$, by Definition 2.1
$= R_*(\, F(x \cup G(y)\,)$
$\supseteq R_*(F(x)) \cup R_*(G(y))$, by the property of lower rough approximation
$= F_*(x) \cup G_*(y) = \underline{H}(x, y)$.
So, $R_*(\Omega \vee \Psi) \supseteq R_*(\Omega) \vee R_*(\Psi)$. □

4. Rough soft groups

Throughout this section, G (as universal set) is supposed to be an additive group and (G, R) is P where R is a congruence relation on G. At first we describe the definition of congruence relation on G and state important lemmas as in [3],[6].

Definition 4.1. An equivalence relation R on G is called a congruence relation if $(a, b) \in R$ implies $(a + x, b + x)$ and $(x + a, x + b) \in R$ for all $x \in G$. Then $[x]_R$ denotes the congruence class containing the element $x \in G$.

Lemma 4.2. *Let R be a congruence relation on G, then $(a, b) \in R$ and $(c, d) \in R$ imply $(a + c, b + d) \in R$ and $(-a, -b) \in R$ for all $a, b, c, d \in G$.*

Lemma 4.3. *Let R be a congruence relation on G. If $a, b \in G$, then*
(1) $[a]_R + [b]_R = [a + b]_R$,
(2) $[-a]_R = -[a]_R$.

Following the definition of soft semiring as in [5], we give the definition of soft group. The definition of soft group was given also in [1].

Definition 4.4. A non null soft set (F, A) over G is said to be a soft group over G if $F(x)$ is a subgroup of G for all $x \in Supp(F, A)$.

Definition 4.5. A non null soft set $\Omega = (F, A)$ over G is said to be an upper rough soft group over G if $R^*(\Omega)$ is a soft group over G.

Theorem 4.6. *Let (G, R) be P. If $\Omega = (F, A)$ is a soft group over G, then Ω is an upper rough soft group over G.*

Proof. Since $\Omega = (F, A)$ is a soft group over G, then $F(x)$ is a subgroup of G for all $x \in Supp(F, A)$, by Definition 4.4. Now $R^*(\Omega) = (F^*, A)$, where $\forall\, x \in A$,

$$F^*(x) = R^*(\, F(x)\,) = \{\, y \in G : [y]_R \cap F(x) \neq \phi\,\}.$$

Let $a, b \in F^*(x)$, this implies $[a]_R \cap F(x) \neq \phi$ and $[b]_R \cap F(x) \neq \phi$. So there exist $c \in [a]_R \cap F(x)$, $d \in [b]_R \cap F(x)$. Since $F(x)$ is a subgroup of G, $c - d \in F(x)$. By Lemma 4.3, we have $c - d \in [a]_R - [b]_R = [a - b]_R$. Hence $[a - b]_R \cap F(x) \neq \phi$. This implies $a - b \in F^*(x)$. So, $F^*(x)$ is a subgroup of $G, \forall\, x \in Supp(F^*, A)$. Therefore $R^*(\Omega)$ is a soft group over G. Hence by Definition 4.5, Ω is an upper rough soft group over G. □

Example 4.7. *Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be the group of integers module 6 w.r.t $+_6$, taken as universal set and R be the congruence relation on $\mathbb{Z}_6$ with the congruence classes $[0]_R = \{0, 2, 4\}$, $[1]_R = \{1, 3, 5\}$. Let $\Omega = (F, A)$ be a soft set over $\mathbb{Z}_6$, where $A = \mathbb{Z}_6$ and $F : A \to \mathcal{P}(\mathbb{Z}_6)$ is a set valued function defined by*

$$F(x) = \{\, y \in \mathbb{Z}_6 \,:\, x \,\rho\, y \,\Leftrightarrow x +_6 y \,\in\, \{0, 1, 2\}\,\}, \forall\, x \in A.$$

