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COMMON COINCIDENCE AND FIXED POINTS FOR SEVERAL FUNCTIONS IN b-METRIC SPACES

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Abstract. The purpose of this research paper is to show the existence and uniqueness of coincidence and fixed points for more then two functions on new contraction mapping and expansive mapping in b-metric spaces. Also we give several corollaries in such spaces. These results are extend many of the past results and will help to research scholar in their advance works.

Key Words: Coincidence point, fixed point, contraction mapping, expansive mapping, b-metric space.
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1. Introduction

Foundational result of fixed point theory is Banach Contraction Principle, is proposed by Banach[4] in 1922. Many researchers and scholars studied it very deeply and gave its detailed and generalized form in front of the Mathematics, Science and technical world. In this sequel, I.A. Bakhtin[3] introduced and defined the concept of b-metric space in 1989, as a generalization of metric space and proved the analogue of this fundamental result in b-metric space. While Stafen Czerwic[6] used it extensively in 1993. Since then till now many analogue results have been made in b-metric spaces. A lot of fixed point results were presented for one function, two functions and three functions with different contraction mappings by many authors(see,.e.g.,[1],[2],[5], [8-10]. A detailed

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study is being done on various types of contractions.

In this sequence, we are inspired to prove the existence and uniqueness of fixed point for three functions on contraction and expansive mapping in $b$metric spaces. Also we present an analogue result of Budi Nurwahyu[7] in b-metric spaces for three functions.

2. Preliminaries

Definition 2.1.[1]. Let $X(\neq \emptyset)$ be a set with a fixed real number $s \geq 1$, then a function d is called b-metric if function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:
(bm_1) $d(u, v) = 0$ iff $u = v$ for all $u, v \in X$.
(bm_2) $d(u, v) = d(v, u)$ for all $u, v \in X$.
(bm_3) $d(u, w) \leq s\Big[d(u, v) + d(v, w)\Big]$ *for all distinct* $u, v, w \in X$.
Then the pair (X, d) is called a b-metric space.

Every metric space is a b-metric space but converse is not necessary true. We validate this by some following examples.
Example 2.2. $X(\neq \Phi)$ be a set of natural numbers and define a function $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v)= \begin{cases} 0, \text{ if } u = v \\ c\mu, \text{ if } (u, v) \in \{3, 4\} \text{ and } u \neq v \\ \mu, \text{ otherwise.} \end{cases}$$

where $\mu > 0$ and $c > 2$, $\forall\ u, v \in X$.

Hence (X, d) is a b-metric space with coefficient $s = \frac{c}{2} > 1$.

But (X, d) is not a metric space, as

$$d(3, 4) = c\mu > 2\mu = d(3, 2) + d(2, 4).$$

Example 2.3. Let $X(\neq \Phi)$be a set of natural numbers and define a function $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v)= \begin{cases} 0, \text{ if } u = v \\ 5\mu, \text{ if } u = 1, v = 4 \\ 2\mu, \text{ if } (u, v) \in \{1, 2, 3\} \text{ and } u \neq v \\ \mu, \text{ otherwise.} \end{cases}$$

where $\mu > 0$ is a constant.

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{3} > 1$.

But (X, d) is not a metric space, as

$$d(1,4) = 5\mu > 3\mu = d(1,2) + d(2,4).$$

Definition 2.4.[7] Let X be a non-empty set and let f be a self map on X, then f has a fixed point at $u \in X$ if $fu = u$.

Definition 2.5.[7] Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let f be a self map onto X itself then f is called a contraction mapping, if it satisfies for $0 < \lambda < 1$

$$d(fu, fv) \leq \lambda d(u, v) \; for \; all \; u, v \in X.$$

Definition 2.6.[7] Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let f be a self map onto X itself then f is called a expansive mapping, if it satisfies $0 < \lambda < 1$

$$d(fu, fv) \geq \lambda d(u, v) \; for \; all \; u, v \in X.$$

Definition 2.7.[7] Let X be a non-empty set and let f and g be two self maps on to X itself, then f and g have a common coincidence point at $u \in X$ if $fu = gu = v$ and $v \in X$ is called a coincidence point of f and g.

Definition 2.8.[7] Let X be a non-empty set and let f and g be two self maps on to X itself, then $\{f, g\}$ is called weakly compatible, if $fu = gu$, then $gfu = fgu$ for all $u \in X$.

We now state our main results.

