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ON THE GEOMETRY OF WARPED PRODUCT SUBMANIFOLDS OF A QUASI-HEMI SLANT SUBMANIFOLDS WITH TRANS PARA SASAKIAN MANIFOLDS

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ABSTRACT. The existence or non-existence of warped product quasi-hemi slant submanifolds in trans para-sasakian manifolds is defined. Then we obtain that there are no proper warped product quasi-hemi slant submanifolds in trans para-sasakian manifolds such that totally geodesic and totally umbilical submanifolds of warped product are proper semi-slant and invariant (or anti-invariant).

Key Words: Warped product, quasi hemi-slant submanifolds, trans para Sasakian manifolds.

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1. Introduction

Bishop and Neill [16], introduced the notion of warped product manifolds and it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The study of warped product submanifolds of Kaehler manifolds was introduced by Chen [7]. Similar notation have been studied in ([17], [21]). Also the differential geometry of slant submanifolds of almost Hermitian manifolds has shown an increasing development since B.Y. Chen [6] defined slant submanifolds as natural generallization of both holomorphic and totally real immersions. The idea of slant submanifolds of a

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Riemannian manifold into an almost contact metric manifold was introduced by A. Lotta [3]. Cabrerizo et al., [13] were defined slant submanfolds of Sasakian manifolds. Bejancu et al., [1] studied the semi-invariant submanifolds of Sasakian manifold. Carriazo [2] defined hemi-slant submanifolds. Since then, this interesting subject has been studied broadly by several geometers during last two decades (for instance we refer Blair [8]; Chen [6]; Rahman et al., [18]; Sahin [4]. The slant submanifolds were generalized as semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds, and hemi-slant submanifolds etc. in different kinds of differentiable manifolds (see, Shahid et al.,[14], Papaghuic,[15], Rahman et al., [20], Uddin et al., [22]). Obina [12] introduced a new class of almost contact manifold namely Trans-Sasakian manifolds which includes the Sasakian, Kenmostu and Cosymplectic structures. It is known that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosympletic (see Blair [8]), β- Kenmotsu (see Janssens et al., [10]) and α-Sasakian (see Janssens et al., ([10]), respectively. It is known that trans-para-Sasakian manifolds are generalized trans-Sasakian manifolds. A trans para-Sasakian manifold is a trans-para-Sasakian structure of type (α,β), where α and β are smooth functions. The trans-para-Sasakian manifolds of types (α,β), and are respectively the para-cosympletic, para-Sasakian and and para-Kenmotsu.

2. Preliminaries

Let $\widehat{\mathcal{M}}$ be a $(2n+1)$-dimensional trans-para-Sasakian manifold with structure tensor $(\phi,\xi,\upsilon,<,>)$, then they satisfy

$$\phi\xi=0,\quad \phi^2=I-\upsilon\otimes\xi,\quad \upsilon(\xi)=1, \tag{2.1}$$

$$\upsilon\circ\phi=0,\quad \upsilon(\mathcal{X})=<\mathcal{X},\xi>,$$

$$<\phi\cdot,\phi\cdot>=-<,>+\upsilon\otimes\upsilon \tag{2.2}$$

$$\begin{aligned}(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} &= \alpha\{-<\mathcal{X},\mathcal{Y}>\xi+\upsilon(\mathcal{Y})\mathcal{X}\}\\ &\quad +\beta\{<\mathcal{X},\phi\mathcal{Y}>\xi+\upsilon(\mathcal{Y})\phi\mathcal{X}\}\end{aligned} \tag{2.3}$$

$$\bar{\nabla}_{\mathcal{X}}\xi=-\alpha\phi\mathcal{X}-\beta(\mathcal{X}-\upsilon(\mathcal{X})\xi), \tag{2.4}$$

for all $\mathcal{X},\mathcal{Y}\in T\widehat{\mathcal{M}}$ and some smooth functions α and β.

We denote by $<,>$ the metric tensor on $\widehat{\mathcal{M}}$ as well as that induced on $\mathcal{M}$. Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the

Levi-Civita connection on $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$). The Gauss and Weingarten formulas for $\mathcal{M}$ are respectively given by

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X},\mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y} \tag{2.5}$$

$$\bar{\nabla}_{\mathcal{X}}\lambda = -\Lambda_{\lambda}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}\lambda \tag{2.6}$$

for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$ where σ (resp.Λ) is the second fundamental form (resp.tensor) of $\mathcal{M}$ in $\widehat{\mathcal{M}}$ and $\nabla^{\perp}$ denotes the operator of the normal connection.
From above, we have

$$< \sigma(\mathcal{X},\mathcal{Y}), \lambda >=< \Lambda_{\lambda}\mathcal{X}, \mathcal{Y} > \tag{2.7}$$

for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$.

