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SOME CURVATURE PROPERTIES OF PARA-KENMOTSU MANIFOLD WITH RESPECT TO ZAMKOVOY CONNECTION

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Abstract. In the present paper we study some properties of the para-Kenmotsu manifold with respect to Zamkovoy connection. We discuss locally ϕ-symmetric para-Kenmotsu manifold with respect to the Zamkovoy connection. Also, we study Ricci Soliton on para-Kenmotsu manifold with respect to Zamkovoy connection. Besides these, we discuss W_i-curvature tensor (i=0,1,2...9) with respect to Zamkovoy connection on para-Kenmotsu manifold.

Key words and phrases : Para-Kenmotsu manifold, Zamkovoy connection, Ricci soliton, W_i-curvature tensor.
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1. Introduction

The notion of para-Kenmotsu manifold analogous to the structure of Kenmotsu manifold [7] was introduced by Welyczko [23]. Also, Sinha and Sai Prasad [19] introduced para-Kenmotsu manifolds as a subclass of para-contact manifold. Further, para-Kenmotsu manifolds have been studied by many researcher. For instance, we see ([4], [12], [13], [17], [18]) and the references therein.

In 2008, the notion of Zamkovoy canonical connection (briefly, Zamkovoy connection) on para contact manifold was introduced by S. Zamkovoy

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[24]. Zamkovoy connection was defined as a canonical para contact connection whose torsion is the obstruction of paracontact manifold to be a para-Sasakian manifold. This connection was further studied by many authors. For instance, we see ([1], [2], [4], [5], [8], [9] [10], [11]). For an n-dimensional almost para-contact metric manifold M equipped with an almost para-contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection (∇^*) in terms of Levi-Civita connection (∇) is defined as

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{1.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes the set of all vector fields on M.

The concept of Ricci flow and its existence was introduced by R. S. Hamilton [6] in the year 1982. Hamilton observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifolds admits a geometric decomposition. By positive curvature operator, Hamilton also classified all compact manifolds of dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S, \tag{1.2}$$

where g is Riemannian metric, S is Ricci tensor and t is the time. A Ricci soliton is a self similar solution of the Ricci flow equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation:

$$L_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where, S is Ricci tensor, $L_V g$ denotes the Lie derivative of g along the vector field V. The Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. If the vector field V is gradient of a smooth function h, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the function h is called the potential function. Ricci soliton was further studied by many researchers. For more details, we refer ([14], [16], [20], [21]) and their references.

Definition 1.1. A Riemannian manifold M is said to be symmetric if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X, Y) Z = 0,$$

for all vector fields X, Y, Z, W on M.

Definition 1.2. A Riemannian manifold M is called locally ϕ-symmetric if its curvature tensor R satisfies the condition

$$\phi^2 (\nabla_W R)(X, Y) Z = 0,$$

for all vector fields X, Y, Z, W on M which are orthogonal to the structure tensor field of the manifold.

Definition 1.3. A non-flat Riemannian manifold M $(n > 2)$ is said to be ϕ-pseudo symmetric if its curvature tensor R satisfies
$\phi^2 (\nabla_W R)(X, Y) Z = 2A(W) R(X, Y) Z + A(X) R(W, Y) Z$
$+A(Y) R(X, W) Z + A(Z) R(X, Y) W + g(R(X, Y) Z, W) \rho,$
where A is a non-zero associated 1-form, ρ is a vector field defined by $g(W, \rho) = A(W)$ for every vector field W and ∇ denotes the operator of covariant differentiation with respect to the metric g.

Definition 1.4. A non-flat Riemannian manifold M $(n > 2)$ is called generalized Ricci-recurrent manifold if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X) S(Y, Z) + B(X) g(Y, Z),$$

where A and B are two non-zero 1-forms. Such a manifold shall be denoted by GR_n.

Definition 1.5. A Riemannian manifold M is said to be pseudo Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the relation

$$(\nabla_X S)(Y, Z) = 2A(X) S(Y, Z) + A(Y) S(X, Z) + A(Z) S(X, Y),$$

where A is a non-zero associated 1-form, ρ is a vector field defined by $g(X, \rho) = A(X)$ for every vector field X on M.

