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ON SPECIAL WEAK FREE (SEMI)HYPERGROUPS

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Abstract. In this paper, we study some properties of special weak free (semi)hypergroups and we generalize the Nielsen-Schreier theorem for the class of special weak free hypergroups.

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1. Introduction

The theory of hyperstructure was rose in 1934 at the 8th congress of Scandinavian Mathematicions, where Marty [7] introduced the hypergroup notion as a generalization of groups and then he proved its utility in solving some problems of groups, algebraic functions and rational fractions. Nowadays hyperstructures are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and coding theory, graphs and hy- pergraphs, probability theory, binary relations, theory of fuzzy and rough sets and automata theory, hyperalgebras and hyper-coalgebras, etc. (see [2, 6]). Surveys of the theory can be found in the books of Corsini [1], Davvaz and Leoreanu-Fotea [3], Corsini and Leoreanu [2] and Vougiouklis [11]. In this paper first we recall some basic notions of (semi)hypergroup theory and then we study the notion of special weak free (semi)hypergroup which was introduced by S. Sh. Mousavi and M. Jafarpour in [8]. Particulary, we generalize the Nielsen-Schreier theorem for the class of special weak free hypergroups.

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Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Let $(\cdot)$ be a *hyperoperation* (or *join operation*) on H, that is $(\cdot)$ is a function from $H \times H$ into $\mathcal{P}^*(H)$. If $(a,b) \in H \times H$, its image under $(\cdot)$ in $\mathcal{P}^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H, $A \cdot B = \cup\{ab \mid a \in A, b \in B\}$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called a *semihypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v,$$

and is called a *hypergroup* if it is a semihypergroup and $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset K of H is called subhypergroupoid of H if $K \cdot K \subseteq K$ and the subhypergroupoid K is called sub(semi)hypergroup of H if $(K, \cdot)$ is a (semi)hypergroup. $(H, \cdot)$ is called regular if it has at least an identity, that is, there exists an element e in H, such that for all $x \in H$, $x \in e \cdot x \cap x \cdot e$ and each element has at least one inverse, that is, if $x \in H$, then there exists $x_0 \in H$ such that $e \in x \cdot x_0 \cap x_0 \cdot x$, the set of all identities of H is denoted by $E(H)$.

Definition 1.1. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semihypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$.

Definition 1.2. Suppose that $(H, \cdot)$ is a semihypergroup. Define the relation β on H as follows:

$$a \beta b \Leftrightarrow \exists n \in \mathbb{N}^*, \exists (x_1, ..., x_n) \in H^n \text{ such that } \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

where $\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot ... \cdot x_n$.

Definition 1.3. A semihypergroup $(H, \cdot)$ is called *complete* if for all $n, m \geqslant 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ and $(y_1, y_2, ..., y_m) \in H^m$ we have the following implication:

$$\prod_{i=1}^{n} x_i \bigcap \prod_{j=1}^{m} y_j \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j.$$

Definition 1.4. If $(H,\cdot)$ is a hypergroup and $R \subseteq H \times H$ is an equivalence relation, we set

$$A \overline{\overline{R}} B \Leftrightarrow a\, R\, b, \quad \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H. The relation R is called *strongly regular on the left* (*on the right*) if $x\, R\, y \Rightarrow a \circ x \overline{\overline{R}}\, a \circ y$ ($x\, R\, y \Rightarrow x \circ a \overline{\overline{R}}\, y \circ a$ respectively), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular on the right and on the left

Let us denote by β^* the transitive closure of β. The following results hold:

Theorem 1.5. *(see* [1]*) If $(H,\cdot)$ is a semihypergroup, then β^* is the smallest equivalence strongly regular on H, with respect to the inclusion. Also $H^* :\overset{def}{=} \frac{H}{\beta^*}$ is a semigroup and the mapping $\varphi_H : H \longrightarrow H^*$, defined by $\varphi_H(x) :\overset{def}{=} \bar{x}$ is a homomorphism that we say canonical homomorphism. Moreover if $(H,\cdot)$ is a hypergroup then H^* is a group and $\omega_H = \{h|\varphi_H(h) = e_{H^*}\}$ is a subsemihypergroup of H, which is called core of H.*

Theorem 1.6. *(See* [2]*). A semihypergroup $(H, \circ)$ is complete if and only if $H = \cup A_s(s \in S)$, where S, A_s the following conditions are satisfied:*

(i) *$(S,\cdot)$ is semigroup;*
(ii) *for all $(s,t) \in S^2$ such that $s \neq t$ have $A_s \cap A_t = \oslash$;*
(iii) *if $(a, b) \in A_s \times A_t$ then $a \circ b = A_{s\cdot t}$.*