3. Warped product manifolds

If $(N_1, <,>_1)$ and $(N_2, <,>_2)$ are two Riemannian manifolds and δ, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_{\delta} N_2 = (N_1 \times N_2, <,>)$, where

$$<,>=<,>_1 + \delta^2 <,>_2 \tag{3.1}$$

More explicitly, if vector fields $\mathcal{X}$ and $\mathcal{Y}$ tangent to $N_1 \times_{\delta} N_2$ at $(\mathcal{X},\mathcal{Y})$, then

$$\begin{aligned} < \mathcal{X},\mathcal{Y} > \; &= \; <,>_1 (\pi_1 * \mathcal{X}, \pi_1 * \mathcal{Y}) \\ &\quad + \delta^2(x) <,>_2 (\pi_2 * \mathcal{X}, \pi_2 * \mathcal{Y}) \end{aligned} \tag{3.2}$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $N_1 \times_{\delta} N_2$ onto N_1 and N_2, respectively, and $*$ stands for derivative map.

If $\widehat{\mathcal{N}} = N_1 \times_{\delta} N_2$ is a warped product manifold, this means that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively.
For warped product manifolds, we have the following proposition [16].

Proposition 3.1. *On a warped product manifold $\widehat{\mathcal{N}} = N_1 \times_{\delta} N_2$, we have*

(1) $\nabla_{\mathcal{X}}\mathcal{Y} \in \Gamma(TN_1)$ *is the lift of* $\nabla_{\mathcal{X}}\mathcal{Y}$ *on* N_1
(2) $\nabla_{\mathcal{U}}\mathcal{X} = \nabla_{\mathcal{X}}\mathcal{U} = \mathcal{X}(ln\delta)\mathcal{U}$
(3) $\nabla_{\mathcal{U}}\mathcal{V} = \nabla'_{\mathcal{U}}\mathcal{V} - < \mathcal{U},\mathcal{V} > \nabla ln_{\delta}$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(TN_1)$ and $\mathcal{U},\mathcal{V} \in \Gamma(TN_2)$, where ∇ and ∇' denote the Levi-Civita connections on N and N_2, respectively.

Throughout this paper, let us suppose that $\widehat{\mathcal{M}}$ be a trans-para-Sasakian manifold and $\widehat{\mathcal{N}} = N_1 \times_\delta N_2$ be a warped product quasi hemi-slant submanifolds of a trans-para-Sasakian manifold $\widehat{\mathcal{M}}$. Such submanifolds are always tangent to the structure vector field ξ. If the manifolds N_θ and N_T (resp. $N^\perp$) are slant and invariant(resp. anti-invariant) submanifolds of a trans-para-Sasakian manifold $\widehat{\mathcal{M}}$, then their warped product quasi hemi-slant submanifolds may be given by one of the following forms:

(1) $N_\theta \times_\delta N_T$
(2) $N_\theta \times_\delta N_\perp$
(3) $N_T \times_\delta N_\theta$
(4) $N_\perp \times_\delta N_\theta$.

In this paper we are concerned with cases (1) and (2).

Now, the submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T_\mathcal{X}\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$ and carring a Riemannian manifold $\mathcal{M}$ isometrically absorbed in an almost contact metric manifold $\widehat{\mathcal{M}}$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is anti-invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T^\perp_\mathcal{X}\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$.

If ξ is tangential in $\mathcal{M}$ for a submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ then, the submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is slant (Cabrerizo et al., [13]) for each non zero vector $\mathcal{X}$ tangent to $\mathcal{M}$ at $\mathcal{X} \in M$ such that $\mathcal{X}$ is linearly independent to $\xi_\mathcal{X}$, the angle $\theta(\mathcal{X})$ between $\phi\mathcal{X}$ and $T_\mathcal{X}\mathcal{M}$ is constant i.e. it does not depend on the choice of the point $\mathcal{X} \in M$ and $\mathcal{X} \in T_\mathcal{X}\mathcal{M} - \{\xi\}$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold $\mathcal{M}$ is proper slant submanifold for niether $\theta = 0$ nor $\theta = \pi/2$. Here $T\mathcal{M} = \mathcal{D}_\theta \oplus \{\xi\}$, where $\mathcal{D}_\theta$ is slant distribution with slant angle θ.

If $\theta = 0$, then the slant submanifolds is said to be an invariant submanifolds and if $\theta = \pi/2$, then slant submanifolds is said to be anti-invariant submanifolds.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is semi-invariant if there exist two orthogonal complementary distributions $\mathcal{D}$

and $\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}^{\perp}$ is anti -invariant i.e. $\phi\mathcal{D}^{\perp} \subset (T^{\perp}\mathcal{M})$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is semi-slant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}_\theta$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \{\xi\}$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}_\theta$ is slant with semi slant angle θ. The submanifold $\mathcal{M}$ of an almost contact metric manifold $\widehat{\mathcal{M}}$ is hemi-slant (Sahin, [4]) if there exist two orthogonal complementary distributions $\mathcal{D}_\theta$ and $\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D}_\theta \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$$

where $\mathcal{D}^{\perp}$ is anti- invariant i.e. $\phi\mathcal{D}^{\perp} \subset (T^{\perp}\mathcal{M})$ and $\mathcal{D}_\theta$ is slant with hemi slant angle θ.