The paper is organized as follows:

Section-1 and **Section-2** are kept for indroduction and preliminaries. In **Section-3** we introduce Zamkovoy connection on para-Kenmotsu manifold**.** In **Section-4**, we have discussed para-Kenmotsu manifold admitting Zamkovoy connection and obtained Riemannian curvature tensor R^*, Ricci tensor S^*, scalar curvature r^*, Ricci operator Q^* with respect to Zamkovoy connection. **Section-5** concerns with locally ϕ-symmetric para-Kenmotsu manifold with respect to the connection ∇^*. **Section-6** contains the study of Ricci Soliton on para-Kenmotsu manifold with respect to Zamkovoy connection. In **Section-7**, we have discussed ϕ-pseudo-symmetric para-Kenmotsu manifold with respect to Zamkovoy connection. **Section-7** concerns with W_i-curvature tensors with respect to Zamkovoy connection on para-Kenmotsu manifold.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with an almost para-contact metric structure (ϕ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a a 1-form and g is a pseudo-Riemannian metric such that

$$\varphi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, \eta(\varphi X) = 0, \varphi\xi = 0, \tag{2.1}$$
$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$
$$g(X, \varphi Y) = -g(\varphi X, Y), g(X,\xi) = \eta(X), \tag{2.3}$$

for all vector fields X, Y on M.

If an almost paracontact metric manifold satisfies

$$(\nabla_X \varphi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for all vector fields X, Y on M, then M is called almost para-Kenmotsu manifold. A normal almost para-Kenmotsu manifold is said to be para-Kenmotsu manifold. The para-Kenmotsu structure for 3-dimensional normal almost para-contact metric structures was introduced by J. Welyczko [23].

Also for an n-dimensional para-Kenmotsu manifold M, following relations hold

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$
$$(\nabla_X \eta) Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$
$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$
$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$
$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.9}$$
$$S(X,\xi) = -(n-1)\eta(X), \tag{2.10}$$
$$Q\xi = -(n-1)\xi, \tag{2.11}$$

where R is the Riemannian curvature tensor, S is Ricci tensor and Q is Ricci operator.

3. Zamkovoy connection on para-Kenmotsu manifold

Lemma 3.1. *The relation between Zamkovoy connection* (∇^*) *and Levi-Civita connection* (∇) *on para-Kenmotsu manifold is given by*

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X + \eta(X)\phi Y, \tag{3.1}$$

with torsion tensor

$$T^*(X,Y) = \eta(X)Y - \eta(Y)X + \eta(X)\phi Y - \eta(Y)\phi X, \tag{3.2}$$

Proof. In view of (1.1) and (2.6), we have

$$(\nabla^*_X g)(Y,Z) = 0. \tag{3.3}$$

Suppose that the Zamkovoy connection ∇^* defined on an n-dimensional para-Kenmotsu manifold M is connected with the Levi-Civita connection ∇ by the relation

$$\nabla^*_X Y = \nabla_X Y + P(X,Y), \tag{3.4}$$

where $P(X,Y)$ is a tensor field of type $(1,1)$. Then, by definition of torsion tensor we have

$$T^*(X,Y) = P(X,Y) - P(Y,X). \tag{3.5}$$

Due to (3.3), Zamkovoy connection is a metric connection and hence from (3.5), we get

$$g(P(X,Y),Z) + g(P(X,Z),Y) = 0. \tag{3.6}$$

In view of (3.5) and (3.6), we get

$$\begin{aligned} & g(T^*(X,Y),Z) + g(T^*(Z,X),Y) + g(T^*(Z,Y),X) \\ = \; & g(P(X,Y),Z) - g(P(Y,X),Z) + g(P(Z,X),Y) \\ & - g(P(X,Z),Y) + g(P(Z,Y),X) - g(P(Y,Z),X) \\ = \; & 2g(P(X,Y),Z). \end{aligned} \tag{3.7}$$

