

ON COINCIDENCE AND FIXED-POINT THEOREMS IN FUZZY SYMMETRIC SPACES

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ABSTRACT. In this paper, common fixed point theorems have been studied in fuzzy symmetric space instead of fuzzy metric space. Using weakly compatibility, property (E.A.), we have generalized the common fixed point theorems for a pair of weakly compatible self mappings, for four self mappings in fuzzy symmetric space. Also we have established the unique common fixed point for four self mappings in this space.

Key Words: Fuzzy symmetric spaces, property (E.A.), weakly compatible mappings, occasionally weakly compatible maps, contractive modulus function, common fixed point theorem.

2010 Mathematics Subject Classification: Primary: 03E72, Secondary: 47H10, 54H25.

1. INTRODUCTION

The notion of compatible maps was introduced by Jungck [5] and the study of common fixed point theorems for contractive maps has centered around the study of compatible maps and its weaker forms. On the other hand, the study of noncompatible maps is also equally interesting. Pant [14], Aamri and Moutawkil [9] and others have initiated wonderful works in this field. In [10], the authors gave a notion of the property (E.A.) which generalizes the concept of noncompatible mappings in

Received: 21 April 2012, Accepted: 11 June 2012. Communicated by Yuming Feng

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metric spaces, and they proved some common fixed point theorems for noncompatible mappings under strict contractive conditions. Recently, in [11] the authors extended the results of [10, 14] to symmetric spaces under tight conditions, where the metric does not satisfy the triangular inequality.

Symmetric spaces were introduced in 1931 by Wilson [17], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained, for example, see [7, 11, 16]. Hicks and Rhoades [6] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metrics are not required in the proofs of certain metric theorems.

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. But, in real life situation, the problem in economics, engineering, environment, social science, medical science, etc. does not always involve crisp data. Consequently, we can not successfully use the traditional classical methods because of various type of uncertainties. To deal with these uncertainties, fuzzy set theory [8] can be considered as one of the mathematical tool. Kramosil and Michalek [12] introduced the concept of fuzzy metric spaces (briefly , FM-spaces) in 1975 , which opened an avenue for further development of analysis in such spaces. Later on it is modified that a few concepts of mathematical analysis have been developed by George and Veeramani [1, 2] and also they have developed the fixed point theorem in fuzzy metric space [15]. In fuzzy metric space, the notion of compatible maps under the name of asymptotically commuting maps was introduced in the paper [13] and then in the paper [3], the notion of weak compatibility has been studied in fuzzy metric space. Later on Pant and Pant[18] studied the common fixed points of a pair of non-compatible maps in fuzzy metric space.

In this paper, we have studied the common fixed point theorems in fuzzy symmetric space. Here, our target is to generalize the common fixed point theorems for a pair of weakly compatible self mappings, for

four self mappings in fuzzy symmetric space. Using weakly compatibility, property (E.A.), we have established the unique common fixed point for four self mappings in fuzzy symmetric space.

2. PRELIMINARIES

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1. [4]. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if $*$ satisfies the following conditions :

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a, \quad \forall a \in [0, 1]$;
- (iv) $a * b \leq c * d$, whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2. [1] The 3-tuple $(X, \mu, *)$ is called a **fuzzy metric space** if X is an arbitrary non-empty set, $*$ is a continuous t -norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ which satisfying the following conditions:

- (i) $\mu(x, y, t) > 0$;
 - (ii) $\mu(x, y, t) = 1$ if and only if $x = y$;
 - (iii) $\mu(x, y, t) = \mu(y, x, t)$;
 - (iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s + t)$;
 - (v) $\mu(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous;
- for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3. The pair (X, μ) is called a **fuzzy symmetric space** if X is an arbitrary non-empty set and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions :

- (i) $\mu(x, y, t) > 0$;
- (ii) $\mu(x, y, t) = 1$ if and only if $x = y$;
- (iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous for all $x, y \in X$ and $t > 0$.

If (X, μ) is a fuzzy symmetric space, then μ is called fuzzy symmetric for X .