We say that $\mathcal{M}$ is quasi hemi-slant submanifold of a trans para-Sasakian manifold $\widehat{\mathcal{M}}$ if there exist three orthogonal complementary distributions $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that

(1) $T\mathcal{M}$ admits the orthogonal direct decomposition

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \mathcal{D}^{\perp}\{\xi\}, \quad \xi \in \Gamma(\mathcal{D}_\theta) \tag{3.3}$$

(2) $\phi\mathcal{D} = \mathcal{D}$
(3) $\phi\mathcal{D}^{\perp} \subseteq T^{\perp}\mathcal{M}$
(4) The distribution $\mathcal{D}_\theta$ is a slant with slant constant angle θ, where $\theta =$ slant angle.

In this case, θ is said to be quasi hemi- slant angle of $\mathcal{M}$. If the dimension of distributions $\mathcal{D}, \mathcal{D}_\theta$ and $\mathcal{D}^{\perp}$ are m_1, m_2 and m_3 respectively, then

(1) $\mathcal{M}$ is a hemi-slant submanifold for $m_1 = 0$.
(2) $\mathcal{M}$ is a semi-invariant submanifold for $m_2 = 0$.
(3) $\mathcal{M}$ is a semi-slant submanifold for $m_3 = 0$.

The quasi hemi-slant submanifold $\mathcal{M}$ is proper if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_\theta \neq \{0\}$, $\mathcal{D}^{\perp} = \{0\}$ and $\theta \neq 0, \pi/2$.

It represents that quasi hemi-slant submanifols is a generalization of invariant, anti-invariant, semi-invarint, slant, hemi-slant, semi-slant submanifolds.

It is clear from the above definition that if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_\theta \neq \{0\}$ and $\mathcal{D}^\perp = \{0\}$, then $dim\mathcal{D} \geq 2, dim\mathcal{D}_\theta \geq 2$ and $\mathcal{D}^\perp \geq 1$. So for proper quasi hemi slant subanifold $\mathcal{M}$, the $dim\mathcal{M} \geq 6$.

Suppose $\mathcal{M}$ be a quasi hemi-slant submanifold of trans para-Sasakian manifold $\widehat{\mathcal{M}}$ and the projections on $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively, then for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$\mathcal{X} = \mathcal{R}\mathcal{X} + \mathcal{Q}\mathcal{X} + \mathcal{P}\mathcal{X} + \upsilon(\mathcal{X})\xi \tag{3.4}$$

Now put

$$T\mathcal{X} + N\mathcal{X} = \phi\mathcal{X} \tag{3.5}$$

where $T\mathcal{X}$ and $N\mathcal{X}$ are tangential and normal part of $\phi\mathcal{X}$ on $\mathcal{M}$. From (3.4) and (3.5), we derive

$$\begin{aligned} \phi\mathcal{X} &= N\mathcal{R}\mathcal{X} + T\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} \\ &\quad + N\mathcal{P}\mathcal{X} + T\mathcal{P}\mathcal{X} \end{aligned} \tag{3.6}$$

As $\phi\mathcal{D} = \mathcal{D}$ and $\phi\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain $N\mathcal{P}\mathcal{X} = 0$, and $T\mathcal{R}\mathcal{X} = 0$ and

$$\phi\mathcal{X} = N\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} + T\mathcal{P}\mathcal{X} \tag{3.7}$$

For all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$T\mathcal{X} = T\mathcal{P}\mathcal{X} + T\mathcal{Q}\mathcal{X}$$

and

$$N\mathcal{X} = N\mathcal{Q}\mathcal{X} + N\mathcal{R}\mathcal{X}$$

Using (3.7), we deduce the following decompositiona,

$$\phi(T\mathcal{M}) = \mathcal{D} \oplus T\mathcal{D}_\theta \oplus N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp \tag{3.8}$$