Setting

$$\begin{aligned} g(T^*(Z,X),Y) &= g\left(\overline{T}(X,Y),Z\right), \\ g(T^*(Z,Y),X) &= g\left(\overline{T}(Y,X),Z\right), \end{aligned} \tag{3.8}$$

and using (3.8) in (3.7), we get

$$g(T^*(X,Y),Z) + g\left(\overline{T}(X,Y),Z\right) + g\left(\overline{T}(Y,X),Z\right) = 2g(P(X,Y),Z) \tag{3.9}$$

which implies that

$$P(X,Y) = \frac{1}{2}\left[T^*(X,Y) + \overline{T}(X,Y) + \overline{T}(Y,X)\right] \tag{3.10}$$

From (3.2) and (3.8), we have

$$\overline{T}(X,Y) = g(X,Y)\xi - \eta(X)Y - g(X,\phi Y)\xi + \eta(X)\phi Y. \tag{3.11}$$

$$\overline{T}(Y,X) = g(Y,X)\xi - \eta(Y)X - g(Y,\phi X)\xi + \eta(Y)\phi X. \tag{3.12}$$

Using (3.2), (3.11) and (3.12) in (3.10), we have

$$P(X,Y) = g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{3.13}$$

In view of (3.4) and (3.13), we can easily bring out the equation (3.1). Hence the linear connection ∇^* defined on an n-dimensional para-Kenmotsu manifold is a metric connection with torsion tensor given by equation (3.2). □

Proposition 3.2. *Zamkovoy connection on para-Kenmotsu manifold is a metric compatible linear connection and its torsion is of the form*

$$T^*(X,Y) = \eta(X)Y - \eta(Y)X + \eta(X)\phi Y - \eta(Y)\phi X.$$

Proposition 3.3. *In a para-Kenmotsu manifold , the structure vector field ξ, 1-form η and the metric g are parallel with respect to Zamkovoy connection.*

Proof. From the equation (3.3), it is obvious that

$$\nabla^*_X\xi = 0, (\nabla^*_X\eta)Y = 0. \tag{3.14}$$

□

Proposition 3.4. *In a para-Kenmotsu manifold, the integral curve of ξ is a geodesic with respect to Zamkovoy connection.*

4. Some properties of para-Kenmotsu manifold with respect to Zamkovoy connection

Let R^* be the Riemannian curvature tensor with respect to Zamkovoy connection and it is defined as

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_YZ - \nabla^*_Y\nabla^*_XZ - \nabla^*_{[X,Y]}Z. \tag{4.1}$$

By the help of (2.4), (2.5), (2.6), (3.1) and (3.14) we get the followings:

$$\nabla^*_X (\phi Z) = g(\phi X, Z)\xi - \eta(Z)\phi X + \phi(\nabla_X Z) + g(X, \phi Z)\xi + \eta(X) Z - \eta(X)\eta(Z)\xi. \tag{4.2}$$

$$\nabla^*_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{4.3}$$

$$\nabla^*_X \eta(Y) = g(X, \phi Y) + \eta(\nabla_X Y) \tag{4.4}$$

In reference to (3.1), (4.2), (4.3) and (4.4) we have

$$\begin{aligned} & \nabla^*_X \nabla^*_Y Z \\ = & \nabla_X \nabla_Y Z + g(X, \nabla_Y Z)\xi - \eta(\nabla_Y Z) X \\ & + \eta(X)\phi\nabla_Y Z + g(\nabla_X Y, Z)\xi + g(Y, \nabla_X Z)\xi \\ & - g(X, Z) Y - \eta(\nabla_X Z) Y + \eta(X)\eta(Z) Y - \nabla_X Y \eta(Z) \\ & - g(X, Y)\eta(Z)\xi + \eta(Y)\eta(Z) X - \eta(X)\eta(Z)\phi Y \\ & + g(X, Y)\phi Z + \eta(\nabla_X Y)\phi Z - \eta(X)\eta(Y)\phi Z + \phi(\nabla_X Z)\eta(Y) \\ & - \eta(Z)\eta(Y)\phi X + \eta(X)\eta(Y) Z - \eta(X)\eta(Y)\eta(Z)\xi. \end{aligned} \tag{4.5}$$