3. Main Result

Theorem 3.1. *Let (X, d) be a b-metric space with $s \geq 1$ and f, g, h be three functions defined onto X itself such that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. If (hX, d) is a complete b-metric space and suppose the following condition holds for all $x, y \in X$:*

$$d(fx, gy) \leq \alpha d(fx, hy) + \beta d(gy, hy) + \gamma d(hx, hy) + \eta d(fy, hy) \tag{3.1}$$

where $\alpha, \beta, \eta \in [0, s^{-1})$ and $\gamma > 0$ such that $\gamma + \beta + \eta < 1$, $s\alpha + \beta + \eta < 1$ and $\alpha + \beta + \gamma < 1$. Then f, g and h have common unique coincidence point in X.

Moreover, if $\{f, h\}$, $\{g, h\}$ be weakly compatible, then f, g and h have

a common unique fixed point.
Proof. Firstly we show the existence of fixed point of f, g and h.
Let $x_0 \in X$. Therefore $fx_0 \in f(X)$. Given that $f(X) \subseteq h(X)$ then there exists $x_1 \in X$ such that $fx_0 = hx_1$.

Since $x_1 \in X$, we have $gx_1 \in g(X)$. Given that $g(X) \subseteq h(X)$ then there exists $x_2 \in X$ such that $gx_1 = hx_2$.

Again $x_2 \in X$, we have $fx_2 \in f(X)$. Since given that $f(X) \subseteq h(X)$ then there exists $x_3 \in X$ such that $fx_2 = hx_3$.

Now again $x_3 \in X$, we have $gx_3 \in g(X)$. Since given that $g(X) \subseteq h(X)$ then there exists $x_4 \in X$ such that $gx_3 = hx_4$.

Therefore we can take $fx_n = hx_{n+1}$ and $gx_{n+1} = hx_{n+2}$ for all $n = 0, 1, 2, ...$

Choose a sequence $\{u_n\}$ such that $u_n = fx_n = hx_{n+1}$ and $u_{n+1} = gx_{n+1} = hx_{n+2}$ for all $n = 0, 1, 2, ...$

Now from inequality (3.1), we have

$$\begin{aligned} d(u_n, u_{n+1}) &= d(fx_n, gx_{n+1}) \\ &\leq \alpha d(fx_n, hx_{n+1}) + \beta d(gx_{n+1}, hx_{n+1}) + \gamma d(hx_n, hx_{n+1}) \\ &+ \eta d(fx_{n+1}, hx_{n+1}) \\ &= \alpha d(u_n, u_n) + \beta d(u_{n+1}, u_n) + \gamma d(u_{n-1}, u_n) + \eta d(u_{n+1}, u_n). \end{aligned}$$

Therefore

$$\begin{aligned} (1 - \beta - \eta) d(u_n, u_{n+1}) &\leq \gamma d(u_{n-1}, u_n). \\ d(u_n, u_{n+1}) &\leq \frac{\gamma}{1 - \beta - \eta} d(u_{n-1}, u_n). \\ &= \mu d(u_{n-1}, u_n) \end{aligned}$$

where $\mu = \frac{\gamma}{1-\beta-\eta} < 1$. Thus, we get

$$d(u_n, u_{n+1}) \leq \mu d(u_{n-1}, u_n).$$

Continuing the above process, we receive

$$d(u_n, u_{n+1}) \leq \mu^n d(u_0, u_1). \tag{3.2}$$

Letting $n \to \infty$, we obtain that

$$lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.3}$$

Now we claim that $\{u_n\}$ is a b-Cauchy sequence in X.

Let $m, n \in \mathbb{N}, m > n$. Setting $d_n = \mu^n d(x_0, x_1)$.
Then by b-inequality, we obtain

$$\begin{aligned}
d(u_n, u_m) &\leq s[d(u_n, u_{n+1}) + d(u_{n+1}, u_m)]\\
&\leq sd(u_n, u_{n+1}) + s^2[d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_m)]\\
&\leq sd(u_n, u_{n+1}) + s^2 d(u_{n+1}, u_{n+2}) + s^3[d(u_{n+2}, u_{n+3}) + d(u_{n+3}, u_m)]\\
&\leq s\mu^n d_0 + s^2\mu^{n+1} d_0 + s^3\mu^{n+2} d_0 + ... + s^{m-n}\mu^{m-1} d_0\\
&\leq s\mu^n\{1 + s\mu + s^2\mu^2 + ... + (s\mu)^{m-n-1}\} d_0\\
&\leq \frac{1}{1 - s\mu} s\mu^n d_0\\
&= \frac{s\mu^n}{1 - s\mu} d_0.
\end{aligned}$$