Note 2.4. Every fuzzy metric space is a fuzzy symmetric space but the converse is not necessarily true. For example, consider $X = [0, \infty)$ and $\mu(x, y, t) = \frac{t}{t+|x-y|}$ if $x \neq 0, y \neq 0$ and $\mu(x, y, t) = \frac{t}{t+\frac{1}{x}}$ if $x \neq 0$. It is easy to see that (X, μ) is a fuzzy symmetric space. Let $x = 1, y = \frac{1}{2}, z = 0, s = 1, t = 0$ and $a * b = \max\{a, b\}$. Then (iv) of definition(2.2) is not satisfied and hence (X, μ) is not a fuzzy metric space but it is a fuzzy symmetric space.

Definition 2.5. A subset S of a fuzzy symmetric space (X, μ) is said to be μ -closed if for a sequence $\{x_n\}$ in S and a point $x \in X$,

$$\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1 \implies x \in S.$$

Definition 2.6. Two self mappings f and g of a fuzzy symmetric space (X, μ) are called **compatible** if $\lim_{n \rightarrow \infty} \mu(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$$

for some x in X , where fg denotes the composition of f and g .

Definition 2.7. Let X be a set and f, g be self mappings of X . A point x in X is called a **coincidence** point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a **point of coincidence** of f and g .

Definition 2.8. A pair of self mappings S and T is called **weakly compatible** if they commute at their coincidence points.

Definition 2.9. We say that a pair of self mappings S and T satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} \mu(Sx_n, l, t) = \lim_{n \rightarrow \infty} \mu(Tx_n, l, t) = 1,$$

for some $l \in X$.

Definition 2.10. The mappings $A, B, S, T : X \rightarrow X$ of a fuzzy symmetric space (X, μ) satisfy a common property (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n \rightarrow \infty} \mu(Ax_n, l, t) = \lim_{n \rightarrow \infty} \mu(Sx_n, l, t) = 1$$

$$\lim_{n \rightarrow \infty} \mu(By_n, l, t) = \lim_{n \rightarrow \infty} \mu(Ty_n, l, t) = 1$$

for some $l \in X$.

We denote Φ by the class of continuous function $\phi : [0, 1] \rightarrow [0, 1]$ satisfying:

$$(\phi_1) \quad \phi(l) > l \text{ for all } l \in [0, 1),$$

$$(\phi_2) \quad \phi(1) = 1.$$

We denote Λ by the class of continuous function $\alpha : [0, 1] \rightarrow [0, 1]$ satisfying :

$$(\alpha_1) \quad \alpha(1) = 1,$$

$$(\alpha_2) \quad \alpha(s) < 1 \text{ for all } s \in [0, 1).$$

3. AXIOMS ON FUZZY SYMMETRIC SPACES

Through out our discussion, (X, μ) stands for fuzzy symmetric space. We now state the following axioms :

(W_3) For a sequence $\{x_n\}$ in X , $x, y \in X$, $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1$ and $\lim_{n \rightarrow \infty} \mu(x_n, y, t) = 1$ imply $x = y$.

(W_4) For sequences $\{x_n\}, \{y_n\}$ in X , $x \in X$, $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1$ and $\lim_{n \rightarrow \infty} \mu(y_n, x_n, t) = 1 \implies \lim_{n \rightarrow \infty} \mu(y_n, x, t) = 1$.

(H_E) For sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1$ and $\lim_{n \rightarrow \infty} \mu(y_n, x, t) = 1 \implies \lim_{n \rightarrow \infty} \mu(x_n, y_n, t) = 1$.

(C_C) For a sequence $\{x_n\} \in X$ and $x, y \in X$, $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1 \implies \lim_{n \rightarrow \infty} \mu(x_n, y, t) = \mu(x, y, t)$.

Proposition 3.1. *For axioms in symmetric space (X, μ) , one has*

- (1) $(W_4) \implies (W_3)$,
- (2) $(C_C) \implies (W_3)$.

Proof. Let $\{x_n\}$ be a sequence in X and $x, y \in X$ with
 $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1$ and $\lim_{n \rightarrow \infty} \mu(x_n, y, t) = 1$
 (1). By putting $y_n = y$ for each $n \in N$, we have

$$\lim_{n \rightarrow \infty} \mu(x_n, x, t) = \lim_{n \rightarrow \infty} \mu(x_n, y_n, t) = 1.$$

By (W_4) , we have

$$1 = \lim_{n \rightarrow \infty} \mu(y_n, x, t) = \lim_{n \rightarrow \infty} \mu(y, x, t) \implies x = y.$$

(2). By (C_C) , $\lim_{n \rightarrow \infty} \mu(x_n, x, t) = 1$

$$\implies \mu(x, y, t) = \lim_{n \rightarrow \infty} \mu(x_n, y, t) = 1 \implies x = y.$$

Following the examples of [16], we now establish a few examples, which would show that other relationships in Proposition(3.1) do not hold.