Also,

$$\nabla^*_{[X,Y]} Z = \nabla_{[X,Y]} Z + g(\nabla_X Y, Z)\xi - g(\nabla_Y X, Z)\xi - \eta(Z)\nabla_X Y + \eta(Z)\nabla_Y X + \eta(\nabla_X Y)\phi Z - \eta(\nabla_Y X)\phi Z. \tag{4.6}$$

Interchanging X and Y in (4.5) and using it along with the equations (4.5) and (4.6) in (4.1), we get

$$R^*(X, Y) Z = R(X, Y) Z - g(X, Z) Y + g(Y, Z) X. \tag{4.7}$$

Taking inner product of (4.7) with V, we obtain

$$R^*(X, Y, Z, V) = R(X, Y, Z, V) - g(X, Z) g(Y, V) + g(Y, Z) g(X, V). \tag{4.8}$$

Taking an orthnornal frame of M and contracting (4.8) over X and V, we get

$$S^*(Y, Z) = S(Y, Z) + (n-1) g(Y, Z). \tag{4.9}$$

Consequently, one can easily bring out the followings:

$$S^*(\xi, Z) = S^*(Z, \xi) = 0, \tag{4.10}$$

$$Q^* Y = QY + (n-1) Y, Q^*\xi = 0, \tag{4.11}$$

$$R^*(X, Y)\xi = R^*(\xi, Y) Z = R^*(X, \xi) Z = 0, \tag{4.12}$$

$$r^* = r + n(n-1). \tag{4.13}$$

Proposition 4.1. *Let M be an n-dimensional para-Kenmotsu manifold admitting Zamkovoy connection ∇^*, Then*

(i) The curvature tensor R^* with respect to ∇^* is given by (4.7),
(ii) The Ricci tensor S^* with respect to ∇^* is given by (4.9),
(iii) The scalar curvature r^* with respect to ∇^* is given by (4.13)
(iv) The Ricci tensor S^* with respect to ∇^* is symmetric.
(v) $R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0$.

Proposition 4.2. *The sectional curvature of a flat para-Kenmotsu manifold with respect to Zamkovoy connection is (-1).*

Proof. Let M be flat with respect to ∇^*, then (4.7) gives

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y].$$

which shows that M is a para-Kenmotsu manifold of sectional curvature (-1). □

Proposition 4.3. *The para-Kenmotsu manifold M is flat with respect to Zamkovoy connection iff M is locally isometric to the hyperbolic space $H^n(-1)$.*

Proposition 4.4. *If the para-Kenmotsu manifold M is Ricci flat with respect to Zamkovoy connection then M is an Einstein manifold.*

Proof. Let M be Ricci flat with respect to ∇^*, then (4.9) gives

$$S(Y,Z) = -(n-1)g(Y,Z),$$

which shows that M is an Einstein manifold. □

5. Locally ϕ-symmetric para-Kenmotsu manifold with respect to the Zamkovoy connection

Theorem 5.1. *An n-dimensional para-Kenmotsu manifold is locally ϕ-symmetric with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection.*

Proof. Let M be an n-dimensional generalized ϕ-recurrent para-Kenmotsu manifold with respect to the Zamkovoy connection, then curvature tensor R^* satisfies the condition

$$\phi^2 (\nabla^*_W R^*)(X,Y)Z = 0, \tag{5.1}$$

for all horizontal vector fields X, Y, Z, W of M.

By virtue of (3.1), we have

$$\begin{aligned}(\nabla^*_W R^*)(X,Y)Z &= \nabla^*_W R^*(X,Y)Z - R^*(\nabla^*_W X,Y)Z \\ &\quad - R^*(X,\nabla^*_W Y)Z - R^*(X,Y)\nabla^*_W Z.\end{aligned} \tag{5.2}$$

