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NUMERICAL SOLUTION OF FREDHOLM FUZZY INTEGRAL EQUATIONS OF THE SECOND KIND VIA DIRECT METHOD USING TRIANGULAR FUNCTIONS

F. MIRZAEE*, M. PARIPOUR AND M. KOMAK YARI

ABSTRACT. In this paper, we present an efficient numerical method to solve linear Fredholm fuzzy integral equations of the second kind based on two m-sets of triangular functions. This approach needs no integration, so all calculations can be easily implemented. Moreover, the error estimate of the proposed method is given. The proposed method is discussed in details and illustrated by solving some numerical examples.

Key Words: Fuzzy number; Fredholm fuzzy integral equations; Two m-sets of triangular functions; Error estimation.
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1. Introduction

Fuzzy integral equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics, in particular in relation to fuzzy control. Usually, in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp state, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5]. Alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [10], Nanda [12] and others. While

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*Address correspondence to F. Mirzaee; E-mail: f.mirzaee@malayeru.ac.ir

Goetschel and Voxman [8] preferred a Rimann integral type approach, Kalva [10] chose to define the integral of fuzzy function, by using the Lebesgue type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma [15] who investigated the fuzzy Fredholm integral equation of second kind FFIE-2. This work which established the existence of a unique solution to FFIE-2 was followed by other work on Fredholm integral equation [14] where a fuzzy integral equation replaced an original fuzzy differential equation.
Recently, Molabahrami et al. [11] have used the Homotopy analysis method to solve fuzzy Fredholm integral equation of second kind.
In 2006, Deb et al. [4] introduced a new set of orthogonal function; these functions have been applied for solving variational problem and integral equation by Babolian et al. [2,3]. The aim of this paper is to apply, for the first time, the triangular functions to obtain approximate solutions for the linear Fredholm fuzzy integral equations of the second kind. Also, we present the error estimate for approximating the solution of FFIE-2.

2. Preliminaries

Definition 2.1. Two m-sets of triangular functions (TFs) are defined over the interval $[0, T)$ as:

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (1+i)h \\ 0, & o.w \end{cases}, \tag{2.1}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t < (1+i)h \\ 0, & o.w \end{cases}, \tag{2.2}$$

where $i = 0, 1, \cdots, m-1$ and m has a positive integer value. Also, consider $h = \frac{T}{m}$, and $T1_i$ as the ith left-handed triangular function and $T2_i$ as the ith right-handed triangular function. In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$ and $h = \frac{1}{m}$. From the definition of TFs, it is clear that triangular functions are disjoint, orthogonal and complete [4]. We can write

$$\int_0^1 T1_i(t)T1_j(t)dt = \int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \frac{h}{3}, & i = j \\ 0, & i \ne j \end{cases}. \tag{2.3}$$

Consider the first m terms of the left-hand triangular functions and the first m terms of the right-handed triangular functions and write them concisely as m-vectors:

$$T1(t) = [T1_0(t), T1_1(t), \cdots, T1_{m-1}(t)]^T, \tag{2.4}$$

$$T2(t) = [T2_0(t), T2_1(t), \cdots, T2_{m-1}(t)]^T, \tag{2.5}$$

where $T1(t)$ and $T2(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively. The following properties of the product of two TFs vectors are presented by [3]:

$$T1(t)T1^T(t) \simeq \begin{pmatrix} T1_0(t) & 0 & \dots & 0 \\ 0 & T1_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T1_{m-1}(t) \end{pmatrix}, \tag{2.6}$$

$$T2(t)T2^T(t) \simeq \begin{pmatrix} T2_0(t) & 0 & \dots & 0 \\ 0 & T2_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T2_{m-1}(t) \end{pmatrix}, \tag{2.7}$$

and

$$T1(t)T2^T(t) \simeq 0, \tag{2.8}$$

$$T2(t)T1^T(t) \simeq 0, \tag{2.9}$$

where 0 is the zero $m \times m$ matrix. Also,

$$\int_0^1 T1(t)T1^T(t)dt = \int_0^1 T2(t)T2^T(t)dt \simeq \frac{h}{3}I, \tag{2.10}$$

$$\int_0^1 T1(t)T2^T(t)dt = \int_0^1 T2(t)T1^T(t)dt \simeq \frac{h}{6}I. \tag{2.11}$$

in which I is an $m \times m$ identity matrix.
The expansion of a function $f(t)$ over [0,1) with respect to TFs, may be

compactly written as

$$(2.12) \qquad f(t) \simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = c^T T1(t) + d^T T2(t),$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \ldots, m-1$.