Example 3.2. *If $(W_4) \not\Rightarrow (H_E)$ and $(W_4) \not\Rightarrow (C_C)$ then $(W_3) \not\Rightarrow (H_E)$ and $(W_3) \not\Rightarrow (C_C)$ by Proposition(3.1).*

Let $X = [0, \infty)$ and $\mu(x, y, t) = \frac{t}{t + |x-y|}$ if $x \neq 0, y \neq 0$ and $\mu(x, y, t) = \frac{t}{t + \frac{1}{x}}$ if $x \neq 0$. By note(2.4), we see that (X, μ) is a fuzzy symmetric space which satisfies (W_4) but does not satisfy (H_E) for $x_n = n, y_n = n + 1$. Also (X, μ) does not satisfy (C_C) .

Example 3.3. *If $(H_E) \not\Rightarrow (W_3)$ then $(H_E) \not\Rightarrow (W_4)$ and also $(H_E) \not\Rightarrow (C_C)$.*

Let $X = [0, 1] \cup \{2\}$ and $\mu(x, y, t) = \frac{t}{t + |x-y|}$ if $0 \leq x \leq 1, 0 \leq y \leq 1$ and $\mu(x, y, t) = \frac{t}{t+x}$ if $0 < x \leq 1, y = 2$ and $\mu(0, 2, t) = \frac{t}{t+1}$. Then (X, μ) is a fuzzy symmetric space which satisfies (H_E) . Let $x_n = \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} \mu(x_n, 0, t) = \lim_{n \rightarrow \infty} \mu(x_n, 2, t) = 1$. But $\mu(0, 2, t) \neq 1$ and hence the symmetric space (X, μ) does not satisfies (W_3) .

Example 3.4. If $(C_C) \not\Rightarrow (W_4)$ then $(W_3) \not\Rightarrow (W_4)$ by Proposition(3.1).

Let $X = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$ and let $\mu(0, \frac{1}{n}, t) = \frac{t}{t + \frac{1}{n}}$ if n is odd, $\mu(0, \frac{1}{n}, t) = \frac{t}{t+1}$ if n is even, $\mu(\frac{1}{m}, \frac{1}{n}, t) = \frac{t}{t + |\frac{1}{m} - \frac{1}{n}|}$ if $m + n$ is even, $\mu(\frac{1}{m}, \frac{1}{n}, t) = \frac{t}{t + |\frac{1}{m} - \frac{1}{n}|}$ if $m + n$ is odd and $|m - n| = 1$, $\mu(\frac{1}{m}, \frac{1}{n}, t) = \frac{t}{t+1}$ if $m + n$ is odd and $|m - n| \geq 2$. Then the symmetric space (X, μ) satisfies (C_C) but does not satisfy (W_4) for $x_n = \frac{1}{2n+1}$ and $y_n = \frac{1}{2n}$.

Example 3.5. $(C_C) \not\Rightarrow (H_E)$.

Let $X = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$ and $\mu(\frac{1}{m}, \frac{1}{n}, t) = \frac{t}{t + |\frac{1}{m} - \frac{1}{n}|}$ if $|m - n| \geq 2$, $\mu(\frac{1}{m}, \frac{1}{n}, t) = \frac{t}{t+1}$ if $|m - n| = 1$ and $\mu(0, \frac{1}{n}, t) = \frac{t}{t + \frac{1}{n}}$. Then (X, μ) is a symmetric space which satisfies (C_C) . Let $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$. Then $\lim_{n \rightarrow \infty} \mu(x_n, 0, t) = \lim_{n \rightarrow \infty} \mu(y_n, 0, t) = 1$. But $\lim_{n \rightarrow \infty} \mu(x_n, y_n, t) \neq 1$. Hence the symmetric space (X, μ) does not satisfy (H_E) .

