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STRONGLY S-DENSE MONOMORPHISMS

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Abstract. Let $\mathcal{M}$ be a class of (mono)morphisms in a category $\mathcal{A}$. To study mathematical notions, such as injectivity, tensor products and flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$). In this paper we take $\mathcal{A}$ to be the category **Act-S** of S-acts, for a semigroup S, and $\mathcal{M}_{sd}$ to be the class of strongly s-dense monomorphisms and study the categorical properties, such as limits and colimits, of this class.

Key Words: Strongly s-dense, Semigroup, Limit, Colimit.

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1. Introduction and Preliminaries

To study mathematical notions in a category $\mathcal{A}$ with respect to a class $\mathcal{M}$ of its morphisms, one should know some of the categorical properties of the pair ($\mathcal{A}$, $\mathcal{M}$). In this paper we take $\mathcal{A}$ to be the category **Act-S** and $\mathcal{M}_{sd}$ to be a particular interesting class of monomorphisms, to be called *strongly-s-dense(st-s-dense)* monomorphisms, and investigate its categorical properties.

Let us first recall the definition and some ingredients of the category **Act-S** needed in the sequel. For more information and the notions not mentioned here see, for example, [8].

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Recall that, for a semigroup S, a set A is a right S-*act* (or an S-*set*) if there is, a so called, an *action* $\mu : A \times S \to A$ such that, denoting $\mu(a, s) := as$, $a(st) = (as)t$ and, if S is a monoid with 1, $a1 = a$.

Each semigroup S can be considered as an S-act with the action given by its multiplication. Notice that, adjoining an external left identity 1 to a semigroup S, an S-act $S^1 := S \cup \{1\}$ is obtained.

The definitions of a *subact* B of A, written as $B \subseteq A$, an *extension* of A, a *congruence* ρ on A, a *quotient* A/ρ of A, and a *homomorphism* between S-acts are clear. The category of all (right) S-acts and homomorphisms between them is denoted by **Act-S**.

The class of all S-acts is an equational class, and so the category **Act-S** is complete (has all products and equalizers). In fact, limits in this category are computed as in the category **Set** of sets and equipped with a natural action. In particular, the terminal object of **Act-S** is the singleton $\{0\}$, with the obvious S-action. Also, for S-acts A, B, their cartesian product $A \times B$ with the S-action defined by $(a, b)s = (as, bs)$ is the *product* of A and B in **Act-S**.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

in **Act-S** is the subact $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps. Notice that for the case where g is an inclusion, P can be taken as $f^{-1}(C)$.

All colimits in **Act-S** exist and are calculated as in **Set** with the natural action of S on them. In particular, $\emptyset$ with the empty action of S on it, is the initial object of **Act-S**. Also, the *coproduct* of S-acts A, B is their disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ with the obvious action, and coproduct injections are defined naturally.

The pushout of a given diagram

$$\begin{array}{ccc} & A & \xrightarrow{g} & C \\ f & \downarrow & & \\ & B & & \end{array}$$

in **Act-S** is the factor act $Q = (B \sqcup C)/\theta$ where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $q_1 = \pi u_C : C \to (B \sqcup C)/\theta,\ q_2 = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Act-S** are constructed analogously.

Recall that for a family $\{A_i : i \in I\}$ of S-acts, each with a unique fixed element 0, the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number of indices.

Let **I** be a small category and $\mathcal{A} : \mathbf{I} \to$ **Act-S** be a diagram in **Act-S** determining the acts A_α, for $\alpha \in I = Obj\mathbf{I}$, and S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor\mathbf{I}$. Recall that the limit of this diagram is $\varprojlim_\alpha A_\alpha := \bigcap_{\alpha \in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha \in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta} p_\alpha(a) = p_\beta(a)\}$ and p_α, p_β are the α, βth projection maps of the product. The limit S-maps are $q_\alpha : \varprojlim_\alpha A_\alpha \to A_\alpha$. Also the limit has the universal property which is, if $\{f_\alpha : A \to A_\alpha\}$ is a family of morphisms such that $g_{\alpha\beta} f_\alpha(a) = f_\beta(a)$, then there is a morphism $f : A \to \varprojlim_\alpha A_\alpha$ such that $q_\alpha f = f_\alpha$.