Using (3.1), (4.7), in (5.2), we get

$$\begin{aligned}&(\nabla^*_W R^*)(X,Y)Z \\ = \; &(\nabla_W R)(X,Y)Z + g(W,R(X,Y)Z)\xi \\ &-g(X,Z)g(W,Y)\xi + g(Y,Z)g(W,X)\xi - \eta(R(X,Y)Z)W \\ &-g(X,Z)\eta(Y)W + g(Y,Z)\eta(X)W + \eta(W)\phi R(X,Y)Z \\ &-\eta(W)g(X,Z)\phi Y + \eta(W)g(Y,Z)\phi X + \eta(X)R(W,Y)Z \\ &-g(W,Z)\eta(X)Y + g(Y,Z)\eta(X)W - \eta(W)R^*(\phi X,Y)Z \\ &+g(\phi X,Z)\eta(W)Y - g(Y,Z)\eta(W)\phi X + \eta(Y)R^*(X,W)Z \\ &-g(X,Z)\eta(Y)W + g(W,Z)\eta(Y)X - \eta(W)R^*(X,\phi Y)Z \\ &+g(X,Z)\eta(W)\phi Y - g(\phi Y,Z)\eta(W)X + \eta(Z)R(X,Y)W \\ &-g(X,W)\eta(Z)Y + g(Y,W)\eta(Z)X - \eta(W)R(X,Y)\phi Z \\ &+g(X,\phi Z)\eta(W)Y - g(Y,\phi Z)\eta(W)X.\end{aligned} \tag{5.3}$$

Applying ϕ^2 on both sides of (5.3) and using (2.1) and cosidering X,Y,Z,W to be horizontal vector fields, i.e., orthogonal to ξ, we get

$$\phi^2((\nabla^*_W R^*)(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z),$$

which shows that M is locally ϕ-symmetric with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection. □

6. Ricci Soliton on para-Kenmotsu manifold with respect to Zamkovoy connection.

Theorem 6.1. *A Ricci soliton (g,V,λ) with respect to Zamkovoy connection and Levi-Civita connection is equivalent if and only if the relation*

$$\begin{aligned}2g(Y,Z)\eta(V) &= g(\phi V,Z)\eta(Y) \\ &\quad + g(\phi V,Y)\eta(Z) + 2(n-1)g(Y,Z),\end{aligned}$$

holds for arbitrary vector fields $Y,Z,V \in \chi(M)$.

Proof. For a Ricci soliton (g, V, λ), the equation (1.3) can be written in terms of Zamkovoy connection as

$$(L^*_V g)(Y, Z) + 2S^*(Y, Z) + 2\lambda g(Y, Z) = 0, \tag{6.1}$$

for all $Y, Z, V \in \chi(M)$, where L^*_V denotes the Lie derivative operator with respect to ∇^* along the vector field V.

Using (3.1) and (4.9) in (6.1), we get

$$\begin{aligned} & (L^*_V g)(Y, Z) + 2S^*(Y, Z) + 2\lambda g(Y, Z) \\ = \; & g(\nabla^*_Y V, Z) + g(\nabla^*_Z V, Y) + 2S^*(Y, Z) + 2\lambda g(Y, Z) \\ = \; & (L_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) - 2g(Y, Z)\eta(V) \\ & + g(\phi V, Z)\eta(Y) + g(\phi V, Y)\eta(Z) + 2(n-1)g(Y, Z). \end{aligned} \tag{6.2}$$

This gives the theorem. □

Theorem 6.2. *If a para-Kenmotsu manifold M is Ricci flat with respect to Zamkovoy connection then the Ricci soliton (g, ξ, λ) is always steady.*

Proof. Considering a Ricci soliton (g, ξ, λ) on M it follows from (6.1) that

$$\begin{aligned} 0 &= (L^*_\xi g)(Y, Z) + 2S^*(Y, Z) + 2\lambda g(Y, Z) \\ &= g(\nabla^*_Y \xi, Z) + g(\nabla^*_Z \xi, Y) + 2S^*(Y, Z) + 2\lambda g(Y, Z) \\ &= S^*(Y, Z) + \lambda g(Y, Z). \end{aligned} \tag{6.3}$$

Now, if M is Ricci flat with respect to Zamkovoy connection then (6.3) gives

$$\lambda = 0.$$

Therefore, the Ricci soliton (g, ξ, λ) is steady on M. □

7. ϕ-pseudo symmetric para-Kenmotsu manifold with respect to Zamkovoy connection.