4. COMMON FIXED POINT THEOREMS

Theorem 4.1. Let (X, μ) be a fuzzy symmetric space that satisfies (W_3) and (H_E) and let A, B, S and T be self-mappings of X such that

- (i) $AX \subset TX$ and $BX \subset SX$;
- (ii) the pair (B, T) satisfies property $(E.A.)$, (resp. (A, S) satisfies property $(E.A.)$);
- (iii) for any $x, y \in X$, $\mu(Ax, By, t) \geq \nu(x, y, t)$, where $\nu(x, y, t) = \min \{ \mu(Sx, Ty, t), \max \{ \mu(Ax, Sx, t), \mu(By, Ty, t) \} \}$,

$$\max \{ \mu(Ax, Ty, t), \mu(By, Sx, t) \};$$

- (iv) SX is a μ -closed subset of X (resp. TX is a μ -closed subset of X).

Then there exist $u, w \in X$ such that $Au = Su = Bw = Tw$.

Proof. From (ii), there exist a sequence $\{x_n\}$ in X , and a point $l \in X$ such that

$$\lim_{n \rightarrow \infty} \mu(Tx_n, l, t) = \lim_{n \rightarrow \infty} \mu(Bx_n, l, t) = 1.$$

From (i), there exist a sequence $\{y_n\}$ in X , such that $Bx_n = Sy_n$ and hence $\lim_{n \rightarrow \infty} \mu(Sy_n, l, t) = 1$. By (H_E) ,

$$\lim_{n \rightarrow \infty} \mu(Bx_n, Tx_n, t) = \lim_{n \rightarrow \infty} \mu(Sy_n, Tx_n, t) = 1.$$

From (iv), there exists a point $u \in X$ such that $Su = l$.

From (iii), we have

$$\begin{aligned} & \mu(Au, Bx_n, t) \\ & \geq \min \{ \mu(Su, Tx_n, t), \max \{ \mu(Au, Su, t), \mu(Bx_n, Tx_n, t) \}, \\ & \quad \max \{ \mu(Au, Tx_n, t), \mu(Bx_n, Su, t) \} \}. \end{aligned}$$

By taking $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \mu(Au, Bx_n, t) = 1$.

By (W_3) , we get $Au = Su$. Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$. We show that $Tw = Bw$.

From (iii), we have $\mu(Au, Bw, t)$

$$\begin{aligned} & \geq \min \{ \mu(Su, Tw, t), \max \{ \mu(Au, Su, t), \mu(Bw, Tw, t) \}, \\ & \quad \max \{ \mu(Au, Tw, t), \mu(Bw, Su, t) \} \} \\ & = \min \{ \mu(Tw, Tw, t), \max \{ \mu(Au, Au, t), \mu(Bw, Tw, t) \}, \\ & \quad \max \{ \mu(Au, Au, t), \mu(Bw, Su, t) \} \}, \end{aligned}$$

$$\implies \mu(Au, Bw, t) = 1.$$

Hence, $Au = Bw$ and hence

$$Au = Su = Bw = Tw.$$

Theorem 4.2. Let (X, μ) be a fuzzy symmetric space that satisfies (W_3) and (H_E) and let A, B, S and T be self-mappings of X such that

(i) $AX \subset TX$ and $BX \subset SX$;

- (ii) the pair (B, T) satisfies property $(E.A.)$, (resp. $(A.S)$ satisfies property $(E.A.)$);
- (iii) the pairs (A, S) and (B, T) are weakly compatible;
- (iv) for any $x, y \in X$ ($x \neq y$), $\mu(Ax, By, t) > \nu(x, y, t)$, where $\nu(x, y, t) = \min \{ \mu(Sx, Ty, t), \max \{ \mu(Ax, Sx, t), \mu(By, Ty, t) \}, \max \{ \mu(Ax, Ty, t), \mu(By, Sx, t) \} \}$;
- (v) SX is a μ -closed subset of X (resp. TX is a μ -closed subset of X).

Then A, B, S , and T have a unique common fixed point in X .