Remind that a directed system of S-acts and S-maps is a family $(B_\alpha)_{\alpha \in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha \leq \beta \in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, also $g_{\alpha\alpha} = id$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha \in I}, (g_{\alpha\beta})_{\alpha \leq \beta \in I})$ in **Act-S** is given as $\varinjlim_\alpha B_\alpha = \coprod_\alpha B_\alpha / \rho$ where the congruence ρ is given by $b_\alpha \rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$, in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for

$\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \varinjlim_\alpha B_\alpha$ is the natural S-map. Also directed colimit has a dual universal property of limit.

2. C^{sd}-Closure operator

In this section, we introduce and briefly study a closure operator, so called C^{sd}-*Closure operator*. Let us denote the finite subset T of S by $T \subset\!\!\cdot\, S$. First recall the following definition of C^{sd}-closure operator.

Definition 2.1. A family $C^{sd} = (C^{sd}_B)_{B \in \mathbf{Act-S}}$, with $C^{sd}_B : sub(B) \to Sub(B)$, is defined as

$C^{sd}_B(A) = \{b \in B : bS \subseteq A \text{ and } \forall T \subset\!\!\cdot\, S,\ \exists a_T \in A,\ a_T t = bt (t \in T)\}.$

It is easy to show that C^{sd} is a closure operator on **Act-S** in the sense of [5]. This means that $C^{sd}_B(A)$ is a subact of B and,

(i) $A \subseteq C^{sd}_B(A)$,

(ii) $A_1 \subseteq A_2 \subseteq B$ implies $C^{sd}_B(A_1) \subseteq C^{sd}_B(A_2)$,

(iii) for every homomorphism $f : B \to D$ and each subact A of B, $f(C^{sd}_B(A)) \subseteq C^{sd}_D(f(A))$.

We just prove (iii). Let $f : B \to C$ be a homomorphism and $b \in C^{sd}_B(A)$. For every $s \in S, f(b)s = f(bs) \in f(A)$, then $f(b)S \subseteq f(A)$. If T is a finite subset of S, there is an element $a_T \in A$ such that $a_T t = bt(t \in T)$. Hence $f(a_T)t = f(b)t$, which means that $f(b) \in C^{sd}_B(f(A))$.

Notice that in the case where S is a monoid, $C^{sd}_B(A) = A$ for every $A \subseteq B$. So, it is more interesting to consider the closure operator C^{sd} only for semigroups, or for semigroup part S of monoids of the form $T = S^1$.

Dikranjan and Tholen in [5] state some properties of a closure operator in general. Here we are going to investigate those for the closure operator C^{sd} satisfy or not.

Definition 2.2. The closure operator C^{sd} is said to be:

(1) idempotent, if for $A \subseteq B$, $C^{sd}_B(A) = C^{sd}_B(C^{sd}_B(A))$.

(2) hereditary, if for $A_1 \subseteq A_2 \subseteq B$, $C^{sd}_{A_2}(A_1) = C^{sd}_B(A_1) \cap A_2$.

(3) weakly hereditary, if for every $A \subseteq B$, $C^{sd}_{C^{sd}_B(A)}(A) = C^{sd}_B(A)$.

(4) grounded, if $C^{sd}_B(\emptyset) = \emptyset$.

(5) additive, if for subacts A, C of B, $C^{sd}_B(A \bigcup C) = C^{sd}_B(A) \bigcup C^{sd}_B(C)$.

(6) productive, if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C^{sd}_B(A) = \prod_i C^{sd}_{B_i}(A_i)$.

(7) fully additive, if for $A_i \subseteq B$, $C^{sd}_B(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} C^{sd}_B(A_i)$.

(8) discrete, if $C^{sd}_B(A) = A$ for every S-act B and $A \subseteq B$.

(9) trivial, if $C^{sd}_B(A) = B$ for every B and $A \subseteq B$.

(10) minimal, if for $C \subseteq A \subseteq B$ one has $C^{sd}_B(A) = A \cup C^{sd}_B(C)$.