As $N\mathcal{D}_\theta \subseteq T^\perp\mathcal{M}$ and $N\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain

$$T^\perp\mathcal{M} = N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp \oplus \kappa \tag{3.9}$$

where κ denotes the orthogonal component of $N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp$ in $\Gamma(T^\perp\mathcal{M})$ and invariant with respect to ϕ
For all non-zero vector field λ normal to $\mathcal{M}$, we infer

$$\phi\lambda = r\lambda + s\lambda \tag{3.10}$$

where $r\lambda$ tangent to $\mathcal{M}$ and $s\lambda$ normal to $\mathcal{M}$. If r and s are the endomorphism defined by (3.5), then

$$< T\mathcal{Z}, \mathcal{W} > + < \mathcal{Z}, T\mathcal{W} >= 0 \tag{3.11}$$

$$< N\mathcal{Z}, \mathcal{W} > + < \mathcal{Z}, N\mathcal{W} >= 0 \tag{3.12}$$

for any $\mathcal{Z}, \mathcal{W} \in \Gamma(TN)$. On the other hand, making use of (2.6) and (2.7) with (2.2), (2.3), (3.5) and (3.10), we have

$$\begin{aligned}(\nabla_{\mathcal{Z}} N)\mathcal{W} &= \beta\upsilon(\mathcal{W})N\mathcal{Z} + s\sigma(\mathcal{Z}, \mathcal{W}) \\ &\quad -\sigma(\mathcal{Z}, T\mathcal{W}) + \alpha < \mathcal{Z}, \mathcal{W} > \xi\end{aligned} \tag{3.13}$$

$$\begin{aligned}(\nabla_{\mathcal{Z}} T)\mathcal{W} &= \Lambda_{N\mathcal{W}}\mathcal{Z} + t\sigma(\mathcal{Z}, \mathcal{W}) \\ &\quad -\beta < \phi\mathcal{Z}, \mathcal{W} > \xi + \alpha\upsilon(\mathcal{W})\mathcal{Z} + \beta\upsilon(\mathcal{W})T\mathcal{Z}\end{aligned} \tag{3.14}$$

for any $\mathcal{Z}, \mathcal{W} \in \Gamma(TN)$, where the covariant derivatives of N and T are, respectively, defined by

$$\nabla^{\perp}_{\mathcal{Z}} N\mathcal{W} - N\nabla_{\mathcal{Z}}\mathcal{W} = (\nabla_{\mathcal{Z}} N)\mathcal{W} \tag{3.15}$$

$$\nabla_{\mathcal{Z}} T\mathcal{W} - T(\nabla_{\mathcal{Z}}\mathcal{W}) = (\nabla_{\mathcal{Z}} T)\mathcal{W} \tag{3.16}$$

Proposition 3.2. *On a submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\widehat{\mathcal{M}}$, we have*

$$\begin{aligned}\nabla_{\mathcal{X}} T\mathcal{Y} &- \Lambda_{N\mathcal{Y}}\mathcal{X} - T\nabla_{\mathcal{X}}\mathcal{Y} - r\sigma(\mathcal{X}, \mathcal{Y}) \\ &= -\alpha < \mathcal{X}, \mathcal{Y} > \xi + \alpha\upsilon(\mathcal{Y})\mathcal{X} + \beta\upsilon(\mathcal{Y})\phi X \\ &\quad +\beta < \mathcal{X}, \phi\mathcal{Y} > \xi\end{aligned}$$

$$\sigma(\mathcal{X}, T\mathcal{Y}) + \nabla^{\perp}_{\mathcal{X}} N\mathcal{Y} - N\nabla_{\mathcal{X}}\mathcal{Y} - s\sigma(\mathcal{X}, \mathcal{Y}) = 0$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$.

Proposition 3.3. *On a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\widehat{\mathcal{M}}$, we have*

$$\begin{aligned}&T\mathcal{D} = \mathcal{D}, \quad T\mathcal{D}_\theta = \mathcal{D}_\theta, \quad T\mathcal{D}^{\perp} = \{0\}, \\ &rN\mathcal{D}_\theta = \mathcal{D}_\theta, \quad rN\mathcal{D}_\theta = \mathcal{D}^{\perp}\end{aligned} \tag{3.17}$$

From (3.5), (3.10) and $\phi^2 = I - \upsilon \otimes \xi$, we get

Proposition 3.4. *If $\mathcal{M}$ is a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\widehat{\mathcal{M}}$, then the endomorphism T and N, t and f in the tangent bundle of $\mathcal{M}$, holds the following identities:*

(1) $T^2 + rN = I - \upsilon \otimes \xi$ *on tangent* $\mathcal{M}$

(2) $NT + sN = 0$ *on tangent* $\mathcal{M}$
(3) $Nr + s^2 = I$ *on normal* $\mathcal{M}$
(4) $Tr + rs = 0$ *on on normal* $\mathcal{M}$.