Proof. From Theorem 4.1, there exist $u, w \in X$ such that $Au = Su = Bw = Tw$. From (iii), $ASu = SAu$,

$$AAu = ASu = SAu = SSu \text{ and } BTw = TBw = TTW = BBw.$$

If $Au \neq w$, then from (iv), we have

$$\begin{aligned} & \mu(Au, AAu, t) \\ &= \mu(AAu, Bw, t) \\ &> \min \{ \mu(SAu, Tw, t), \max \{ \mu(AAu, SAu, t), \\ & \quad \mu(Bw, Tw, t) \}, \max \{ \mu(AAu, Tw, t), \mu(Bw, SAu, t) \} \} \\ &= \min \{ \mu(AAu, Au, t), 1, \mu(AAu, Au, t) \} = \mu(AAu, Au, t), \\ &\implies \mu(Au, AAu, t) > \mu(Au, AAu, t), \end{aligned}$$

which is a contradiction and this contradiction proves that $Au = w$. Similarly, we have $Bw = u$, which implies that,

$$Au = w = Su = Bw = Tw = u,$$

that is, w is a common fixed point of A, B, S and T . For the uniqueness, let z be another common fixed point of A, B, S and T . If $w \neq z$, then from (iv) we get

$$\mu(z, w, t)$$

$$\begin{aligned}
&= \mu(Az, Bw, t) \\
&> \min \{ \mu(Sz, Tw, t), \max \{ \mu(Az, Sz, t), \\
&\quad \mu(Bw, Tw, t) \}, \max \{ \mu(Az, Tw, t), \mu(Bw, Sz, t) \} \} \\
&= \min \{ \mu(z, w, t), \max \{ \mu(z, z, t), \mu(w, w, t) \}, \\
&\quad \max \{ \mu(z, w, t), \mu(w, z, t) \} \}, \\
&\implies \mu(z, w, t) > \mu(z, w, t),
\end{aligned}$$

which is a contradiction . Hence $w = z$.

Theorem 4.3. *Let (X, μ) be a fuzzy symmetric space that satisfies (C_C) and (H_E) and let A, B, S and T be self-mappings of X and $\alpha \in \Lambda$ and $\phi \in \Phi$ satisfying*

- (i) $AX \subset TX$ and $BX \subset SX$;
- (ii) the pair (B, T) satisfies property $(E.A.)$, (resp. $(A.S)$ satisfies property $(E.A.)$) ;
- (iii) the pairs (A, S) and (B, T) are weakly compatible ;
- (iv) for any $x, y \in X, \alpha(\mu(Ax, By, t)) \geq \phi(\alpha(\nu(x, y, t)))$, where
$$\nu(x, y, t) = \min \{ \mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(By, Ty, t), \\ \mu(Ax, Ty, t), \mu(By, Sx, t) \};$$
- (v) SX is a μ -closed subset of X (resp., TX is a μ -closed sub set of X).

Then A, B, S and T have a unique common fixed point in X .

Proof. From (ii), there exist a sequence $\{x_n\}$ in X and a point $l \in X$ such that

$$\lim_{n \rightarrow \infty} \mu(Tx_n, l, t) = \lim_{n \rightarrow \infty} \mu(Bx_n, l, t) = 1.$$

From (i), there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ and hence

$$\lim_{n \rightarrow \infty} \mu(Sy_n, l, t) = 1. \text{ By } (H_E),$$

$$\lim_{n \rightarrow \infty} \mu(Bx_n, Tx_n, t) = \lim_{n \rightarrow \infty} \mu(Sy_n, Tx_n, t) = 1.$$

From (v), there exists a point $u \in X$ such that $Su = l$. We show $Au = Su$. From (iv) we have

$$\begin{aligned} & \alpha(\mu(Au, Bx_n, t)) \\ & \geq \phi(\alpha(\min\{\mu(Su, Tx_n, t), \mu(Au, Su, t), \\ & \quad \mu(Bx_n, Tx_n, t), \mu(Au, Tx_n, t), \mu(Bx_n, Su, t)\})) \end{aligned}$$

In the above inequality, we take $n \rightarrow \infty$ then by (C_C) and (H_E) we have

$$\begin{aligned} & \alpha(\mu(Au, Su, t)) \\ & \geq \phi(\alpha(\min\{1, \mu(Au, Su, t), 1, \mu(Au, Su, t), 1\})) \\ & = \phi(\alpha(\mu(Au, Su, t))), \text{ which implies } \alpha(\mu(Au, Su, t)) = 1. \end{aligned}$$

By (α_1) , we have $\mu(Au, Su, t) = 1$. Hence $Au = Su$.

Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$.

Thus we get $Au = Su = Tw$.