Then $F(0) = \{0, 1, 2\}$, $F(1) = \{0, 1, 5\}$, $F(2) = \{0, 4, 5\}$, $F(3) = \{3, 4, 5\}$, $F(4) = \{2, 3, 4\}$, $F(5) = \{1, 2, 3\}$ are not subgroups of G. Hence Ω is not a soft group over $\mathbb{Z}_6$. But by Definition 2.1, $R^*(\Omega) = (F^*, A)$, where $\forall\, x \in A$,

$$F^*(x) = R^*(\, F(x)\,) = \{\, y \in \mathbb{Z}_6 : [y]_R \cap F(x) \neq \phi\,\} = \mathbb{Z}_6.$$

So, $F^*(x)$ is a subgroup of $\mathbb{Z}_6, \forall\, x \in A$. Hence $R^*(\Omega)$ is a soft group over $\mathbb{Z}_6$.

Theorem 4.8. *Let (G, R) be P and $\Omega = (F, A)$ be a soft group over G. If $R_*(\Omega)$ is non null, then $R_*(\Omega) = \Omega$.*

Proof. Clearly, by Theorem 2.2, $R_*(\Omega) \subseteq \Omega$. To prove the theorem we have only to prove $\Omega \subseteq R_*(\Omega)$. Suppose $R_*(\Omega)$ is non null. This implies $Supp(F_*, A) \neq \phi$. Let $F_*(x) \neq \phi$, then there exists $c \in F_*(x)$. So, $[0]_R = [c+(-c)]_R = [c]_R + [-c]_R = [c]_R + (-[c]_R) \subseteq F(x)+F(x) = F(x)$. Let a be an arbitrary element of $F(x)$. Since $F(x)$ is a subgroup of G, then $a + [0]_R \subseteq a + F(x) = F(x)$. Again

$$y \in a + [0]_R \Leftrightarrow y - a \in [0]_R \Leftrightarrow (y-a, 0) \in R \Leftrightarrow (y, a) \in R \Leftrightarrow y \in [a]_R.$$

So, we get $[a]_R \subseteq F(x)$, which implies $a \in F_*(x)$. Therefore $F(x) \subseteq F_*(x)$ for all $x \in Supp(F_*, A)$. Hence $\Omega \subseteq R_*(\Omega)$. □

Definition 4.9. A non null soft set $\Omega = (F, A)$ over G is said to be a rough soft group over G if $R_*(\Omega)$ and $R^*(\Omega)$ are both soft groups over G.

Corollary 4.10. *If $\Omega = (F, A)$ is a soft group over G and $R_*(\Omega)$ is non null, then Ω is a rough soft group over G.*

Some parts of the following theorem are proved earlier in [1] and in other research articles by some authors, we shall give a proof for completeness.

Theorem 4.11. *Let (F, A) and (H, B) be two soft group over G. Then*
(1) $(F, A) \Cap (H, B)$ is a soft group over G if it is non null.
(2) $(F, A) \sqcap_{\mathcal{E}} (H, B)$ is a soft group over G if it is non null.
(3) $(F, A) \cup_{\mathcal{R}} (H, B)$ is a soft group over G if either $(F, A) \subseteq (H, B)$ or $(H, B) \subseteq (F, A)$.
(4) $(F, A) \,\widetilde{\cup}\, (H, B)$ is soft group over G if either $A \cap B = \phi$ or $(F, A) \subseteq (H, B)$ or $(H, B) \subseteq (F, A)$.
(5) $(F, A) \wedge (H, B)$ is a soft group over G if it is non null.

Proof. (1) From Definition 2.3, we can write $(F, A) \Cap (H, B) = (K, C)$, where $C = A \cap B$ and $K(e) = F(e) \cap H(e)$, $\forall\, e \in C$. By hypothesis (K, C) is non null. If $e \in Supp(K, C)$, then $K(e) = F(e) \cap H(e) \neq \phi$. As $F(e)$ and $H(e)$ are subgroups of G, $K(e)$ is a subgroup of G. Hence $(F, A) \Cap (H, B)$ is a soft group over G.
(2) From Definition 2.3, we can write $(F, A) \sqcap_{\mathcal{E}} (H, B) = (K, C)$, where $C = A \cup B$ and $K(e) = F(e)$ if $e \in A \setminus B$, $K(e) = H(e)$ if $e \in B \setminus A$, $K(e) = F(e) \cap H(e)$ if $e \in A \cap B$. By hypothesis $Supp(K, C) \neq \phi$. If $e \in Supp(K, C)$, then $e \in A \setminus B$ or $B \setminus A$ or $A \cap B$. For all of the cases respectively $F(e)$, $H(e)$, $F(e) \cap H(e)$ are nonempty. As $F(e)$ and $H(e)$ are subgroups of G, $K(e)$ is a subgroup of G for all of the cases.