3. Expanding two variable function by TFs

We can expand each $f(t,s) \in L^2([0,1) \times [0,1))$ by two TFs vectors, with m_1 and m_2 components, respectively. For convenience, consider $m_1 = m_2 = m$. To obtain desired results, we first fix the independent variable s. Then, we expand $f(t,s)$ by TFs with respect to independent variable t as follows:

$$(3.1) \qquad f(t,s) \simeq T1^T(t) \begin{pmatrix} f(0,s) \\ f(h,s) \\ \vdots \\ f((m-1)h,s) \end{pmatrix} + T2^T(t) \begin{pmatrix} f(h,s) \\ f(2h,s) \\ \vdots \\ f(mh,s) \end{pmatrix}.$$

Now, each of $f(ih,s)$'s for $i = 0, 1, \cdots, m-1$ can be expanded by TFs with respect to independent variable s. Hence, the expansion of $f(t,s)$ can be written as

$$T1^T(t) \begin{pmatrix} F11_1^T T1(s) + F12_1^T T2(s) \\ F11_2^T T1(s) + F12_2^T T2(s) \\ \vdots \\ F11_m^T T_1(s) + F12_m^T T2(s) \end{pmatrix} + T2^T(t) \begin{pmatrix} F21_1^T T1(s) + F22_1^T T2(s) \\ F21_2^T T1(s) + F22_2^T T2(s) \\ \vdots \\ F21_m^T T1(s) + F22_m^T T2(s) \end{pmatrix}$$

$$= T1^T(t) \left(\begin{pmatrix} F11_1^T \\ F11_2^T \\ \vdots \\ F11_m^T \end{pmatrix} T1(s) + \begin{pmatrix} F12_1^T \\ F12_2^T \\ \vdots \\ F12_m^T \end{pmatrix} T2(s) \right) + T2^T(t) \left(\begin{pmatrix} F21_1^T \\ F21_2^T \\ \vdots \\ F21_m^T \end{pmatrix} T1(s) + \begin{pmatrix} F22_1^T \\ F22_2^T \\ \vdots \\ F22_m^T \end{pmatrix} T2(s) \right)$$

$$= T1^T(t) F11 T1(s) + T1^T(t) F12 T2(s) + T2^T(t) F21 T1(s) + T2^T(t) F22 T2(s),$$

in which,

(3.2)
$$F11 = \begin{pmatrix} f(0,0) & f(0,h) & \dots & f(0,(m-1)h) \\ f(h,0) & f(h,h) & \dots & f(h,(m-1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f((m-1)h,0) & f((m-1)h,h) & \dots & f((m-1)h,(m-1)h) \end{pmatrix},$$

(3.3)
$$F12 = \begin{pmatrix} f(0,h) & f(0,2h) & \dots & f(0,mh) \\ f(h,h) & f(h,2h) & \dots & f(h,mh) \\ \vdots & \vdots & \ddots & \vdots \\ f((m-1)h,h) & f((m-1)h,2h) & \dots & f((m-1)h,mh) \end{pmatrix},$$

$$F21 = \begin{pmatrix} f(h,0) & f(h,h) & \dots & f(h,(m-1)h) \\ f(2h,o) & f(2h,h) & \dots & f(2h,(m-1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f(mh,0) & f(mh,h) & \dots & f(mh,(m-1)h) \end{pmatrix}, \tag{3.4}$$

$$F22 = \begin{pmatrix} f(h,h) & f(h,2h) & \dots & f(h,mh) \\ f(2h,h) & f(2h,2h) & \dots & f(2h,mh) \\ \vdots & \vdots & \ddots & \vdots \\ f(mh,h) & f(mh,2h) & \dots & f(mh,mh) \end{pmatrix}. \tag{3.5}$$

Let T(t) be a $2m - vector$ defined as

$$T(t) = \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix};\ 0 \leq t < 1, \tag{3.6}$$

where $T1(t)$ and $T2(t)$ have been defined in Eqs. (2.4) and (2.5). Now, assume that $f(s,t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$f(s,t) \simeq T^T(s)FT(t), \tag{3.7}$$

where $T(s)$ and $T(t)$ are $2m_1$ and $2m_2$ dimensional TFs and F is a $2m_1 \times 2m_2$ TFs coefficient matrix.