2. Special weak free (semi)hypergroups

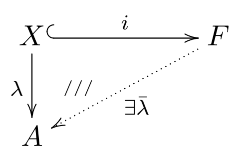
In [8], Mousavi and Jafarpour studied the notion of being free for the categories of hypergroups and semihypergroups first they showed that the concrete categories, semihypergroups (**SHypgrp**) and hypergroups (**Hypgrp**) have not free objects in the sense of universal property. Then they introduced the notion of weak free semihypergroup for the classes of semihypergroups and extension complete semihypergroups (**ECS**). In the following first we recall the definition of weak free (semi)hypergroup and then some of those theorems are mentioned. We also study some new results concerning this notion.

Definition 2.1. (See [5]). The free abelian group on a set S, is defined via its universal property. Consider a pair (F, ϕ), where F is an abelian group and $\phi : S \longrightarrow F$ is a function. F is said to be a free abelian group on S with respect to ϕ if for any abelian group G and any function $\psi : S \longrightarrow G$, there exists an unique homomorphism $f : F \longrightarrow G$ such that $\forall s \in S$, f$(\phi(s)) = \psi(s)$.

Definition 2.2. (See [8]). Suppose that $(F, \circ)$ is an object in the category **SHypgrp** and $i : X \hookrightarrow F$ is an inclusion map of sets. We say F is *weak* free on the subset X provided that:

(i) $F = \langle X \rangle = \cup\{x_1 \circ ... \circ x_k \mid x_i \in X, k \in \mathbb{N}\}$;

(ii) for all object $(K, \bullet)$ in **SHypgrp** and any map $\lambda : X \longrightarrow K$ there is a homomorphism $\bar{\lambda} : F \longrightarrow K$ such that for all $x \in X$ we have $\bar{\lambda}(x) = \lambda(x)$, that is the following diagram commutes:



Theorem 2.3. *(See [8]). Suppose that F is weak free on the non-empty subset X in the category* **SHypgrp**. *Then for all $a \in F$ there are unique elements $x_1, ..., x_n$ in X such that $a \in x_1...x_n$.*

Theorem 2.4. *(See [8]). Suppose X is a non-empty subset of F and $f : F \longrightarrow X^*$ is a surjection which is not an injection map, where X^* is the free semigroup on X in the category of semigroups such that $f(x) = x$ for every $x \in X$, then there exists a hyperoperation $*$ on F such that $(F, *)$ is a weak free semihypergroup.*

In the previous theorem the hyperoperation $*$ defined as $w_1 * w_2 = f^{-1}(f(w_1)f(w_2))$ and from now on we call the weak free semihypergroup $(F, *)$ special weak free semihypergroup on X with f or for simplicity, special weak free semihypergroup.

Proposition 2.5. *The special weak free semihypergroup $(F, *)$ is complete.*

Proof. Let $n, m \geq 2$ and $(x_1, x_2, ..., x_n) \in F^n$ and $(y_1, y_2, ..., y_m) \in F^m$. If $a \in x_1 \cdot x_2 \cdot ... \cdot x_n \cap y_1 \cdot y_2 \cdot ... \cdot y_m$ then $f(a) = f(x_1)f(x_2)...f(x_n) = f(y_1)f(y_2)...f(y_m)$. Now let $b \in x_1 \cdot x_2 \cdot ... \cdot x_n$ so $f(b) = f(a)$ and hence

$b \in f^{-1}(f(y_1)f(y_2)...f(y_m)) = y_1 \cdot y_2 \cdot ... \cdot y_m$ therefore $x_1 \cdot x_2 \cdot ... \cdot x_n \subseteq y_1 \cdot y_2 \cdot ... \cdot y_m$. Similarly we have $y_1 \cdot y_2 \cdot ... \cdot y_m \subseteq x_1 \cdot x_2 \cdot ... \cdot x_n$. Thus $x_1 \cdot x_2 \cdot ... \cdot x_n = y_1 \cdot y_2 \cdot ... \cdot y_m$.

Join spaces have been introduced by Jantosciak and Prenwitz [9]. The topic is covered in considerable detail in [10]. Jantosciak has noted that join spaces are a special case of hypergroups [4].

Definition 2.6. A semihypergroup $(H, \cdot)$ is called join space if the following conditions are satisfied:

(i) H is abelian (i.e. $x \cdot y = y \cdot x$, for every $(x, y) \in H^2$);
(ii) if $a/b \cap c/d \neq \oslash$ then $a \cdot d \cap b \cdot c \neq \oslash$, where $a/b := \{x | a \in x \cdot b\}$.