Theorem 3.5. *If* $\mathcal{M}$ *be a quasi hemi-slant submanifold* $\mathcal{M}$ *of a trans para-Sasakian manifolds* $\widehat{\mathcal{M}}$ *such that* $\xi \in T\mathcal{M}$, *then* $\mathcal{M}$ *is slant submanifold if and only if there exists a constant* $\lambda \in [0,1]$ *such that*

$$T^2\mathcal{X} = (\cos^2\theta)\mathcal{X}, \tag{3.18}$$

Furthermore, if θ *is the slant angle of* N, *then* $\mu = \cos^2\theta$

The following relations are straight forward consequence of equations (3.18):

$$< T\mathcal{Z}, T\mathcal{W} >= (\cos^2\theta) < \mathcal{Z}, \mathcal{W} > \tag{3.19}$$

$$< N\mathcal{Z}, N\mathcal{W} >= (\sin^2\theta) < \mathcal{Z}, \mathcal{W} > \tag{3.20}$$

for all $\mathcal{Z}, \mathcal{W} \in \Gamma(T\mathcal{M})$.
Next we state, for later use

Proposition 3.6. *On a quasi hemi- slant submanifold* $\mathcal{M}$ *of a trans para-Sasakian manifolds* $\widehat{\mathcal{M}}$, *we have*

$$\begin{aligned}(\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y} &= \Lambda_{N\mathcal{Y}}\mathcal{X} + r\sigma(\mathcal{X},\mathcal{Y}) - \alpha < \mathcal{X},\mathcal{Y} > \xi \\ &\quad + \alpha\upsilon(\mathcal{Y})\mathcal{X} + \beta < \mathcal{X}, T\mathcal{Y} > \xi + \beta\upsilon(\mathcal{Y})T\mathcal{X}\end{aligned} \tag{3.21}$$

$$(\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y} = \beta\upsilon(\mathcal{Y})N\mathcal{X} + s\sigma(\mathcal{X},\mathcal{Y}) - \sigma(\mathcal{X}, T\mathcal{Y}) \tag{3.22}$$

$$(\bar{\nabla}_{\mathcal{X}}r)\lambda = \Lambda_{s\lambda}\mathcal{X} - T\Lambda_{\lambda}\mathcal{X} \tag{3.23}$$

and

$$(\bar{\nabla}_{\mathcal{X}}s)\lambda = -\sigma(\mathcal{X}, r\lambda) - N\Lambda_{\lambda}\mathcal{X} \tag{3.24}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ *tangent to* $\mathcal{M}$ *and vector fields* λ *normal to* $\mathcal{M}$.

Proposition 3.7. *On a quasi hemi-slant submanifold* $\mathcal{M}$ *of a trans para-Sasakian manifolds* $\widehat{\mathcal{M}}$, *we have*

$$\nabla_{\mathcal{X}}\xi = -\alpha T\mathcal{X} - \beta\mathcal{X}$$

and

$$\sigma(\mathcal{X},\xi) = -\alpha N\mathcal{X} + \beta\upsilon(\mathcal{X})\xi$$

for all vector fields $\mathcal{X}$ *tangent to* $\mathcal{M}$.

Lemma 3.8. *On a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\widehat{\mathcal{M}}$, we have*

$$\sigma_{\phi\mathcal{Z}}\mathcal{W} = \sigma_{\phi\mathcal{W}}\mathcal{Z}$$

for all $\mathcal{Z}, \mathcal{W} \in \mathcal{D}^{\perp}$.

Lemma 3.9. *On a quasi hemi- slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\widehat{\mathcal{M}}$, we have*

$$< [\mathcal{Y}, \mathcal{X}], \xi > -2\alpha < T\mathcal{Y}, \mathcal{X} >= 0$$

$$\begin{aligned} < \bar{\nabla}_{\mathcal{Y}}\mathcal{X}, \xi > -\alpha < T\mathcal{Y}, \mathcal{X} > \quad - \quad & \beta < \mathcal{Y}, \mathcal{X} > \\ & +\beta\upsilon(\mathcal{Y})\upsilon(\mathcal{X}) = 0 \end{aligned}$$

for all $\mathcal{Y}, \mathcal{X} \in \Gamma(\mathcal{D} \oplus \mathcal{D}_{\theta} \oplus \mathcal{D}^{\perp})$.

In the following section, we shall investigate warped product quasi hemi-slant submanifolds of a trans para Sasakian manifold.

4. Warped product quasi hemi-slant submanifolds of a trans para Sasakian manifold

Theorem 4.1. *If $\widehat{\mathcal{M}}$ is a trans para Sasakian manifold, then there do not exist proper warped product quai hemi-slant submanifolds $\widehat{\mathcal{N}} = N_{\theta}\times_{\delta}N_T$ such that N_{θ} is a proper slant submanifold, N_T is an invariant submanifold of $\widehat{\mathcal{M}}$ and ξ is tangent to $\widehat{\mathcal{N}}$.*

Proof. If $\widehat{\mathcal{N}} = N_{\theta}\times_{\delta}N_T$ is a proper warped product quai hemi-slant submanifolds of a trans para Sasakian manifold $\widehat{\mathcal{M}}$, then for any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_{\theta})$ and $\mathcal{U}, \mathcal{V} \in \Gamma(TN_T)$, we have