Theorem 7.1. *A ϕ-pseudo-symmetric para-Kenmotsu manifold with respect to Zamkovoy connection is pseodo-Ricci symmetric with respect to Zamkovoy connection if and only if*

$$A(R^*(W, Y)Z) + A(R^*(Z, W)Y) = 0.$$

Proof. Let M be ϕ-pseudo symmetric para-Kenmotsu manifold with respect to Zamkovoy connection, then

$$\begin{aligned}\phi^2\left(\nabla_W^* R^*\right)(X,Y)Z &= 2A(W)R^*(X,Y)Z \\ &\quad +A(X)R^*(W,Y)Z+A(Y)R^*(X,W)Z \\ &\quad +A(Z)R^*(X,Y)W+g\left(R^*(X,Y)Z,W\right)\rho, \qquad (7.1)\end{aligned}$$

where A is a non zero associated 1-form, ρ is a vector field defined by $g(W,\rho)=A(W)$ for every vector field W and ∇ denotes the operator of covariant differentiation with respect to the metric g.

Using (2.1) in (7.1), we get

$$\begin{aligned}\left(\nabla_W^* R^*\right)(X,Y)Z &= \eta\left(\left(\nabla_W^* R^*\right)(X,Y)Z\right)\xi+2A(W)R^*(X,Y)Z \\ &\quad +A(X)R^*(W,Y)Z+A(Y)R^*(X,W)Z \\ &\quad +A(Z)R^*(X,Y)W+g\left(R^*(X,Y)Z,W\right)\rho. \qquad (7.2)\end{aligned}$$

Taking inner product of (7.2) with a vector field V, we obtain

$$\begin{aligned}&g\left(\left(\nabla_W^* R^*\right)(X,Y)Z,V\right) \\ &= \eta\left(\left(\nabla_W^* R^*\right)(X,Y)Z\right)\eta(V)+2A(W)g\left(R^*(X,Y)Z,V\right) \\ &\quad +A(X)g\left(R^*(W,Y)Z,V\right)+A(Y)g\left(R^*(X,W)Z,V\right) \\ &\quad +A(Z)g\left(R^*(X,Y)W,V\right)+g\left(R^*(X,Y)Z,W\right)g(\rho,V). \qquad (7.3)\end{aligned}$$

Let $\{e_i\}\,(1\le i\le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X=V=e_i$ in (7.3) and taking summation over i $(1\le i\le n)$ and then using (2.1) in (7.3), we get

$$\begin{aligned}&\left(\nabla_W^* S^*\right)(Y,Z) \\ &= g\left(\left(\nabla_W^* R^*\right)(\xi,Y)Z,\xi\right) \\ &\quad +2A(W)S^*(Y,Z)+A\left(R^*(W,Y)Z\right)+A(Y)S^*(W,Z) \\ &\quad +A(Z)S^*(W,Y)+A\left(R^*(Z,W)Y\right). \qquad (7.4)\end{aligned}$$

By virtue of (4.12) it follows from (7.4) that

$$\begin{aligned}\left(\nabla_W^* S^*\right)(Y,Z) &= 2A(W)S^*(Y,Z) \\ &\quad +A(Y)S^*(W,Z)+A(Z)S^*(W,Y) \\ &\quad +A\left(R^*(W,Y)Z\right)+A\left(R^*(Z,W)Y\right). \qquad (7.5)\end{aligned}$$

Therefore, M is pseudo-Ricci-symmetric with respect to Zamkovoy connection if and only if

$$A\left(R^*(W,Y)Z\right)+A\left(R^*(Z,W)Y\right)=0.$$

□

8. W_i-curvature tensor with respect to Zamkovoy connection on para-Kenmotsu manifold.

The W_i-curvature tensors $(i = 0, 1, 2...9)$ are defined as a particular case of τ-Tensor introduced by M. M. Tripathi and P. Gupta [22]. Some of the W_i-curvature tensors were formerly introduced by Pokhariyal [15]. The W_i-curvature tensor $(i = 1, 2...9)$ of rank three is defined as

$$\begin{aligned} W_i(X,Y)Z &= a_0R(X,Y)Z + a_1S(Y,Z)X \\ &+a_2S(X,Z)Y + a_3S(X,Y)Z + a_4g(Y,Z)QX \\ &+a_5g(X,Z)QY + a_6g(X,Y)QZ, \end{aligned} \tag{8.1}$$

for all $X, Y, Z \in \chi(M)$, where R, S and Q are Riemannian curvature tensor, Ricci tensor and Ricci operator respectively. The expressions for $W_0, W_1....W_9$ curvature tensors are given by