Hence $(F,A)\sqcap_{\mathcal{E}}(H,B)$ is a soft group over G.
(3) From Definition 2.3, we can write $(F,A)\cup_{\mathcal{R}}(H,B)=(K,C)$, where $C=A\cap B$ and $K(e)=F(e)\cup H(e)$, $\forall\, e\in C$. By hypothesis if $(F,A)\subseteq(H,B)$, then $A\subseteq B$ and $F(e)\subseteq H(e), \forall\, e\in A$. Hence $C=A$ and $K(e)=H(e), \forall\, e\in C$. As $H(e)$ is a subgroups of G, $K(e)$ is a subgroup of G. Therefore $(F,A)\cup_{\mathcal{R}}(H,B)$ is a soft group over G. Similarly the result follows if $(H,B)\subseteq(F,A)$.
(4) From Definition 2.3, we can write $(F,A)\,\widetilde{\cup}\,(H,B)=(K,C)$ where $C=A\cup B$ and $K(e)=F(e)$ if $e\in A\setminus B$, $K(e)=H(e)$ if $e\in B\setminus A$, $K(e)=F(e)\cup H(e)$ if $e\in A\cap B$. By hypothesis if $A\cap B=\phi$, then $\forall\, e\in Supp(K,C)$, either $e\in A\setminus B$ or $e\in B\setminus A$. For both cases $K(e)$ is a subgroup of G, as $F(e)$, $H(e)$ are subgroups of G. By hypothesis if $(F,A)\subseteq(H,B)$, then $A\subseteq B$ and $F(e)\subseteq H(e), \forall\, e\in A$. Hence $C=B$ and $K(e)=H(e), \forall\, e\in C$. Therefore $K(e)$ is a subgroup of G, $\forall\, e\in Supp(K,C)$. Similar result follows if $(H,B)\subseteq(F,A)$. Hence $(F,A)\,\widetilde{\cup}\,(H,B)$ is soft group over G.
(5) From Definition 3.1, we can write $(F,A)\wedge(H,B)=(K,A\times B)$, where $K(x,y)=F(x)\cap H(y), \forall\,(x,y)\in A\times B$. By hypothesis, $Supp(K, A\times B)\neq\phi$. Then $\forall\,(x,y)\in Supp(K,A\times B)$, $K(x,y)=F(x)\cap H(y)\neq\phi$. As $F(x)$ and $H(y)$ are subgroups of G, $K(x,y)$ is a subgroup of G. Hence $(F,A)\wedge(H,B)$ is a soft group over G. $\square$

Theorem 4.12. *Let (G,R) be P. Also let $\Omega=(F,A)$, $\Psi=(H,B)$ are soft sets over G such that $R_*(\Omega)$, $R_*(\Psi)$ are soft group over G.*

1. *If $R_*(\Omega)\Cap R_*(\Psi)$ is non null then $R_*(\Omega\Cap\Psi)$ is a soft group over G.*
2. *If $R_*(\Omega)\sqcap_{\mathcal{E}}R_*(\Psi)$ is non null then $R_*(\Omega\sqcap_{\mathcal{E}}\Psi)$ is a soft group over G.*