Theorem 2.3. *The closure operator C^{sd} is hereditary, weakly hereditary, grounded and productive.*

Proof. It is easy to check that the closure operator C^{sd} is hereditary, weakly hereditary and grounded. We just prove productivity. Let $b \in C^{sd}_B(A)$, $b = \{b_i\}$. For every $s \in S, bs \in A$, then for each $i \in I$, $b_i s \in A_i$. Let T be a finite subset of S. So there exists $\{a_i\} \in \prod A_i$ such that $\{a_i\}t = \{b_i\}t (t \in T)$. Thus for every $i \in I, a_i t = b_i t$ and hence for each $i \in I, b_i \in C^{sd}_{B_i}(A_i)$. It deduces that $b \in \prod C^{sd}_{B_i}(A_i)$. The converse is done in a similar way. $\square$

Theorem 2.4. *The closure operator C^{sd} is idempotent.*

Proof. By definition of the closure operator for each $A \subseteq B$ we see that $C^{sd}_B(A) \subseteq C^{sd}_B(C^{sd}_B(A))$. *Conversly*, let $b \in C^{sd}_B(C^{sd}_B(A))$. For each $t \in S$, there exists $a_1 \in C^{sd}_B(A)$ such that $bt = a_1 t$, and since $a_1 \in C^{sd}_B(A)$, there exists $a \in A$ such that $a_1 t = at$, which implies $bt \in A$. Thus $bS \subseteq A$. Now let T be a finite subset of S. There is an element $a_T \in C^{sd}_B(A)$ such that $a_T t = bt (t \in T)$. Since $a_T \in C^{sd}_B(A)$, there exists an element $a'_T \in A$ such that $a'_T t = a_T t$ and hence $a'_T t = bt$. Therefore $b \in C^{sd}_B(A)$. $\square$

Theorem 2.5. *Let A and C be two subacts of B such that $A \bigcap C = \emptyset$. Then $C^{sd}_B(A \bigcup C) = C^{sd}_B(A) \bigcup C^{sd}_B(C)$.*

Proof. By definition of the closure operator, $C^{sd}_B(A)\bigcup C^{sd}_B(C) \subseteq C^{sd}_B(A\bigcup C)$. Consider $x \in C^{sd}_B(A\bigcup C)$. So $xS \subseteq A\bigcup C$. Let $xS\bigcap A \neq \emptyset$ and $xS\bigcap C \neq \emptyset$. Then there exist elements $t_1, t_2 \in S$ such that $xt_1 \in A \setminus C$ and $xt_2 \in C \setminus A$. Set $T_1 = \{t_1, t_2\}$. There is an element $y \in A \cup C$ such that $xt_1 = yt_1$ and $xt_2 = yt_2$. If $y \in A$, then $xt_2 \in A$ and if $y \in C$, then $xt_1 \in C$ which are contradictions. Suppose that $xS \cap C = \emptyset$, so $xS \subseteq A$. For every finite subset $T \subseteq S$, there exists $x_T \in A \cup C$ such that $xt = x_T t$. It is clear that $x_T \notin C$, thus $x_T \in A$ and hence $x \in C^{sd}_B(A)$. □

Now we show that some of the properties of closure operator do not satisfy in general. But first recall another closure operator C^d defined by

$$C^d_B(A) = \{b \in B \mid bS \subseteq A\}.$$

A subact A is, by definition, s-dense in B if $C^d_B(A) = B$.

Lemma 2.6. *The closure operator C^{sd} is not necessary fully additive.*

Proof. Let $S = (\mathbb{N}, min)$ be a semigroup, $B = \mathbb{N}^\infty$ and $A = \mathbb{N}$. Set $A_n = \{m \in \mathbb{N} \mid m \leq n\}$ for each $n \in \mathbb{N}$. It is easy to check that $C^{sd}_{\mathbb{N}^\infty}(A_n) = A_n$ and hence $\bigcup C^{sd}_{\mathbb{N}^\infty}(A_n) = \bigcup(A_n) = \mathbb{N}$, but $C^{sd}_{\mathbb{N}^\infty}(\cup A_n) = C^{sd}_{\mathbb{N}^\infty}(\mathbb{N}) = \mathbb{N}^\infty$. □

Lemma 2.7. (i) *Let $\{A_i \mid i \in I\}$ be a family of subacts of A. If $C^{sd}_A(\bigcap A_i) = C^d_A(\bigcap A_i)$, then $C^{sd}_A(\bigcap A_i) = \bigcap C^{sd}_A(A_i)$.*