For convenience, we put $m_1 = m_2 = m$, so matrix F can be written as

$$F = \begin{pmatrix} (F11)_{m\times m} & (F12)_{m\times m} \\ (F21)_{m\times m} & (F22)_{m\times m} \end{pmatrix}, \tag{3.8}$$

where $F11, F12, F21$ and $F22$ in above-stated Eq., are previously defined in Eqs.(3.2)-(3.5).

Definition 3.1. A fuzzy number is a fuzzy set $u : R^1 \to [0, 1]$ which satisfies following conditions
a: u is upper semicontinuous.
b: $u(x) = 0$ outside some interval $[c, d]$.
c: There are real numbers a and b, $c \le a \le b \le d$, for which

i) $u(x)$ is monotonicly increasing on $[c, a]$,

ii) $u(x)$ is monotonicly decreasing on $[b, d]$,

iii) $u(x) = 1$ for $a \le x \le b$.

The set of all fuzzy numbers, as given by definition 3.1 is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [10].

Definition 3.2. A fuzzy number u is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, satisfying the following requirements:
a: $\underline{u}(r)$ is abounded monotonic increasing left continuous function,
b: $\overline{u}(r)$ is abounded monotonic decreasing left continuous function,
c: $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we define addition $(u + v)$ and multiplication by k as:

$$\begin{aligned} (\underline{u+v})(r) &= \underline{u}(r) + \underline{v}(r), \\ (\overline{u+v})(r) &= \overline{u}(r) + \overline{v}(r), \end{aligned} \tag{3.9}$$

$$\begin{aligned} (\underline{ku})(r) &= k\underline{u}(r), \\ (\overline{ku})(r) &= k\overline{u}(r). \end{aligned} \tag{3.10}$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (3.9) and (3.10) is denoted by E^1 and is u convex cone. it can be shown that Eqs. (3.9) and (3.10) are equivalent to the addition and multiplication as defined by using the $\alpha - cut$ approach [8] and the extension principles [13]. We will next define the fuzzy function notation and a metric D in E^1 [8].

Definition 3.3. For arbitrary numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$,

$$D(u, v) = \max\{\sup_{0\le r\le 1} |\overline{u}(r) - \overline{v}(r)|, \sup_{0\le r\le 1} |\underline{u}(r) - \underline{v}(r)|\},$$

is the distance between u and v [8].

Definition 3.4. Suppose $f : [a, b] \to E^1$ for each partition $p = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and for arbitrary $\varepsilon_i;x_{i-1} \le \varepsilon_i \le x_i$, $1 \le i \le n$, take

$$\lambda = \max_{1\le i\le n} |x_i - x_{i-1}| \ ,$$

and $R_p = \sum_{i=1}^{n} f(\varepsilon_i)(x_i - x_{i-1})$. The definition integral of $f(x)$ over [a,b] is

$$\int_a^b f(x)dx = \lim_{\lambda\to 0} R_p \ ,$$

provided that this *limit* exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, the definite integral exists [8]. Furthermore,

$$(3.11) \quad (\underline{\int_a^b f(x, r)dx}) = \int_a^b \underline{f}(x, r)dx, \quad (\overline{\int_a^b f(x, r)dx}) = \int_a^b \overline{f}(x, r)dx,$$

where $(\underline{f}(x, r), \overline{f}(x, r))$ is the parametric form of $f(x)$. It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [10]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eq. (3.10) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [8,10].

Lemma 3.5. *([1]) If f and $g : [a, b] \subseteq R \to E^1$ are fuzzy continuous function, then the function $F : [a, b] \to R_+$ by $F(x) = D(f(x), g(x))$ is* *continuous on* $[a, b]$*, and*

$$D\left(\int_a^b f(x)dx, \int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx. \tag{3.12}$$

4. Solving linear fuzzy Fredholm integral equation

In this section, we present a TFs method to solve linear FFIE-2. First consider the following equation:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)u(t)dt, \tag{4.1}$$

where $k(x, t)$ is an arbitrary kernel function over the square $0 \leq x, t \leq 1$, and $u(x)$ is a fuzzy real valued function. In [15], the authors presented sufficient conditions for the existence and unique solution of (4.1) as the following theorem:

Theorem 4.1. *([15]) Let* $k(x, t)$ *be continuous for* $a \leq x, t \leq b$*, and* $f(x)$ *a fuzzy continuous of* x*,* $a \leq x \leq b$*. If* $\lambda < \frac{1}{M(b-a)}$*, where*

$$M = \max_{a \leq x,t \leq b} |k(x, t)|,$$

then the iterative procedure

$$u_0(x) = f(x),$$

$$u_k(x) = f(x) + \lambda \int_a^b k(x,t)u_{k-1}(t)dt, \ \ k \geq 1$$

converges to the unique solution of (4.1). Specially,

$$\sup_{a \leq x \leq b} D(u(x), u_k(x)) \leq \frac{L^k}{1-L} \sup_{a \leq x \leq b} D(u_0(x), u_1(x)),$$

where $L = \lambda M(b - a)$*.*

Throughout this paper, we consider fuzzy Fredholm integral equation (4.1) with $a = 0, b = 1$ and $\lambda > 0$, where $u(x)$ and $f(x)$ are in $L^2([0, 1))$ and $k(x, t)$ belongs to $L^2([0, 1) \times [0, 1))$. Our problem is to determine

TFs pair coefficients of $u(x)$ in the interval $[0,1)$ from the know functions $f(x)$ and kernel $k(x,t)$.
So, we introduce the parametric form of a $FFIE-2$ with respect to definition 3.1. Let $(\underline{f}(x,r), \overline{f}(x,r))$ and $(\underline{u}(x,r), \overline{u}(x,r))$, $0 \leq r \leq 1$ and $x \in [0,1)$ be parametric forms of $f(x)$ and $u(x)$, respectively.
Therefore, we rewrite system (4.1) in the following form

$$\underline{u}(x,r) = \underline{f}(x,r) + \lambda \int_0^1 k(x,t)\underline{u}(t,r)dt, \tag{4.2}$$

$$\overline{u}(x,r) = \overline{f}(x,r) + \lambda \int_0^1 k(x,t)\overline{u}(t,r)dt. \tag{4.3}$$

Let us expand $\underline{u}(x,r)$, $\underline{f}(x,r)$ and $k(x,t)$ by TFs ($LHTF$ and $RHTF$) as follows:

$$\begin{aligned}\underline{u}(x,r) &\simeq T1^T(x)U11T1(r) + T1^T(x)U12T2(r) + T2^T(x)U21T1(r)\\ &\quad +T2^T(t)U22T(r) = T^T(x)UT(r),\\ f(x,r) &\simeq T1^T(x)F11T(r) + T1^T(x)F12T2(r) + T2^T(x)F21T1(r)\\ &\quad +T2^T(x)F22T2(r) = T^T(x)FT(r),\end{aligned}$$

and

$$\begin{aligned}k(x,t) &\simeq T1^T(x)K11T(r) + T1^T(x)K12T2(r) + T2^T(x)K21T1(r)\\ &\quad +T2^T(x)K22T2(r) = T^T(x)KT(r).\end{aligned}$$

with

$$U = \begin{pmatrix} U11 & U12 \\ U21 & U22 \end{pmatrix}, F = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix} \text{ and } K = \begin{pmatrix} K11 & K12 \\ K21 & K22 \end{pmatrix}.$$

substituting in Eq.(4.2):

$$T^T(x)UT(r) \simeq T^T(x)FT(r) + \lambda \int_0^1 T^T(x)KT(t)T^T(t)UT(r)dt,$$
$$T^T(x)UT(r) \simeq T^T(x)FT(r) + \lambda T^T(x)K\Big(\int_0^1 T(t)T^T(t)dt\Big)UT(r),$$

with the equation

$$\begin{aligned}\int_0^1 T(t)T^T(t)dt &= \int_0^1 \begin{pmatrix} T1(t)T2(t) \end{pmatrix} \begin{pmatrix} T1^T(t) & T2^T(t) \end{pmatrix} dt\\ &= \int_0^1 \begin{pmatrix} T1(t)T1^T(t) & T1(t)T2^T(t) \\ T2(t)T1^T(t) & T2(t)T2^T(t) \end{pmatrix} dt\\ &\simeq \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix} = D_{2m\times 2m}\end{aligned}$$

,

we have

$$T^T(x)UT(r) \simeq T^T(x)FT(r) + \lambda T^T(x)KDUT(r),$$

then

$$U = F + \lambda KDU \Rightarrow (I - \lambda KD)U = F,$$

thus

$$U = (I - \lambda KD)^{-1}F.$$

By solving this matrix system we can find matrix $U_{2m\times 2m}$ so $\underline{u}(x,r) \simeq T^T(x)UT(r)$. The same trend holds for Eq. (4.3).