We show that $Tw = Bw$. From (iv) we have

$$\begin{aligned} & \alpha(\mu(Tw, Bw, t)) = \alpha(\mu(Au, Bw, t)) \\ & \geq \phi(\alpha(\min\{\mu(Su, Tw, t), \mu(Au, Su, t), \mu(Bw, Tw, t), \\ & \quad \mu(Au, Tw, t), \mu(Bw, Su, t)\})) \\ & = \phi(\alpha(\min\{\mu(Tw, Tw, t), \mu(Au, Au, t), \mu(Bw, Tw, t), \\ & \quad \mu(Au, Au, t), \mu(Bw, Tw, t)\})) \\ & = \phi(\alpha(\mu(Bw, Tw, t))). \end{aligned}$$

Thus we get $\alpha(\mu(Bw, Tw, t)) = 1$. Hence $\mu(Bw, Tw, t) = 1$, i.e., $Tw = Bw$. Therefore we have

$$Au = Su = Bw = Tw = z \text{ (say)} \quad (1)$$

From (iv), we have

$$AAu = ASu = SAu = SSu \quad (2)$$

and

$$BTw = TBw = TTW = BBW \quad (3)$$

We show $z = Az$. From (iv), (1), (2) we have

$$\alpha(\mu(z, Az, t)) = \alpha(\mu(Au, AAu, t)) = \alpha(\mu(AAu, Bw, t))$$

$$\begin{aligned}
&\geq \phi(\alpha(\min\{\mu(SAu, Tw, t), \mu(AAu, SAu, t), \mu(Bw, Tw, t), \\
&\quad \mu(AAu, Tw, t), \mu(Bw, SAu, t)\})) \\
&= \phi(\alpha(\min\{\mu(AAu, Au, t), 1, 1, \mu(AAu, Au, t), \mu(AAu, Au, t)\})) \\
&= \phi(\alpha(\mu(AAu, Au, t))) = \phi(\alpha(\mu(z, Az, t))),
\end{aligned}$$

which implies $\alpha(\mu(z, Az, t)) = 1$. Thus we have $\alpha(\mu(z, Az, t)) = 1$, i.e., $z = Az$. From (1) and (2) we get

$$z = Az = Sz \quad (4)$$

Next, we show $z = Bz$. Again from (iv), (1), (3) we have

$$\begin{aligned}
&\alpha(\mu(z, Bz, t)) = \alpha(\mu(Bw, BBw, t)) = \alpha(\mu(Au, BBw, t)) \\
&\geq \phi(\alpha(\min\{\mu(Su, TBw, t), \mu(Au, Su, t), \\
&\quad \mu(BBw, TBw, t), \mu(Au, TBw, t), \mu(BBw, Su, t)\})) \\
&= \phi(\alpha(\min\{\mu(Bw, BBw, t), \mu(Bw, Bw, t), \\
&\quad \mu(BBw, BBw, t), \mu(Bw, BBw, t), \mu(BBw, Bw, t)\})) \\
&= \phi(\alpha(\min\{\mu(Bw, BBw, t), 1, 1, \mu(Bw, BBw, t), \mu(Bw, BBw, t)\})) \\
&= \phi(\alpha(\mu(Bw, BBw, t))) = \phi(\alpha(\mu(z, Bz, t)))
\end{aligned}$$

which implies $\alpha(\mu(z, Bz, t)) = 1$. Thus we have $\mu(z, Bz, t) = 1$ i.e., $z = Bz$. Thus from (1) and (3) we get

$$z = Bz = Tz.$$

Therefore, by (4), we have

$$z = Az = Sz = Tz = Bz.$$

For the uniqueness, let w be an another common fixed point of A, B, S and T . Now from (4) we get

$$\begin{aligned}
&\alpha(\mu(z, w, t)) = \alpha(\mu(Az, Bw, t)) \\
&\geq \phi(\alpha(\min\{\mu(Sz, Tw, t), \mu(Az, Sz, t), \mu(Bw, Tw, t), \\
&\quad \mu(Az, Tw, t), \mu(Bw, Sz, t)\})) \\
&= \phi(\alpha(\min\{\mu(z, w, t), \mu(z, z, t), \mu(w, w, t), \mu(z, w, t), \mu(w, z, t)\}))
\end{aligned}$$

$$\begin{aligned}
&= \phi(\alpha(\min\{\mu(z, w, t), 1, 1, \mu(w, z, t), \mu(w, z, t)\})) \\
&= \phi(\alpha(\mu(z, w, t))),
\end{aligned}$$

which implies that $\alpha(\mu(z, w, t)) = 1$ and so $\mu(z, w, t) = 1$.