(ii) *If for every $i \in I$, $C^{sd}_A(A_i) = C^d_A(A_i)$ and $C^{sd}_A(\bigcap A_i) = \bigcap C^{sd}_A(A_i)$, then $C^{sd}_A(\bigcap A_i) = C^d_A(\bigcap A_i)$.*

Proof. (i) By the hypothesis we see that $\bigcap C^{sd}_A(A_i) \subseteq \bigcap C^d_A(A_i) = C^d_A(\bigcap A_i) = C^{sd}_A(\bigcap A_i)$. Thus $C^{sd}_A(\bigcap A_i) = \bigcap C^{sd}_A(A_i)$.

(ii) $C^d_A(\bigcap A_i) = \bigcap C^d_A(A_i) = \bigcap C^{sd}_A(A_i) = C^{sd}_A(\bigcap A_i)$. □

Lemma 2.8. *For every semigroup S, the closure operator C^{sd} is not discrete nor trivial and minimal.*

Proof. Let $0 \in A$ be a fixed element of a nonempty S-act A. Adjoin two elements θ, ω to A with actions $\omega s = \omega$ and $\theta s = 0$. Consider

$B = A \cup \{\theta, \omega\}$. It is clear that $C_B^{sd}(A) = A \cup \{\theta\}$. This shows that C^{sd} is neither discrete nor trivial. Also, it is not minimal. Because, adjoining two elements θ, ω to a nonempty S-act C with actions $\omega s = \theta$ and $\theta s = \theta$, and taking $A = C \cup \{\theta\}$, $B = C \cup \{\theta, \omega\}$, we get $C \subset A \subset B$, and $C_B^{sd}(A) = B$ while $C_B^{sd}(C) = C$. □

Theorem 2.9. (i) *The closure C^{sd} is discrete if and only if S has a left identity element.*

(ii) *The closure C^{sd} is trivial if and only if S is the empty set.*

Proof. (i) Let C^{sd} be a discrete closure operator and S do not have a left identity. Consider $t_0 \in S$ and adjoin an element x to S defined by $xs = t_0 s$ for each $s \in S$. It is clear that $C_{S^x}^{sd}(S) = S^x$ and by the hypothesis we have $C_{S^x}^{sd}(S) = S$. So $S^x = S$ which is a contradiction.

(ii) Let the closure C^{sd} be trivial and $S \neq \emptyset$. If B is an S-act such that all of its elements are fixed and A is a subact of B, then $C_B^{sd}(A) = A \neq B$ which is a contradiction. Thus S is the empty set.

Conversly, let $S = \emptyset$. Then it is clear that $C_B^{sd}(A) = B$. □

3. Categorical properties of st-s-dense monomorphisms

In this section we investigate the categorical and algebraic properties of the class $\mathcal{M}_{sd}$ of st-s-dense monomorphisms in the following three subsections.

3.1. **Composition Property.**

In this subsection we investigate some properties of the class $\mathcal{M}_{sd}$ of strongly-s-dense monomorphisms which are mostly related to the composition of st-s-dense monomorphisms. These properties and the ones given in the next two subsections are what normally used to study injectivity with respect to a class of monomorphisms (see [1]). The class $\mathcal{M}_{sd}$ is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.

Definition 3.1. An S-act A is *strongly-s-dense (or simply st-s-dense)* subact of B, if for every $b \in B, bS \subseteq A$ and for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt (t \in T)$. In other word $C_B^{sd}(A) = B$. A monomorphism $A \xrightarrow{f} B$ is an st-s-dense monomorphism, if $f(A)$ is an st-s-dense subact of B.

Lemma 3.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two monomorphisms. The monomorphism gf is an st-s-dense monomorphism if and only if f and g are st-s-dense monomorphisms too.*

Proof. Suppose that gf is an st-s-dense monomorphism. It is clear that f and g are s-dense monomorphisms. Now let $b \in B$, $c \in C$ and T be a finite subset of S. Since gf is st-s-dense, then there exist a_{T_1} and a_{T_2} in A such that $a_{T_1} t = bt$ and $a_{T_2} t = ct$ for each $t \in T$. Thus f and g are st-s-dense monomorphisms.