Hence,

$$d(u_n, u_m) \leq \frac{s\mu^n}{1 - s\mu} d_0. \tag{3.4}$$

Letting $n \to \infty$, we find

$$lim_{n\to\infty} d(u_n, u_m) = 0. \tag{3.5}$$

Thus $\{u_n\}$ is a b-Cauchy sequence in hX.
Since function hX is complete, there exists $u^* \in hX$ such that

$$lim_{n\to\infty} d(u_n, u^*) = 0. \tag{3.6}$$

Since $u^* \in hX$ then there exists $x^* \in X$ such that $u^* = hx^*$.
Now,

$$\begin{aligned}
d(fx^*, u_{n+1}) &= d(fx^*, gx_{n+1}) \\
&\leq \alpha d(fx^*, hx_{n+1}) + \beta d(gx_{n+1}, hx_{n+1}) + \gamma d(hx^*, hx_{n+1}) \\
&+ \eta d(fx_{n+1}, hx_{n+1}) \\
&= \alpha d(fx^*, u_n) + \beta d(u_{n+1}, u_n) + \gamma d(u^*, u_n) + \eta d(u_{n+1}, u_n) \\
&\leq \alpha s[d(fx^*, u_{n+1}) + d(u_{n+1}, u_n)] + (\beta + \eta) d(u_{n+1}, u_n) \\
&+ \gamma d(u^*, u_n).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1 - \alpha s) d(fx^*, u_{n+1}) &\leq (\alpha s + \beta + \eta) d(u_{n+1}, u_n) + \gamma d(u^*, u_n). \\
d(fx^*, u_{n+1}) &\leq \frac{\alpha s + \beta + \eta}{1 - \alpha s} d(u_{n+1}, u_n) + \gamma d(u^*, u_n).
\end{aligned}$$

Letting $n \to \infty$ and using (3.3), (3.6), $1 - \alpha s > 0$, we get

$$lim_{n \to \infty} d(fx^*, u_{n+1}) = 0. \tag{3.7}$$

Now, we show that $\{f, h\}$ has a coincidence point.

$$\begin{aligned}
d(fx^*, hx^*) &\leq s[d(fx^*, u_{n+1}) + d(u_{n+1}, hx^*)] \\
&= s[d(fx^*, u_{n+1}) + d(u_{n+1}, u^*)].
\end{aligned}$$

Letting $n \to \infty$ and using (3.6), (3.7), we get

$$d(fx^*, hx^*) = 0.$$

Therefore, we get

$$fx^* = hx^*. \tag{3.8}$$

Thus $\{f, h\}$ has a coincidence point.

Claim that $d(gx^*, u_n) \to 0$ for $n \to \infty$.

Now from inequality (3.1)), we have

$$\begin{aligned}
d(gx^*, u_n) &= d(gx^*, fx_n) = d(fx_n, gx^*) \\
&\leq \alpha d(fx_n, hx^*) + \beta d(gx^*, hx^*) + \gamma d(hx_n, hx^*) + \eta d(fx^*, hx^*) \\
&= \alpha d(u_n, u^*) + \beta d(gx^*, u^*) + \gamma d(u_{n-1}, u^*) + \eta d(fx^*, hx^*) \\
&\leq \alpha d(u_n, u^*) + \beta s[d(gx^*, u_n) + d(u_n, u^*)] + \gamma d(u_{n-1}, u^*) \\
&+ \eta d(fx^*, hx^*).
\end{aligned}$$

Therefore,

$$(1 - \beta s) d(gx^*, u_n) \leq \alpha d(u_n, u^*) + \beta s d(u_n, u^*) + \gamma d(u_{n-1}, u^*) + \eta d(fx^*, hx^*).$$

Since $1 - \beta s > 0$, letting $n \to \infty$ and using (3.6),(3.8), we get

$$lim_{n \to \infty} d(gx^*, u_n) = o. \tag{3.9}$$

Again by b-inequality

$$\begin{aligned}
d(gx^*, hx^*) &\leq s[d(gx^*, u_n) + d(u_n, hx^*)] \\
&\leq s[d(gx^*, u_n) + d(u_n, u^*)].
\end{aligned}$$

Letting $n \to \infty$ and using (3.6), (3.9), we have

$$d(gx^*, hx^*) = 0. \tag{3.10}$$

Now from (3.8) and (3.10), we get
$fx^* = hx^* = u^*$ and $gx^* = hx^* = u^*$.
Thus,

$$fx^* = gx^* = hx^* = u^*. \tag{3.11}$$

Hence functions f, g and h have common coincidence point u^* for $x^* \in X$.