Proposition 2.7. *If $(H, \cdot)$ be an abelian complete semihyprergroup then $(H, \cdot)$ is a join space.*

Proof. The proof is straightforward. □

Proposition 2.8. *The weak free semihyprgroup $(F, *)$ is a join space if and only if X^* is a free abelian semigroup.*

Proof. Since X^* is abelian resulted F is abelian. F is abelian and complete consequently F is a join space. On the other hand let $x, y \in X^*$ since f is a surjection map there are $w_1 \in F, w_2 \in F$ such that $f(w_1) = x, f(w_2) = y$ and since F is an abelian semihypergroup therefore $w_1 * w_2 = w_2 * w_1$ and so $f^{-1}(f(w_1)f(w_2)) = f^{-1}(f(w_2)f(w_1)$ thus $\{z \in F : f(z) = f(w_1)f(w_2)\} = \{t \in F : f(t) = f(w_2)f(w_1)\}$ and hence $f(w_1)f(w_2) = f(w_2)f(w_1)$ that is $xy = yx$. □

Proposition 2.9. *If X^* is a group then the special weak free semihypergroup $(F, *)$ is regular.*

Proof. Let e be the identity element of the group X^* and $w_1 \in F$, if $y \in f^{-1}(e)$ then $w_1 * y = f^{-1}(f(w_1)e) = f^{-1}(ef(w_1)) = y * w_1$ so $w_1 \in w_1 * y \cap y * w_1$ hence any element of $f^{-1}(e)$ is an identity of the semihypergroup F.

Let $f(w_1) \in X^*$ because X^* is a group $f(w_1)$ has an inverse. let z be the inverse of $f(w_1)$ in X^*. We have $f^{-1}(f(w_1)z) = f^{-1}(zf(w_1)) = f^{-1}(e)$. If $x \in f^{-1}(z)$ then $w_1 * x = f^{-1}(f(w_1)z) = f^{-1}(zf(w_1)) = x * w_1 = f^{-1}(e)$. Thus x is an inverse of w_1 in F. □

Definition 2.10. A semihypergroup $(P, \cdot)$ is called projective if for all surjection good homomorphism $\nu : B \longrightarrow C$ and for all good homomorphism $\varphi : P \longrightarrow C$ there exists a good homomorphism $\mu : P \longrightarrow B$ such that $\varphi(x_1 \cdot ... \cdot x_n) = (\nu\mu)(x_1 \cdot ... \cdot x_n)$, where $\nu\mu$ is the composition of functions ν and μ.

Theorem 2.11. *Any special weak free semihypergroup is projective.*

Proof. Let F be a special weak free semihypergroup on X. Consider semihypergroups B, C, good homomorphism φ and surjection good homomorphism ν. let $y \in F$, $\varphi(y) = a_y$ because ν is a surjection there exists $b_y \in B$ such that $\nu(b_y) = a_y$. Because F is weak free on X therefore there exists a homomorphism μ such that for all $x \in X$ we have $\mu(x) = b_x$. Now let $w \in F$. Since F is weak free on X there exist unique elements $x_1, ..., x_n \in X$ such that $w \in x_1...x_n$. Now we have $\nu\mu(w) \in \nu\mu(x_1...x_n) = \nu\mu(x_1)...\nu\mu(x_n) = a_{x_1}...a_{x_n} = \varphi(x_1...x_n)$. □

In group theory [5], the Nielsen-Schreier theorem states that every non trivial subgroup of a free group is itself free. In the following we extend this theorem for the class of weak free special hypergroups.

Theorem 2.12. *Let $(F, \circ)$ be a special weak free hypergroup and K be a subhypergroup of F such that $K \nsubseteq \omega_F$. Then K is a weak free subhypergroup of F.*

Proof. According Proposition 2.5, F is a complete hypergroup therefore $F = \cup A_s (s \in G)$ and the following conditions valid:

(i) $(G, \cdot)$ is a group;
(ii) for all $(s,t) \in G^2$ such that $s \neq t$ have $A_s \cap A_t = \emptyset$;
(iii) if $(a,b) \in A_s \times A_t$ then $a \circ b = A_{s \cdot t}$.

Theorem 3.10 of [8] states that F is a weak free special hypergroup if and only if G is a free group. So we have G is a free group. Because K^* is a non-trivial subgroup of G (notice that $K \nsubseteq \omega_F$) the Nielsen-Schreier theorem shows that K^* is a free subgroup of G. Hence K is a weak free subhypergroup of F and the proof is complete. □

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