Conversely, assume each f and g is st-s-dense monomorphism. Let T be a finite subset of S and $c \in C$. By the hypothesis there exist $b \in B$ and $a \in A$ such that for every $t \in T$, $ct = bt$ and $bt = at$. If T is a one element subset of S, the above equations show that A is s-dense in C. Therefore A is st-s-dense in C. □

Definition 3.3. The semigroup S locally has left identity element if every finitely generated right ideal of S has a left identity element in S.

In the following lemma we have the characterization of a semigroup S over which all s-dense extensions are st-s-dense. First recall that every st-s-dense monomorphism is an s-dense monomorphism.

Lemma 3.4. *For a semigroup S, the following are equivalent:*

(i) *Every s-dense extension is st-s-dense extension.*

(ii) *The semigroup S is st-s-dense in S^1.*

(iii) *The semigroup S locally has left identity element.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii) Let $I = \bigcup_{i=1}^{n} t_iS^1$ be a finitely generated right ideal of S. Since S is st-s-dense in S^1, then there exists $s_0 \in S$ such that $s_0t_i = 1t_i = t_i(1 \leq i \leq n)$. So S locally has left identity.

(iii)⇒(i) Let B be an s-dense extension of A, T be a finite subset of S and $b \in B$. Then $A \cup bS^1 = A \cup \{b\}$ and since S locally has left identity element, there exists $t_0 \in S$ such that $t_0t = t(t \in T)$. Thus $(bt_0)t = bt$ and so A is st-s-dense in B. □

3.2. **Limits of st-s-dense monomorphisms.**

In this subsection we will investigate the behaviour of st-s-dense monomorphisms with respect to limits. First recall that, we say the class $\mathcal{M}_{sd}$ is closed under products (coproduct, direct sum), if for every family of st-s-dense monomorphisms $\{f_i : A_i \to B_i\}$, $\prod f_i : \prod A_i \to \prod B_i(\coprod f_i, \oplus f_i)$ is st-s-dense monomorphism. The proof of the following is straightforward.

Proposition 3.5. (i) *The class $\mathcal{M}_{sd}$ is closed under products.*

(ii) *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a family of st-s-dense monomorphisms. Then their product homomorphism $h : A \to \prod_{\alpha\in I} B_\alpha$ is also an st-s-dense monomorphism.*

Proposition 3.6. *The class $\mathcal{M}_{sd}$ is closed under direct sums.*

Theorem 3.7. *In the category* **Act-S***, the following are equivalent:*

(i) *pullbacks transfer st-s-dense monomorphisms.*

(ii) *The semigroup S locally has left identity element.*

Proof. (i)⇒(ii) By Lemma 3.4, it is enough to show that S is st-s-dense in S^1. Let E be an injective S-act and 0 be a fixed element of E which exists by [4]. Adjoin an element θ to E by action $\theta s = 0$ for all $s \in S$. Then, E is clearly st-s-dense in $E^\theta = E \cup \{\theta\}$. Taking a homomorphism $f : S^1 \longrightarrow E^\theta$ given by $f(s) = \theta s$ $(s \in S^1)$, one gets the pullback diagram:

$$\begin{array}{ccc} S & \xrightarrow{\tau} & S^1 \\ f\downarrow & & \downarrow f \\ E & \xrightarrow{\tau'} & E^\theta \end{array}$$

where τ, τ' are inclusion maps. By the hypothesis, Since τ' is st-s-dense; so is τ.

(ii)⇒(i) Consider the pullback diagram: $\begin{array}{ccc} P & \xrightarrow{q} & B \\ p\downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$ where $P = \{(a,b) | f(a) = g(b)\}$ and f is an st-s-dense monomorphism. We show that q is a monomorphism. Let $q(a_1, b_1) = q(a_2, b_2)$. So $b_1 = b_2$ and $fp(a_1, b_1) = fp(a_2, b_2)$ which implies $a_1 = a_2$ by using that f is a monomorphism. Thus q is a monomorphism. Now one should show that q is an s-dense monomorphism. Let $b \in B$ and $s \in S$. Since f is s-dense, $g(bs) = g(b)s \in f(A)$. So there exist elements $a_s \in A (s \in S)$, such that $g(bs) = f(a_s)$. Hence $(a_s, bs) \in P$ and $bs = q(a_s, bs)$ which implies that q is s-dense. Now by using Lemma 3.4 the proof is complete. □

3.3. **Colimits of st-s-dense monomorphisms.**

This subsection is devoted to the study of the behaviour of st-s-dense monomorphisms with respect to colimits.