Uniqueness of coincidence point. If possible, suppose that v^* is another coincidence point of functions f, g and h such that

$$fx^* = gx^* = hx^* = v^*.$$

From inequality (3.1), we have

$$\begin{aligned}
d(u^*, v^*) &= d(fx^*, gx^*) \\
&\leq \alpha d(fx^*, hx^*) + \beta d(gx^*, hx^*) + \gamma d(hx^*, hx^*) + \eta d(fx^*, hx^*) \\
&= \alpha d(u^*, v^*) + \beta d(u^*, v^*) + \eta d(u^*, v^*).
\end{aligned}$$

Therefore,

$$(1-\alpha-\beta-\eta)d(u^*,v^*) \le 0.$$

Since $1-\alpha-\beta-\eta > 0$. Thus $d(u^*,v^*)=0$.
Hence,

$$u^* = v^*.$$

Now $fx^* = hx^*$and $gx^* = hx^*$. Given $\{f,h\}$ weakly compatible, then we have $hfx^* = fhx^*$.

Therefore

$$hu^* = hfx^* = fhx^* = fu^*. \tag{3.12}$$

Again since $\{g,h\}$ weakly compatible, then we have $hgx^* = ghx^*$.

Therefore

$$hu^* = hgx^* = ghx^* = gu^*. \tag{3.13}$$

By using inequality (3.1), we have

$$\begin{aligned}
d(u^*,hu^*) &= d(fx^*,gu^*) \\
&\le \alpha d(fx^*,hu^*) + \beta d(gu^*,hu^*) + \gamma d(hx^*,hu^*) + \eta d(fu^*,hu^*) \\
&= \alpha d(u^*,hu^*) + \beta d(gu^*,gu^*) + \gamma d(u^*,hu^*) + \eta d(fu^*,fu^*).
\end{aligned}$$

Therefore

$$(1-\alpha-\gamma)d(u^*,hu^*) \le 0.$$

Since $1-\alpha-\gamma > 0$, then we get $d(u^*,hu^*)=0$,
implies that

$$hu^* = u^*.$$

Thus, we have

$$fu^* = gu^* = hu^* = u^*.$$

Hence functions f, g and h have common fixed point u^*.

Now we show the uniqueness of fixed point.

If possible, suppose that w^* is another fixed point of functions f, g and h such that $fw^* = gw^* = hw^* = w^*$.
From inequality (3.1), we have

$$\begin{aligned} d(u^*, w^*) &= d(fu^*, gw^*) \\ &\leq \alpha d(fu^*, hw^*) + \beta d(gw^*, hw^*) + \gamma d(hu^*, hw^*) + \eta d(fw^*, hw^*) \\ &= \alpha d(u^*, w^*) + \beta d(w^*, w^*) + \gamma d(u^*, w^*) + \eta d(w^*, w^*). \end{aligned}$$

Therefore,

$$(1 - \alpha - \gamma) d(u^*, w^*) \leq 0.$$

Since $1 - \alpha - \gamma > 0$. Thus $d(u^*, w^*) = 0$.
Hence,

$$u^* = w^*$$

.
This completes the proof. □

Corollary 3.1. *Let (X, d) be a b-metric space with $s \geq 1$ and f, g be two functions defined onto X itself such that the following condition holds:*

$$d(fx, gy) \leq \alpha d(fx, y) + \beta d(gy, y) + \gamma d(x, y) + \eta d(fy, y)$$

where $\alpha, \beta, \eta \in [0, s^{-1})$ and $\gamma > 0$ such that $\gamma+\beta+\eta < 1$, $s\alpha+\beta+\eta < 1$ and $\alpha + \beta + \gamma < 1$. Then f, g and h have common unique coincidence point in X.

If f, g are weakly compatible then f and g have common unique fixed point in X .

Proof. In *Theorem*3.1 substitute $h = I$, where I is an identity function on X.

Corollary 3.2. *Let (X, d) be a b-metric space with $s \geq 1$ and f, g, h be three functions defined onto X itself such that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. If (hX, d) is a complete b-metric space and suppose the following condition holds for all $x, y \in X$:*

$$d(fx, gy) \leq \alpha d(fx, hy) + \gamma d(hx, hy) + \eta d(fy, hy)$$

where $\alpha, \eta \in [0, s^{-1})$ and $\gamma > 0$ such that $\alpha + \gamma + \eta < 1$ then f, g and h have common unique fixed point in X.