Proof. (1) By hypothesis $R_*(\Omega)\Cap R_*(\Psi)$ is non null, hence by Theorem 4.11, $R_*(\Omega)\Cap R_*(\Psi)$ is a soft group over G. From Theorem 2.4, $R_*(\Omega)\Cap R_*(\Psi)=R_*(\Omega\Cap\Psi)$, hence $R_*(\Omega\Cap\Psi)$ is a soft group over G.
(2) By hypothesis $R_*(\Omega)\sqcap_{\mathcal{E}}R_*(\Psi)$ is non null, hence by Theorem 4.11, $R_*(\Omega)\sqcap_{\mathcal{E}}R_*(\Psi)$ is a soft group over G. From Theorem 2.4, $R_*(\Omega\sqcap_{\mathcal{E}}\Psi)=R_*(\Omega)\sqcap_{\mathcal{E}}R_*(\Psi)$, hence $R_*(\Omega\sqcap_{\mathcal{E}}\Psi)$ is a soft group over G. $\square$

Theorem 4.13. *Let (G,R) be P. Also let $\Omega=(F,A)$, $\Psi=(H,B)$ are soft sets over G such that $R^*(\Omega)$, $R^*(\Psi)$ are soft group over G.*
(1) If $R^*(\Omega)\subseteq R^*(\Psi)$ or $R^*(\Psi)\subseteq R^*(\Omega)$, then $R^*(\Omega\cup_{\mathcal{R}}\Psi)$ is soft group over G.

(2) *If either* $A \cap B = \phi$ *or* $R^*(\Omega) \subseteq R^*(\Psi)$ *or* $R^*(\Psi) \subseteq R^*(\Omega)$, *then* $R^*(\Omega \,\widetilde{\cup}\, \Psi)$ *is a soft group over* G.

Proof. (1) If $R^*(\Omega) \subseteq R^*(\Psi)$ or $R^*(\Psi) \subseteq R^*(\Omega)$, then by Theorem 4.11, $R^*(\Omega) \cup_{\mathcal{R}} R^*(\Psi)$ is a soft group over G. By Theorem 2.4, we have $R^*(\Omega \cup_{\mathcal{R}} \Psi) = R^*(\Omega) \cup_{\mathcal{R}} R^*(\Psi)$. So, $R^*(\Omega \cup_{\mathcal{R}} \Psi)$ is a soft group over G.
(2) If either $A \cap B = \phi$ or $R^*(\Omega) \subseteq R^*(\Psi)$ or $R^*(\Psi) \subseteq R^*(\Omega)$, then by Theorem 4.11, $R^*(\Omega) \,\widetilde{\cup}\, R^*(\Psi)$ is a soft group over G. By Theorem 2.4, we have $R^*(\Omega \,\widetilde{\cup}\, \Psi) = R^*(\Omega) \,\widetilde{\cup}\, R^*(\Psi)$. So, $R^*(\Omega \,\widetilde{\cup}\, \Psi)$ is a soft group over G. □

Theorem 4.12 may not be true for upper rough approximation as seen in the following example.

Example 4.14. *Let* G *be a Klein's 4-group. Composition table is given by:*

0	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let R *be a congruence relation on* G *given by*

$$R = \{(e,e),\, (e,b),\, (a,a),\, (a,c),\, (b,e),\, (b,b),\, (c,a),\, (c,c)\}.$$

Hence the R*-congruence classes are* $[e]_R = [b]_R = \{e,b\}$, $[a]_R = [c]_R = \{a,c\}$.
Let $\Omega = (F,A)$, $\Psi = (H,B)$ *be two soft set over* G, *where* $A = B = G$, *and*

$$F(x) = \{\ y \in G : x\,\rho\,y \Leftrightarrow xy \in \{e,a\}\ \},$$
$$H(x) = \{\ y \in G : x\,\rho\,y \Leftrightarrow xy \in \{e,c\}\ \},$$