Hence $w = z$.

Corollary 4.4. *Let (X, μ) be a fuzzy symmetric space that satisfies (C_C) and (H_E) and let A, B, S and T be self-mappings of X and $\alpha \in \Lambda$ and $\phi \in \Phi$ satisfying*

- (i) $AX \subset TX$ and $BX \subset SX$;
- (ii) the pair (B, T) satisfies property $(E.A.)$ (resp. (A, S) satisfies property $(E.A.)$);
- (iii) the pairs (A, S) and (B, T) are weakly compatible;
- (iv) for any $x, y \in X$,

$$\alpha(\mu(Ax, By, t) \geq \phi(\alpha(\min\{\mu(Sx, Ty, t), \mu(By, ty, t), \mu(By, Sx, t)\})));$$

- (v) SX is a μ -closed subset of X (resp. TX is a μ -closed sub set of X).

Then A, B, S and T have a unique common fixed point in X .

Theorem 4.5. *Let (X, μ) be a fuzzy symmetric space that satisfies (W_4) and (H_E) and let A, B, S and T be self-mappings of X and $\alpha \in \Lambda$ and $\phi \in \Phi$ satisfying*

- (i) $AX \subset TX$ and $BX \subset SX$;
- (ii) the pair (B, T) satisfies property $(E.A.)$ (resp. (A, S) satisfies property $(E.A.)$);
- (iii) the pairs (A, S) and (B, T) are weakly compatible;
- (iv) for any $x, y \in X$,

$$\begin{aligned}
\alpha(\mu(Ax, By, t)) \geq \phi(\alpha(\min\{\mu(Sx, Ty, t), \mu(Ax, Sx, t), \\
\mu(By, Ty, t), \mu(Ax, Ty, t), \mu(By, Sx, t)\})));
\end{aligned}$$

- (v) one of AX, BX, SX and TX is complete subspace of X .

Then A, B, S and T have a unique common fixed point in X .

Proof. As in proof of Theorem(4.3), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and a point $l \in X$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(Tx_n, l, t) &= \lim_{n \rightarrow \infty} \mu(Bx_n, l, t) = \lim_{n \rightarrow \infty} \mu(Bx_n, Tx_n, t) \\ &= \lim_{n \rightarrow \infty} \mu(Sy_n, Tx_n, t) = 1 \end{aligned}$$

and $Bx_n = Sy_n$. We now show that $\lim_{n \rightarrow \infty} \mu(Ay_n, l, t) = 1$.

From (iv) we have

$$\begin{aligned} &\alpha(\mu(Ay_n, Bx_n, t)) \\ &\geq \phi(\alpha(\min\{\mu(Sy_n, Tx_n, t), \mu(Bx_n, Tx_n, t), \mu(Bx_n, Sy_n, t)\})). \end{aligned}$$

Let $n \rightarrow \infty$. We have

$$\lim_{n \rightarrow \infty} \alpha(\mu(Ay_n, Bx_n, t)) \geq \phi(\alpha(1)) = 1.$$

Thus, $\lim_{n \rightarrow \infty} \mu(Ay_n, Bx_n, t) = 1$. By (W_4) , we get

$$\lim_{n \rightarrow \infty} \mu(Ay_n, l, t) = 1.$$

If SX is complete subspace of X , then there exists $u \in X$ such that $l = Su$. Thus we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(Ay_n, Su, t) &= \lim_{n \rightarrow \infty} \mu(Bx_n, Su, t) \\ &= \lim_{n \rightarrow \infty} \mu(Tx_n, Su, t) = \lim_{n \rightarrow \infty} \mu(Sy_n, Su, t) = 1. \end{aligned}$$