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{U} = \bar{\nabla}_{\mathcal{X}}\phi\mathcal{U} - \phi(\bar{\nabla}_{\mathcal{X}}\mathcal{U}) \tag{4.1}$$

$$\begin{aligned} \alpha < \mathcal{X}, \mathcal{U} > \xi \quad + \quad & \alpha\upsilon(\mathcal{U})\mathcal{X} \\ & +\beta < \mathcal{X}, \phi\mathcal{U} > \xi + \beta\upsilon(\mathcal{U})\phi\mathcal{X} \\ & = \sigma(\mathcal{X}, T\mathcal{U}) - r\sigma(\mathcal{X}, \mathcal{U}) - s\sigma(\mathcal{X}, \mathcal{U}) \end{aligned}$$

From the tangential and normal components of (4.1), respectively, we get

$$\alpha < \mathcal{X}, \mathcal{U} > +\beta < \mathcal{X}, \phi\mathcal{U} >= 0 \tag{4.2}$$

$$\alpha\upsilon(\mathcal{U})\mathcal{X} + \beta\upsilon(\mathcal{U})T\mathcal{X} = -r\sigma(\mathcal{X}, \mathcal{U}) \tag{4.3}$$

and

$$\beta\upsilon(\mathcal{U})N\mathcal{X} = -\sigma(\mathcal{X}, T\mathcal{U}) + s\sigma(\mathcal{X}, T\mathcal{U}) \tag{4.4}$$

Then apply, by interchanging roles of $\mathcal{U}$ and $\mathcal{X}$ in (4.1), we conclude

$$T\mathcal{X}log(\delta)\mathcal{U} = \Lambda_{N\mathcal{X}}\mathcal{U} + \mathcal{X}log(\delta)T\mathcal{U} + r\sigma(\mathcal{U}, \mathcal{X}) \tag{4.5}$$
$$-\alpha\upsilon(\mathcal{X})\mathcal{U} - \beta\upsilon(\mathcal{X})T\mathcal{U}$$

$$T\mathcal{X}log(\delta)\mathcal{U} = \Lambda_{N\mathcal{X}}\mathcal{U} + \mathcal{X}log(\delta)T\mathcal{U} + r\sigma(\mathcal{U}, \mathcal{X}) \tag{4.6}$$

and

$$\alpha < \mathcal{U}, \mathcal{X} >= \nabla^{\perp}_{\mathcal{U}}N\mathcal{X} + \sigma(\mathcal{U}, T\mathcal{X}) - s\sigma(\mathcal{U}, \mathcal{X})$$

From (4.6), we deduce

$$\begin{aligned} T\mathcal{X}log(\delta) < \mathcal{U}, \quad \mathcal{U} >&=< \Lambda_{N\mathcal{X}}\mathcal{U},\mathcal{U} > + < r\sigma(\mathcal{U}, \mathcal{X}),\mathcal{U} > \\ &-\alpha\upsilon(\mathcal{X}) < \mathcal{U},\mathcal{U} > \\ &=< \sigma(\mathcal{U},\mathcal{U}), N\mathcal{X} > + < r\sigma(\mathcal{U}, \mathcal{X}),\mathcal{U} > \\ &-\alpha\upsilon(\mathcal{X}) < \mathcal{U},\mathcal{U} > \\ &=< \sigma(\mathcal{U},\mathcal{U}), N\mathcal{X} > - < \sigma(\mathcal{X},\mathcal{U}), \phi\mathcal{U} > \\ &-\alpha\upsilon(\mathcal{X}) < \mathcal{U},\mathcal{U} > \\ &=< \sigma(\mathcal{U},\mathcal{U}), N\mathcal{X} > -\alpha\upsilon(\mathcal{X}) < \mathcal{U},\mathcal{U} > \end{aligned} \tag{4.7}$$

On the other hand, since the ambient space M is a trans para Sasakian manifold, by using (4.4) and (4.6), we get

$$N\mathcal{X} = s\sigma(\mathcal{X}, \xi) = 0 \tag{4.8}$$

$$T\mathcal{X}log(\delta) < \mathcal{U},\mathcal{U} > +\alpha\upsilon(\mathcal{X}) < \mathcal{U},\mathcal{U} >= 0 \tag{4.9}$$

here if $\upsilon(\mathcal{X}) = 0$, then $T\mathcal{X}log(\delta) < \mathcal{U},\mathcal{U} >= 0$, implies that $T\mathcal{X}log(\delta) = 0$. Since U is non zero vector field and $N_T \neq 0$. Thus $< N\mathcal{X}, N\mathcal{Y} >= sin^2\theta < \mathcal{X}, \mathcal{Y} >= 0$, which implies that the slant angle θ is either 0 or the warping function δ is constant on N_θ.