$\forall\, x \in G$. *Then* $F(e) = F(a) = \{e,a\}$, $F(b) = F(c) = \{b,c\}$ *and* $H(e) = H(c) = \{e,c\}$, $H(a) = H(b) = \{a,b\}$. *So,* Ω, Ψ *are not soft groups over* G.
From Definition 2.1, $R^*(\Omega) = (F^*,A)$ *and* $R^*(\Psi) = (H^*,B)$, *where* $F^*(x) = G\ \ \forall\, x \in A$ *and* $H^*(x) = G\ \ \forall\, x \in B$. *Hence* $R^*(\Omega)$, $R^*(\Psi)$ *are both soft groups over* G. *Also it can be shown from Definition 2.3,* $R^*(\Omega) \Cap R^*(\Psi)$, $R^*(\Omega) \sqcap_{\mathcal{E}} R^*(\Psi)$ *are non null. Hence from Theorem 4.11,* $R^*(\Omega) \Cap R^*(\Psi)$, $R^*(\Omega) \sqcap_{\mathcal{E}} R^*(\Psi)$ *are soft groups over* G. *Now from Definition 2.3, we can write* $\Omega \Cap \Psi = (K,C)$, *where* $C = A \cap B = G$ *and*

$K(e) = \{e\}$, $K(a) = \{a\}$, $K(b) = \{b\}$, $K(c) = \{c\}$. From Definition 2.1, $R^*(\Omega \Cap \Psi) = (K^*, C)$, where $K^*(e) = K^*(b) = \{e, b\}$, $K^*(a) = K^*(c) = \{a, c\}$. Since $K^*(a)$, $K^*(c)$ are not subgroups of G, hence $R^*(\Omega \Cap \Psi)$ is not soft group over G.
Again from Definition 2.3, we can write $\Omega \sqcap_{\mathcal{E}} \Psi = (L, D)$, where $D = A \cup B = G$ and $L(e) = \{e\}$, $L(a) = \{a\}$, $L(b) = \{b\}$, $L(c) = \{c\}$. From Definition 2.1, $R^*(\Omega \sqcap_{\mathcal{E}} \Psi) = (L^*, D)$, where $L^*(e) = L^*(b) = \{e, b\}$, $L^*(a) = L^*(c) = \{a, c\}$. Since $L^*(a)$, $L^*(c)$ are not subgroup of G, hence $R^*(\Omega \sqcap_{\mathcal{E}} \Psi)$ is not soft group over G.

Theorem 4.13 may not be true for lower rough approximation as seen in the following example.

Example 4.15. *Let $\mathbb{Z}_{12}$ be the group of integers module 12 w.r.t. $+_{12}$, taken as universal set and R be the congruence relation on $\mathbb{Z}_{12}$ with the congruence classes $[0]_R = \{0, 4, 8\}$, $[1]_R = \{1, 5, 9\}$, $[2]_R = \{2, 6, 10\}$, $[3]_R = \{3, 7, 11\}$. Let $\Omega = (F, A)$, $\Psi = (H, B)$ be two soft set over $\mathbb{Z}_{12}$, where $A = B = \{0, 4, 8\}$ and $F : A \to \mathcal{P}(\mathbb{Z}_{12})$ is a set valued function defined by*

$$F(0) = \{0, 1, 4, 5, 8\},\ F(4) = \{0, 2, 6, 8\},\ F(8) = \{0, 1, 4, 8, 9\},$$

and $H : B \to \mathcal{P}(\mathbb{Z}_{12})$ is a set valued function defined by

$$H(0) = \{0, 1, 4, 8, 9\},\ H(4) = \{0, 2, 6, 8\},\ H(8) = \{0, 1, 4, 5, 8\}.$$

Obviously, $\Omega = (F, A)$, $\Psi = (H, B)$ are not soft group over $\mathbb{Z}_{12}$. Now from Definition 2.1, we can write $R_*(\Omega) = (F_*, A)$, $R_*(\Psi) = (H_*, B)$, where

$$F_*(0) = F_*(8) = \{0, 4, 8\},\ F_*(4) = \phi;\ \ H_*(0) = H_*(8) = \{0, 4, 8\},\ H_*(4) = \phi.$$