Proposition 3.8. *$\mathcal{M}_{sd}$ is closed under coproducts.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & A_i & \xrightarrow{f_i} & B_i & \\ u_i & \downarrow & & \downarrow & u'_i \\ & \coprod_{i\in I} A_i & \xrightarrow{f} & \coprod_{i\in I} B_i & \end{array}$$

in which $\{f_i : A_i \to B_i : i \in I\}$ is a family of st-s-dense monomorphisms. We want to show that $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ is an st-s-dense monomorphism. It is not difficult to show that f is an s-dense monomorphism. Now let $b \in \coprod_{i\in I} B_i$, and T be a finite subset of S. Then there exists

$i \in I$ and $b_i \in B_i$ such that $b = u'_i(b_i)$. Since f_i is st-s-dense, there exists $a_i \in A_i$ such that $f_i(a_i)t = b_i t$ for $t \in T$. Thus $fu_i(a_i)t = u'_i f_i(a_i)t = bt$ which means that f is an st-s-dense monomorphism. □

Theorem 3.9. *Pushouts transfer st-s-dense monomorphisms. That is, for the following pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

in **Act-S***, If f is st-s-dense, then h is st-s-dense too.*

Proof. Recall that $Q = (B \sqcup C)/\theta$ where $\theta = \rho(H)$ and H consists of all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C$, $u_C : C \to B \sqcup C$ are coproduct injections. And $h = \pi u_C : C \to (B \sqcup C)/\theta$, $h' = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. By [6], pushout transfers s-dense monomorphism. So h is an s-dense monomorphism.

Let T be a finite subset of S and $q \in Q$ be a solution for $\Sigma = \{xt_1 = h(c_1), xt_2 = h(c_2), ..., xt_m = h(c_m)\}$. Two cases may be occur: (i) there exists $c \in C$ such that $q = [u_C(c)]_{\rho(H)}$.Then for all $1 \leq i \leq m$, $h(c)t_i = h(c_i)$. (ii) There exists $b \in B$ such that $q = [u_B(b)]_{\rho(H)}$. For every $1 \leq i \leq m$, $[u_B(bt_i)]_{\rho(H)} = h(c_i) = [u_C(c_i)]_{\rho(H)}$, and so there exist $a_{i1}, a_{i2}, ...a_{in} \in A$, $s_{i1}, s_{i2}, ..., s_{in} \in S^1$ such that $u_B(b)t_1 = u_B f(a_{i1})s_{i1}$, $u_C g(a_{i1})s_{i1} = u_C g(a_{i2})s_{i2}, u_B f(a_{i2})s_{i2} = u_B f(a_{i3})s_{i3}$, ..., $u_B f(a_{i(n-1)})s_{i(n-1)} = u_B f(a_{in})s_{in}$, $u_C g(a_{in})s_{in} = u_C(c_i)$. Since f is a monomorphism, $a_{i2}s_{i2} = a_{i3}s_{i3}, a_{i4}s_{i4} = a_{i5}s_{i5}, \cdots, a_{i(n-1)}s_{i(n-1)} = a_{in}s_{in}$, and hence $u_C g(a_{i1}s_{i1}) = u_C g(a_{i2}s_{i2}) = u_C g(a_{i3}s_{i3}) = \cdots = u_C g(a_{in}s_{in}) = u_C(c_i)$. Thus $g(a_{i1}s_{i1}) = c_i$. Now, for all $1 \leq i \leq m$, $bt_i = f(a_{i1}s_{i1}) \in f(A)$, and hence $\Sigma_1 = \{xt_1 = f(a_{11}s_{11}), \cdots xt_m = f(a_{m1}s_{m1})\}$ has a solution $b \in B$. So Σ_1 has a solution $f(a)$ for some $a \in A$. Then for every $1 \leq i \leq m$, $hg(a)t_i = h'f(a)t_i = h'f(a_{i1}s_{i1}) = hg(a_{i1}s_{i1}) = h(c_i)$ which yields $hg(a)$ is a solution for Σ. □