5. Error Estimation

Now, we obtain the error estimation for given FFIE-2 as (4.1). Suppose that

$$u_n(x) \simeq \sum_{i=0}^{n-1}{}^* \; u(ih)T1_i(x) + \sum_{i=0}^{n-1}{}^* \; u((i+1)h)T2_i(x),$$

is approximate solution of $u(x)$, where $\Sigma\,^*$ denotes the fuzzy summation and $h = \frac{1}{n}$. Therefore, we get:

$$D(u(x), u_n(x))$$

$$= D\left(\int_0^1 k(x,t)u(t)dt, (\int_0^1 k(x,t)(\sum_{i=0}^{n-1}{}^* \; u(ih)T1_i(t) + \sum_{i=0}^{n-1}{}^* \; u((i+1)h)T2_i(t))dt\right)$$

$$\leq M\int_0^1 D\left(u(t), \sum_{i=0}^{n-1}{}^* \; u(ih)T1_i(t) + \sum_{i=0}^{n-1}{}^* \; u((i+1)h)T2_i(t)\right)dt,$$

where

$$M = \max_{0\leq x,t\leq 1}|k(x,t)|.$$

Therefore, we have:

$$D(u(x), u_n(x)) \leq M\int_0^1 D(u(t), u_n(t))dt,$$

$$\sup_{x\in[0,1]} D(u(x),u_n(x)) \leq M \sup_{x\in[0,1]} D(u(x),u_n(x)).$$

Therefore, if $M<1$, we will have:

$$\lim_{n\to\infty}\sup_{x\in[0,1]} D(u(x),u_n(x)) = 0.$$

6. Numerical examples

Here, we consider three examples to illustrate the presented method for FFIE-2.

Example 6.1. ([7]) Consider the following FFIE-2 with

$$\underline{f}(x,r) = -\frac{1}{3}x^2 + x^2 r + \frac{1}{3}x + \frac{1}{4}r - \frac{1}{12},$$

$$\overline{f}(x,r) = \frac{1}{3}x - x^2 r - \frac{1}{4}r + \frac{5}{3}x^2 + \frac{5}{12},$$

and

$$k(x,t) = (2t-1)^2(1-2x), \quad 0 \leq x,t \leq 1 \quad \textit{and } \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,$$

$$\overline{u}(x,r) = (2-r)x.$$

The results are shown in Table 1.

Table 1

Numerical results of Example 1 with presented method and Block-Pulse functions method

r	Exact solution $(\underline{u}(x,r),\overline{u}(x,r))$	Presented method for $x=0.5$ and $m=2$	Method of [7] for $x=0.5$ and $m=32$
0	(0.0000,1.0000)	(0.0000,1.0000)	(0.007956,1.024160)
0.1	(0.0500,0.9500)	(0.0500,0.9500)	(0.056347,0.975770)
0.2	(0.1000,0.9000)	(0.1000,0.9000)	(0.104737,0.927379)
0.3	(0.1500,0.8500)	(0.1500,0.8500)	(0.153128,0.878988)
0.4	(0.2000,0.8000)	(0.2000,0.8000)	(0.201519,0.830598)
0.5	(0.2500,0.7500)	(0.2500,0.7500)	(0.266040,0.766077)
0.6	(0.3000,0.7000)	(0.3000,0.7000)	(0.314430,0.717986)
0.7	(0.3500,0.6500)	(0.3500,0.6500)	(0.362820,0.669290)
0.8	(0.4000,0.6000)	(0.4000,0.6000)	(0.411210,0.630905)
0.9	(0.4500,0.5500)	(0.4500,0.5500)	(0.359603,0.572514)

Example 6.2. ([6]) Consider the following FFIE-2 with

$$\underline{f}(x,r) = rx - x^2[\frac{2}{3}rx^3 - \frac{4}{3}x^3 - \frac{1}{2}rx^2 + x^2 + \frac{1}{12}r - \frac{1}{12}],$$

$$\overline{f}(x,r) = (2-r)x + x^2[\frac{2}{3}rx^3 - \frac{1}{2}rx^2 + \frac{1}{12}r - \frac{1}{12}],$$

and

$$k(x,t) = x^2(1-2t), \quad 0 \le x,t \le 1 \quad and \quad \lambda = 1.$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx,$$

$$\overline{u}(x,r) = (2-r)x.$$

The results are shown in Tables 2 and 3.