So, $R_*(\Omega)$, $R_*(\Psi)$ are soft group over $\mathbb{Z}_{12}$. Also it can be shown from Theorem 4.11, $R_*(\Omega) \cup_{\mathcal{R}} R_*(\Psi)$, $R_*(\Omega) \widetilde{\cup} R_*(\Psi)$ are soft group over $\mathbb{Z}_{12}$. Again from Definition 2.3, we can write $\Omega \cup_{\mathcal{R}} \Psi = (K, C)$, where $C = A \cap B = \{0, 4, 8\}$ and $K(0) = K(8) = \{0, 1, 4, 5, 8, 9\}$, $K(4) = \{0, 2, 6, 8\}$. From Definition 2.1, $R_*(\Omega \cup_{\mathcal{R}} \Psi) = (K_*, C)$, where $K_*(0) = K_*(8) = \{0, 1, 4, 5, 8, 9\}$, $K_*(4) = \phi$. Since $K^*(0)$, $K^*(8)$ are not subgroup of $\mathbb{Z}_{12}$, hence $R^*(\Omega \cup_{\mathcal{R}} \Psi)$ is not soft group over $\mathbb{Z}_{12}$.

Theorem 4.16. *Let (G, R) be P. Also let $\Omega = (F, A)$, $\Psi = (H, B)$ are soft sets over G such that $R_*(\Omega)$, $R_*(\Psi)$ are soft group over G. If $R_*(\Omega) \wedge R_*(\Psi)$ is non null then $R_*(\Omega \wedge \Psi)$ is a soft group over G.*

Proof. By hypothesis $R_*(\Omega) \wedge R_*(\Psi)$ is non null, so by Theorem 4.11, $R_*(\Omega) \wedge R_*(\Psi)$ is a soft group over G. Now from Theorem 3.3, we have $R_*(\Omega \wedge \Psi) = R_*(\Omega) \wedge R_*(\Psi)$. Therefore $R_*(\Omega \wedge \Psi)$ is a soft group over G. $\square$

Theorem 4.16 may not be true for upper rough approximation as seen in the following example.

Example 4.17. *Let (G, R) be P as in Example 4.14, where R-congruence classes are $[e]_R = [b]_R = \{e, b\}$, $[a]_R = [c]_R = \{a, c\}$. Let $\Omega = (F, A)$, $\Psi = (H, B)$ be two soft set over G, defined in Example 4.14. Then $F(e) = F(a) = \{e, a\}$, $F(b) = F(c) = \{b, c\}$ and $H(e) = H(c) = \{e, c\}$, $H(a) = H(b) = \{a, b\}$. So, Ω, Ψ are not soft group over G, but from Example 4.14, $R^*(\Omega)$, $R^*(\Psi)$ are soft group over G. Now from Definition 3.1, it can be shown that $R^*(\Omega) \wedge R^*(\Psi)$ is non null. Hence by Theorem 4.11, $R^*(\Omega) \wedge R^*(\Psi)$ is a soft group over G.*
Now from Definition 3.1, we can write $\Omega \wedge \Psi = (K, A \times B)$, where $A = B = G$ and $K(x, y) = F(x) \cap H(y), \forall (x, y) \in A \times B$. So, we have $K(e, e) = K(e, c) = K(a, e) = K(a, c) = \{e\}$; $K(e, a) = K(e, b) = K(a, a) = K(a, b) = \{a\}$; $K(b, a) = K(b, b) = K(c, a) = K(c, b) = \{b\}$; $K(b, e) = K(b, c) = K(c, e) = K(c, c) = \{c\}$.
From Definition 2.1, $R^*(\Omega \wedge \Psi) = (K^*, A \times B)$, where $K^*(e, e) = K^*(e, c) = K^*(a, e) = K^*(a, c) = K^*(b, a) = K^*(b, b) = K^*(c, a) = K^*(c, b) = \{e, b\}$; $K^*(e, a) = K^*(e, b) = K^*(a, a) = K^*(a, b) = K^*(b, e) = K^*(b, c) = K^*(c, e) = K^*(c, c) = \{a, c\}$. Since $\{a, c\}$ is not a subgroup of G, therefore $R^*(\Omega \wedge \Psi)$ is not a soft group over G.

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