We now show that $Au = Su$. From (iv) we have

$$\begin{aligned} &\alpha(\mu(Au, Bx_n, t)) \\ &\geq \phi(\alpha(\min\{\mu(Su, Tx_n, t), \mu(Bx_n, Tx_n, t), \mu(Bx_n, Su, t)\})). \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} &\lim_{n \rightarrow \infty} \alpha(\mu(Au, Bx_n, t)) \geq \phi(\alpha(1)) = 1, \\ \implies &\lim_{n \rightarrow \infty} \alpha(\mu(Au, Bx_n, t)) = 1. \end{aligned}$$

By Proposition(3.1), (X, μ) satisfies (W_3) and we have $Su = Au = z$ (say). By (iii) we have

$$Az = Sz. \tag{5}$$

From (i) there exists $v \in X$ such that $Au = Tv$. Thus we get $Au = Tv = Su = z$. We claim that $Bv = Tv$. If not, then we have

$$\begin{aligned} \alpha(\mu(Tv, Bv, t)) &= \alpha(\mu(Au, Bv, t)) \\ &\geq \phi(\alpha(\min\{\mu(Su, Tv, t), \mu(Bv, Tv, t), \mu(Bv, Su, t)\})) \\ &= \phi(\alpha(\min\{\mu(Tv, Tv, t), \mu(Bv, Tv, t), \mu(Bv, Tv, t)\})) \\ &= \phi(\alpha(\mu(Bv, Tv, t))) \end{aligned}$$

$$\implies \alpha(\mu(Bv, Tv, t)) > \alpha(\mu(Bv, Tv, t)),$$

which is a contradiction. Thus we have $Bv = Tv$. Therefore, we get

$$Bv = Tv = Su = Au = z. \quad (6)$$

From (iii) we have $Bz = Tz$. We show that $z = Az$. From (iii), (1) and (2) we have

$$\begin{aligned} \alpha(\mu(z, Az, t)) &= \alpha(\mu(Az, Bv, t)) \\ &\geq \phi(\alpha(\min\{\mu(Sz, Tv, t), \mu(Bv, Tv, t), \mu(Bv, Sz, t)\})) \\ &= \phi(\alpha(\min\{\mu(Az, z, t), \mu(z, z, t), \mu(z, Az, t)\})) \\ &= \phi(\alpha(\mu(z, Az, t))), \end{aligned}$$

which implies $\alpha(\mu(z, Az, t)) = 1$ and so $\mu(z, Az, t) = 1$.

Hence $z = Az$. From (5) we have

$$z = Az = Sz. \quad (7)$$

We show that $z = Bz$. Using $Bz = Tz$ and from (iv) and (4) we have

$$\begin{aligned} \alpha(\mu(z, Bz, t)) &= \alpha(\mu(Az, Bz, t)) \\ &\geq \phi(\alpha(\min\{\mu(Sz, Tz, t), \mu(Bz, Tz, t), \mu(Bz, Sz, t)\})) \\ &= \phi(\alpha(\min\{\mu(Bz, z, t), 1, \mu(z, Bz, t)\})) \\ &= \phi(\alpha(\mu(z, Bz, t))), \end{aligned}$$

which implies $\alpha(\mu(z, Bz, t)) = 1$ and so $\mu(z, Bz, t) = 1$.
Hence $z = Bz$ and thus we have

$$z = Bz = Tz. \quad (8)$$

There fore , by (7) and (8) we have

$$z = Az = Bz = Tz = Sz.$$

For the uniqueness , let w be an another common fixed point of A, B, S and T . If $w \neq z$ then from (iv) we get

$$\begin{aligned} \alpha(\mu(z, w, t)) &= \alpha(\mu(Az, Bw, t)) \\ &\geq \phi(\alpha(\min\{\mu(Sz, Tw, t), \mu(Bw, Tw, t), \mu(Bw, Sz, t)\})) \\ &= \phi(\alpha(\min\{\mu(z, w, t), \mu(w, w, t), \mu(w, z, t)\})) \\ &= \phi(\alpha(\min\{\mu(z, w, t), 1, \mu(w, z, t)\})) \\ &= \phi(\alpha(\mu(z, w, t))). \end{aligned}$$

$$\implies \alpha(\mu(z, w, t)) > \alpha(\mu(z, w, t)),$$

which is a contradiction . Thus we have $\alpha(\mu(z, w, t)) = 1$ and
so $\mu(z, w, t) = 1$. Hence $w = z$.

Acknowledgments

The authors wish to thank the chief editor and the reviewers for their
valuable suggestions to rectify the paper.

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