$Value\ of\ a_i$	Expressions for $W_i - curvature\ tensors$
$a_0 = 1, a_1 = -a_5 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_0(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY]$
$a_0 = 1, a_1 = -a_2 = \frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_1(X,Y)Z = R(X,Y)Z$ $+\frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$
$a_0 = 1, a_4 = -a_5 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_2(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY]$
$a_0 = 1, a_2 = -a_4 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_3(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(X,Z)Y - g(Y,Z)QX]$
$a_0 = 1, a_5 = -a_6 = \frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_4(X,Y)Z = R(X,Y)Z$ $+\frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ]$
$a_0 = 1, a_2 = -a_5 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_5(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(X,Z)Y - g(X,Z)QY]$
$a_0 = 1, a_1 = -a_6 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_6(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(Y,Z)X - g(X,Y)QZ]$
$a_0 = 1, a_1 = -a_4 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_7(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(Y,Z)X - g(Y,Z)QX]$
$a_0 = 1, a_1 = -a_3 = -\frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_8(X,Y)Z = R(X,Y)Z$ $-\frac{1}{n-1}[S(Y,Z)X - S(X,Y)Z]$
$a_0 = 1, a_3 = -a_4 = \frac{1}{n-1}$ $all\ other\ a_i = 0$	$W_9(X,Y)Z = R(X,Y)Z$ $+\frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX]$

Theorem 8.1. *An n-dimensional W_i-flat para-Kenmotsu manifold with respect to Zamkovoy connection is an Einstein manifold for $i \neq 6$.*

Proof. The W_i-curvature tensor with respect to Zamkovoy connection is given by

$$\begin{aligned} & W_i^*(X,Y)Z \\ = \; & a_0 R^*(X,Y)Z + a_1 S^*(Y,Z)X \\ & + a_2 S^*(X,Z)Y + a_3 S^*(X,Y)Z + a_4 g(Y,Z)Q^*X \\ & + a_5 g(X,Z)Q^*Y + a_6 g(X,Y)Q^*Z, \end{aligned} \tag{8.2}$$

for all $X, Y, Z \in \chi(M)$, where R^*, S^* and Q^* are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection respectively. If M is W_i-flat with respect ∇^* then (8.2) gives

$$\begin{aligned} 0 \; = \; & a_0 R^*(X,Y)Z + a_1 S^*(Y,Z)X \\ & + a_2 S^*(X,Z)Y + a_3 S^*(X,Y)Z + a_4 g(Y,Z)Q^*X \\ & + a_5 g(X,Z)Q^*Y + a_6 g(X,Y)Q^*Z. \end{aligned} \tag{8.3}$$

Taking inner product of (8.3) with a vector field V, we get

$$\begin{aligned} 0 \; = \; & a_0 g(R^*(X,Y)Z,V) + a_1 S^*(Y,Z)g(X,V) \\ & + a_2 S^*(X,Z)g(Y,V) + a_3 S^*(X,Y)g(Z,V) \\ & + a_4 g(Y,Z)S^*(X,V) + a_5 g(X,Z)S^*(Y,V) \\ & + a_6 g(X,Y)S^*(Z,V). \end{aligned} \tag{8.4}$$

Let $\{e_i\}\,(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X = V = e_i$ in (8.4) and taking summation over i $(1 \le i \le n)$, we get

$$0 = (a_0 + na_1 + a_2 + a_3 + a_5 + a_6)\,S^*(Y,Z) + r^* a_4 g(Y,Z). \tag{8.5}$$

Using (4.9) and (4.13) in (8.5), we obtain

$$S(Y,Z) = -\frac{1}{a}\left[ra_4 + (a + na_4)(n-1)\right] g(Y,Z), \tag{8.6}$$

where, $a = a_0 + na_1 + a_2 + a_3 + a_5 + a_6$ and $a = 0$ if $i = 6$.