Table 2

Numerical results of Example 2 with presented method

r	Exact solution $(\underline{u}(x,r),\overline{u}(x,r))$	Presented method for $x=0.1$ and $m=10$	Absolute error
0	(0.0000,0.2000)	(0.0003,0.1964)	(3.9582e-04,3.6000e-03)
0.1	(0.0100,0.1900)	(0.0102,0.1866)	(1.9745e-04,3.4000e-03)
0.2	(0.0200,0.1800)	(0.0200,0.1768)	(9.0743e-07,3.2000e-03)
0.3	(0.0300,0.1700)	(0.0298,0.1670)	(1.9927e-04,3.0000e-03)
0.4	(0.0400,0.1600)	(0.0396,0.1572)	(3.9763e-04,2.8000e-03)
0.5	(0.0500,0.1500)	(0.0494,0.1474)	(5.9599e-04,2.6000e-03)
0.6	(0.0600,0.1400)	(0.0592,0.1376)	(7.9436e-04,2.4000e-03)
0.7	(0.0700,0.1300)	(0.0690,0.1278)	(9.9272e-04,2.2000e-03)
0.8	(0.0800,0.1200)	(0.0788,0.1180)	(1.2000e-03,2.0000e-03)
0.9	(0.0900,0.1100)	(0.0886,0.1082)	(1.4000e-03,1.8000e-03)

Table 3

Numerical results of Example 2 with Homotopy analysis method [6].

x	h=-1.2	h=-1.1	h=-1	h=-0.9	h=-0.8
0.1	1.4898e -03	1.7651e -04	1.7602e -07	2.2547e -04	1.6846e -03

Example 6.3. Consider the following FFIE-2 with

$$\underline{f}(x,r) = (r^2+r)(sin(\frac{x}{2}) - 0.05sin(x)(1-sin(1))),$$

$$\overline{f}(x,r) = (4-r^3-r)(sin(\frac{x}{2}) - 0.05sin(x)(1-sin(1))),$$

and

$$k(x,t) = 0.1sin(x)sin(\tfrac{t}{2}), \quad 0 \leq x,t \leq 1 \quad and\ \lambda = 1..$$

The exact solution in this case is given by

$$\underline{u}(x,r) = (r^2+r)sin(\frac{x}{2}),$$

$$\overline{u}(x,r) = (4-r^3-r)sin(\frac{x}{2}).$$

The results are shown in Table 4.

Table 4
Numerical results of Example 3 with presented method

r	Exact solution $(\underline{u}(x,r),\overline{u}(x,r))$	Presented method for $x=0.5$ and $m=14$	Absolute error
0	(0.0000,0.9896)	(0.0000,0.9902)	(0.0000000 ,6.0577e-04)
0.1	(0.0272,0.9646)	(0.0275,0.9651)	(3.1979e-04,4.9521e-04)
0.2	(0.0594,0.9382)	(0.0596,0.9386)	(2.3843e-04,4.6168e-04)
0.3	(0.0965,0.9087)	(0.0967,0.9091)	(2.6115e-04,3.6571e-04)
0.4	(0.1385,0.8748)	(0.1389,0.8750)	(3.8794e-04,1.7608e-04)
0.5	(0.1856,0.8350)	(0.1857,0.8355)	(1.1358e-04,5.1112e-04)
0.6	(0.2375,0.7877)	(0.2380,0.7877)	(4.4852e-04,6.7766e-05)
0.7	(0.2944,0.7316)	(0.2948,0.7316)	(3.8230e-04,3.2099e-05)
0.8	(0.3563,0.6650)	(0.3567,0.6649)	(4.2017e-04,8.6587e-05)
0.9	(0.4231,0.5866)	(0.4236,0.5861)	(5.6210e-04,4.5504e-04)

7. Conclusion

In this paper, we considered linear FFIE-2. By the embedding method, the original equation is converted to two crisp Fredholm integral equations of the second kind. Then, we apply the two m-sets of TFs for approximation of the unique solution of FFIE-2. Also, we proved the error estimation for approximated solution of FFIE-2. The main advantage of this method is low cost of setting up the equations without using any projection method and any integration.

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F. Mirzaee
Department of Mathematics, Faculty of Science, Malayer University, Malayer, 65719-95863, Iran.
Email: `f.mirzaee@malayeru.ac.ir`

M. Paripour
Department of Science, Hamedan University of Technology, Hamedan, 65156-579, Iran.
Email: `paripour@hut.ac.ir`

M. Komak Yari
Department of Mathematics, Faculty of Science, Malayer University, Malayer, 65719-95863, Iran.
Email: `morteza_komakyari@yahoo.com`