□

Theorem 4.2. *If $\widehat{\mathcal{M}}$ is a trans para Sasakian manifold, then there do not exist proper warped product quai hemi-slant submanifolds $\widehat{\mathcal{N}} = N_\theta \times_\delta N_\perp$ such that N_θ is a proper slant submanifold, $N_\perp$ is an invariant submanifold of M and ξ is tangent to $\widehat{\mathcal{N}}$.*

Proof. If $\widehat{\mathcal{N}} = N_\theta \times_\delta N_\perp$ is a proper warped product quai hemi-slant submanifolds of a trans para Sasakian manifold $\bar{\mathcal{M}}$ such that ξ is tangent to $\widehat{\mathcal{M}}$, then for any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_\theta)$ and $\mathcal{U}, \mathcal{V} \in \Gamma(TN_\perp)$, we have

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{U} = \bar{\nabla}_{\mathcal{X}}\phi\mathcal{U} - \phi(\bar{\nabla}_{\mathcal{X}}\mathcal{U})$$

$$\begin{aligned}\alpha < \mathcal{X},\mathcal{U} > \xi \quad + \quad & \alpha v(\mathcal{U})\mathcal{X} + \beta < \mathcal{X}, \phi\mathcal{U} > \xi \\ & + \beta v(\mathcal{U})\phi\mathcal{X} = -\Lambda_{N\mathcal{U}}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}} N\mathcal{U} \\ & - \mathcal{X}(log(\delta))N\mathcal{U} - \phi\sigma(\mathcal{X},\mathcal{U})\end{aligned} \tag{4.10}$$

Considering the tangential and normal components of (4.10), respectively, we get

$$\alpha v(\mathcal{U})\mathcal{X} + \beta v(\mathcal{U})T\mathcal{X} = -\Lambda_{N\mathcal{U}}\mathcal{X} - r\sigma(\mathcal{X},\mathcal{U}) \tag{4.11}$$

and

$$\begin{aligned}\alpha < \mathcal{X},\mathcal{U} > \xi \quad + \quad & \beta v(\mathcal{U})N\mathcal{X} = \nabla^{\perp}_{\mathcal{X}} N\mathcal{U} \\ & - \mathcal{X}(logf)N\mathcal{U} - s\sigma(\mathcal{X},\mathcal{U})\end{aligned} \tag{4.12}$$

By interchanging roles of $\mathcal{X}$ and $\mathcal{U}$ in (4.10), we reach

$$\begin{aligned}\alpha < \mathcal{U}, \mathcal{X} > \quad \xi \quad & + \alpha v(\mathcal{X})\mathcal{U} - \beta < \mathcal{U}, \phi\mathcal{X} > \xi \\ & + \beta v(\mathcal{X})\phi\mathcal{U} = T\mathcal{X}log(\delta)\mathcal{U} + \sigma(\mathcal{U}, T\mathcal{X}) \\ & - \Lambda_{N\mathcal{X}}\mathcal{U} + \nabla^{\perp}_{\mathcal{U}} N\mathcal{X} - \mathcal{X}log(\delta)N\mathcal{U} \\ & - r\sigma(\mathcal{U}, \mathcal{X}) - s\sigma(\mathcal{U}, \mathcal{X})\end{aligned} \tag{4.13}$$

From the tangential and normal components of (4.13), respectively, we have

$$\begin{aligned}T\mathcal{X}log(\delta)\mathcal{U} \quad = \quad & \Lambda_{N\mathcal{X}}\mathcal{U} + r\sigma(\mathcal{U}, \mathcal{X}) \\ & + \alpha v(\mathcal{X})\mathcal{U} + \beta v(\mathcal{X})T\mathcal{U}\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\alpha < \mathcal{U}, \mathcal{X} > \xi \quad + \quad & \beta v(\mathcal{X})N\mathcal{U} = \sigma(\mathcal{U}, T\mathcal{X}) \\ & + \nabla^{\perp}_{\mathcal{U}} N\mathcal{X} - \mathcal{X}log(\delta)N\mathcal{U} - s\sigma(\mathcal{U}, \mathcal{X})\end{aligned} \tag{4.15}$$

From (4.14), we deduce

$$\begin{aligned}< \Lambda_{N\mathcal{X}}\mathcal{U}, T\mathcal{Y} > \quad + \quad & < r\sigma(\mathcal{U}, \mathcal{X}), T\mathcal{Y} > \\ & + \beta v(\mathcal{X}) < T\mathcal{U}, T\mathcal{Y} >= 0\end{aligned} \tag{4.16}$$