Theorem 3.10. *The category* **Act-S** *has* $\mathcal{M}_{sd}$*-directed colimits.*

Proof. Let $g_{\alpha\beta} : B_\alpha \to B_\beta$ $(\alpha \leq \beta)$ be a directed system of homomorphisms and $h : A \to \underrightarrow{lim}_\alpha B_\alpha$ be a directed colimit in **Act-S** of st-s-dense monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$, with the colimit maps $g_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$. Since $h = \underrightarrow{lim}_\alpha h_\alpha = g_\alpha h_\alpha$ for each $\alpha \in I$, then h is an s-dense monomorphism because of each h_α. Now we show that h is st-s-dense. Let $b \in \underrightarrow{lim}_\alpha B_\alpha$ and T be a finite subset of S. Since $b \in \underrightarrow{lim}_\alpha B_\alpha$, there exists $x_\alpha \in B_\alpha$ such that $b = [x_\alpha]_\rho$ and since h_α is st-s-dense, there exists an element $a_T \in A$ with $h_\alpha(a_T)t = x_\alpha t$ for all $t \in T$. Then $bt = [x_\alpha]_\rho t = g_\alpha(x_\alpha)t = g_\alpha(x_\alpha t) = g_\alpha h_\alpha(a_T)t = h(a_T)t$. $\square$

We say that *multiple pushouts* transfer st-s-dense monomorphisms if in multiple pushout $(P, A_\alpha \overset{h_\alpha}{\to} P)$ of a family of st-s-dense monomorphisms $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$, every h_α, $\alpha \in I$, is an st-s-dense monomorphism. In multiple pushout diagram for every $\alpha, \beta \in I$, $h_\alpha f_\alpha = h_\beta f_\beta$, which calls diagonal map.

Theorem 3.11. *Multiple pushouts transfer st-s-dense monomorphisms.*

Proof. Let $(P, A_\alpha \overset{h_\alpha}{\to} P)$ be a multiple pushout of the family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of st-s-dense monomorphisms. We know that $P = \coprod A_\alpha/\rho(H)$ where $H = \{(f_\alpha(a), f_\beta(a)) \mid a \in A, \alpha, \beta \in I\}$ (we have taken the image of each element of A_α under coproduct morphisms equal to itself). Let $h_\alpha(a) = h_\alpha(a')$, $a, a' \in A_\alpha$. So there exist $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A, s_1, s_2, ..., s_n \in S^1$ where for $i = 1, ..., n, (p_i, q_i) \in H \cup H^{-1}$ and such that $a = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, ..., q_n s_n = a'$. Then, $a = f_\alpha(a_1)s_1$ and there exists $\beta \in I$ such that $f_\beta(a_1)s_1 = f_\beta(a_2)s_2$. Since f_β is a monomorphism, $a_1 s_1 = a_2 s_2$. Continuing this process, we get that $a_1 s_1 = a_2 s_2 = ... = a_n s_n$, and therefore $a = a'$. Now let $q \in P$ and $s \in S$. There exist $\beta \in I$ and $p \in A_\beta$ such that $q = h_\beta(p)$. Since f_β is s-dense then $ps = f_\beta(a_s)$ and so $qs = h_\beta(ps) = h_\beta(f_\beta(a_s)) = h_\alpha(f_\alpha(a_s))$. Thus h_α is s-dense. Let $x \in P$, $x = [p]_{\rho(H)}$ and T be a finite subset of S. If $p \in A_\alpha$, the result is true. If $p \in A_\beta$, $\beta \neq \alpha$, then for every

$t \in T$, $[pt]_{\rho(H)} = [f_\beta(a_T)t]_{\rho(H)} = h_\beta f_\beta(a_T t) = h_\alpha f_\alpha(a_T t)$ and thus h_α is an st-s-dense monomorphism. □

Corollary 3.12. *In every multiple pushout diagram of st-s-dense monomorphisms the diagonal map is an st-s-dense monomorphism.*

Proof. Apply Lemma 3.2 and Theorem 3.11. □

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