Therefore, M is an Einstein manifold. □

Corollary 8.2. *The expressions for Ricci tensors for different W_i-flat para-Kenmotsu manifolds are as follows:*

Type of flat Manifold	Ricci Tensor
$\mathcal{W}_0^*$-$flat$	$S(Y,Z) = -(n-1)\,g(Y,Z)$,
$\mathcal{W}_1^*$-$flat$	$S(Y,Z) = -(n-1)\,g(Y,Z)$,
$\mathcal{W}_2^*$-$flat$	$S(Y,Z) = \frac{r}{n} g(Y,Z)$,
$\mathcal{W}_3^*$-$flat$	$S(Y,Z) = -\frac{1}{n-2}\left[2(n-1)^2 + r\right] g(Y,Z)$,
$\mathcal{W}_4^*$-$flat$	$S(Y,Z) = -(n-1)\,g(Y,Z)$,
$\mathcal{W}_5^*$-$flat$	$S(Y,Z) = -(n-1)\,g(Y,Z)$,
$\mathcal{W}_6^*$-$flat$	Indeterminate
$\mathcal{W}_7^*$-$flat$	$S(Y,Z) = rg(Y,Z)$,
$\mathcal{W}_8^*$-$flat$	Indeterminate
$\mathcal{W}_9^*$-$flat$	$S(Y,Z) = \frac{r}{n} g(Y,Z)$.

Proof. The above expressions for Ricci tensors are obtained directly from equation (8.6). □

Theorem 8.3. *An n-dimensional W_i-flat symmetric para-Kenmotsu manifold with respect to Zamkovoy connection is of constant scalar curvature for $i = 2, 3, 7, 9$.*

Proof. If M is symmetric with respect to Zamkovoy connection, i.e., $(\nabla_U^* R^*)(X,Y)Z = 0$, then from (8.3) we get

$$\begin{aligned} 0 \;=\; & a_1 (\nabla_U^* S^*)(Y,Z)X + a_2 (\nabla_U^* S^*)(X,Z)Y \\ & + a_3 (\nabla_U^* S^*)(X,Y)Z + a_4 g(Y,Z)(\nabla_U^* Q^*)X \\ & + a_5 g(X,Z)(\nabla_U^* Q^*)Y + a_6 g(X,Y)(\nabla_U^* Q^*)Z. \end{aligned} \tag{8.7}$$

Taking Inner product of (8.7) with a vector field V, we get

$$\begin{aligned} 0 \;=\; & a_1 (\nabla_U^* S^*)(Y,Z)\,g(X,V) + a_2 (\nabla_U^* S^*)(X,Z)\,g(Y,V) \\ & + a_3 (\nabla_U^* S^*)(X,Y)\,g(Z,V) + a_4 g(Y,Z)(\nabla_U^* S^*)(X,V) \\ & + a_5 g(X,Z)(\nabla_U^* S^*)(Y,V) + a_6 g(X,Y)(\nabla_U^* S^*)(Z,V). \end{aligned} \tag{8.8}$$

Let $\{e_i\}\,(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X = V = e_i$ in (8.8) and taking summation over i $(1 \le i \le n)$, we get

$$0 = (a_1 n + a_2 + a_3 + a_5 + a_6)(\nabla_U^* S^*)(Y,Z) + a_4 g(Y,Z)\,\nabla_U^* r^*. \tag{8.9}$$

Setting $Z = \xi$ and using (4.10), (4.13) in (8.9) we get

$$U(r) = 0.$$

for $a_4 \neq 0$, i.e., $i = 2, 3, 7, 9$. Therefore, M is a space of constant curvature. □

9. Conclusion

In this paper, Zamkovoy connection has been introduced and studied on para-Kenmotsu manifold. Some properties of para-kenmotsu manifold by the help of W_i-curvature tensor and Zamkovoy connection has been studied. It is also investigated that the Ricci solition on a Ricci flat para-Kenmotsu manifold with respect to Zamkovoy connection is always steady.

There is a huge scope of further study of para-Kenmotsu manifold by the help of different curvature tensors with respect to Zamkovoy connection.

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