Since the ambient space $\bar{\mathcal{M}}$ is a quai hemi-slant submanifolds of a trans para Sasakian manifold, ξ is tangent to $\mathcal{M}$ and by using (2.1), we obtain

$$\begin{aligned} < r\sigma(\mathcal{U},\mathcal{X}), T\mathcal{Y} > &= < \phi\sigma(\mathcal{U},\mathcal{X}), \phi\mathcal{Y} > \\ &= - < \sigma(\mathcal{U},\mathcal{Y}), \mathcal{X} > + \upsilon(\sigma(\mathcal{U},\mathcal{X}))\upsilon(Y) \\ &= 0 \end{aligned}$$

that is,

$$\begin{aligned} < r\sigma(\mathcal{U},\mathcal{X}), T\mathcal{Y} > &- \beta\upsilon(\mathcal{U})cos^2\theta < \mathcal{X},\mathcal{Y} > \\ &= < \sigma(\mathcal{U},T\mathcal{Y}), N\mathcal{X} > \\ &= 0 \end{aligned} \tag{4.17}$$

Thus we have

$$< \sigma(\mathcal{U},T\mathcal{Y}), \phi\mathcal{X} >= 0 \tag{4.18}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(TN_\theta)$. Moreover, making use of (4.11) and (4.18), we derive

$$\begin{aligned} \alpha \quad \upsilon(\mathcal{U}) & < \mathcal{X}, T\mathcal{Y} > + \beta\upsilon(\mathcal{U}) < T\mathcal{X}, T\mathcal{Y} > \\ &= - < \sigma(\mathcal{X},T\mathcal{Y}), N\mathcal{U} > - < r\sigma(\mathcal{X},\mathcal{U}), T\mathcal{Y} > \\ &= - < \sigma(\mathcal{X},T\mathcal{Y}), \phi\mathcal{U} > \end{aligned} \tag{4.19}$$

By using the Gauss-Weingarten formulas and considering N_θ is totally geodesic in $\widehat{\mathcal{M}}$, we arrive at

$$\begin{aligned} < \sigma(\mathcal{X},T\mathcal{Y}), \phi\mathcal{U} > &= < \bar{\nabla}_{T\mathcal{Y}}\mathcal{X}, \phi\mathcal{U} > \\ &= - < \phi(\bar{\nabla}_{T\mathcal{Y}}\mathcal{X}), \mathcal{U} > \\ &= - < \bar{\nabla}_{T\mathcal{Y}}\phi\mathcal{X} - (\bar{\nabla}_{T\mathcal{Y}}\phi)\mathcal{X}, \mathcal{U} > \\ &= - < \bar{\nabla}_{T\mathcal{Y}}T\mathcal{X}, \mathcal{U} > - < \bar{\nabla}_{T\mathcal{Y}}N\mathcal{X}, \mathcal{U} > \\ &+ < \alpha < T\mathcal{Y}, \mathcal{X} > \xi + \alpha\upsilon(\mathcal{X})T\mathcal{Y} - \beta < \phi T\mathcal{Y}, \mathcal{X} > \xi \\ &+ \upsilon(\mathcal{X})\phi T\mathcal{Y}, \mathcal{U} >=< \Lambda_{N\mathcal{X}}T\mathcal{Y}, \mathcal{U} > + \alpha < T\mathcal{Y}, \mathcal{X})\upsilon(\mathcal{U}) \\ &+ \alpha\upsilon(\mathcal{X}) < T\mathcal{Y}, \mathcal{U} > - \beta < \phi T\mathcal{Y}, \mathcal{X} > \upsilon(\mathcal{U}) \end{aligned}$$

$$\begin{aligned} \alpha\upsilon(\mathcal{U}) < T\mathcal{Y}, \mathcal{X} > &+ \beta\upsilon(\mathcal{U}) < T\mathcal{Y}, T\mathcal{X} > \\ &= < \sigma(\mathcal{X},T\mathcal{Y}), \phi\mathcal{U} > \end{aligned} \tag{4.20}$$

Thus from (4.19) and (4.20), we deduce

$$\begin{aligned} \alpha\upsilon(\mathcal{U}) < \mathcal{X}, T\mathcal{Y} > &+ \beta\upsilon(\mathcal{U}) < T\mathcal{X}, T\mathcal{Y} > \\ &- < \sigma(\mathcal{X},T\mathcal{Y}), \phi\mathcal{U} >= 0 \end{aligned} \tag{4.21}$$

Here, if $\upsilon(\mathcal{U}) = 0$, then by using (2.13) and (4.12), we have

$$\begin{aligned} \mathcal{X}log(\delta)N\mathcal{U} &= \upsilon(\nabla_{\mathcal{X}}\mathcal{U}) \\ &= - < \nabla_{\mathcal{X}}\xi, \mathcal{U} > \\ &= < \alpha\phi\mathcal{X} + \beta(\mathcal{X} - \upsilon(\mathcal{X})\xi), \mathcal{U} > \\ &= 0 \end{aligned}$$

This is impossible. Because U is a nonzero vector field and $N_{\perp} \neq 0$. Thus $< T\mathcal{X}, T\mathcal{Y} >= cos^2\theta < \mathcal{X}, \mathcal{Y} >$, which implies that the slant angle θ is either identically $\pi/2$ or the warping function δ is constant on N_θ . $\square$

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