Theorem 3.2. *Let (X, d) be a b-metric space with $s \geq 1$ and f, g, h be three functions defined onto X itself such that $f(X) \subseteq h(X)$ and*

$g(X) \subseteq h(X)$. If (hX, d) is a complete b-metric space and suppose the following condition holds for all x, $y \in X$:

$$(3.14) \quad d(hx, hy) \geq \alpha d(fx, hy) + \beta d(gx, gy) + \gamma d(fx, hx) + \eta d(fy, hy)$$

where $\alpha, \beta, \eta > 1$ and $0 < \gamma < 1$ such that $s(1-\gamma) < \beta + \eta$ then f, g and h have common unique coincidence point in X.

Moreover, if $\{f, h\}, \{g, h\}$ weakly compatible then f, g and h have common unique fixed point in X.
Proof. Firstly we show the existence of fixed point of f, g and h.

Let $x_0 \in X$. Therefore $fx_0 \in f(X)$. Given that $f(X) \subseteq h(X)$ then there exists $x_1 \in X$such that $fx_0 = hx_1$.

Since $x_1 \in X$, we have $gx_2 \in g(X)$. Given that $g(X) \subseteq h(X)$ then there exists $x_2 \in X$ such that $gx_1 = hx_2$.

Again $x_2 \in X$, we have $fx_2 \in f(X)$. Since given that $f(X) \subseteq h(X)$ then there exists $x_3 \in X$ such that $fx_2 = hx_3$.

Again $x_3 \in X$, we have $gx_3 \in g(X)$. Since given that $g(X) \subseteq h(X)$ then there exists $x_4 \in X$ such that $gx_3 = hx_4$.

Therefore we can take $fx_n = hx_{n+1}$ and $gx_{n+1} = hx_{n+2}$ for all $n = 0, 1, 2, ...$

Choose a sequence $\{u_n\}$ such that $u_n = fx_n = hx_{n+1}$ and $u_{n+1} = gx_{n+1} = hx_{n+2}$ for all $n = 0, 1, 2, ...$

Now, using inequality (3.14), we have

$$\begin{aligned} d(u_{n-1},u_n) = d(hx_n,hx_{n+1}) &\geq \alpha d(fx_n,hx_{n+1}) + \beta d(gx_n,gx_{n+1}) \\ &+ \gamma d(fx_n,hx_n) + \eta d(fx_{n+1},hx_{n+1}) \\ &= \alpha d(u_n,u_n) + \beta d(u_n,u_{n+1}) + \gamma d(u_n,u_{n-1}) \\ &+ \eta d(u_{n+1},u_n). \end{aligned}$$

$$Therefore,\ (1-\gamma)d(u_{n-1},u_n) \geq (\beta+\eta)d(u_n,u_{n+1}).$$

$$d(u_n,u_{n+1}) \leq \frac{1-\gamma}{\beta+\eta}d(u_{n-1},u_n).$$

$$d(u_n,u_{n+1}) \leq \lambda d(u_{n-1},u_n)\ where,\ \lambda = \frac{1-\gamma}{\beta+\eta} < 1.$$

Thus, we get

$$d(u_n,u_{n+1}) \leq \lambda d(u_{n-1},u_n)\ \forall\, n = 0,1,2,...$$

Continuing the above process, then we have

$$d(u_n,u_{n+1}) \leq \lambda^n d(u_0,u_1)\ \forall\, n = 0,1,2,...$$

Letting $n \to \infty$, we obtain that

$$lim_{n\to\infty} d(u_n,u_{n+1}) = 0. \tag{3.15}$$

Now we claim that $\{u_n\}$ is a b-Cauchy sequence in X.

Let $m,n \in \mathbb{N}, m > n$. Setting $d_n = \mu^n d(x_0,x_1)$.
Then by b-inequality, we obtain

$$\begin{aligned} d(u_n,u_m) &\leq s[d(u_n,u_{n+1}) + d(u_{n+1},u_m)] \\ &\leq sd(u_n,u_{n+1}) + s^2[d(u_{n+1},u_{n+2}) + d(u_{n+2},u_m)] \\ &\leq sd(u_n,u_{n+1}) + s^2 d(u_{n+1},u_{n+2}) + s^3[d(u_{n+2},u_{n+3}) + d(u_{n+3},u_m)] \\ &\leq s\mu^n d_0 + s^2\mu^{n+1}d_0 + s^3\mu^{n+2}d_0 + ... + s^{m-n}\mu^{m-1}d_0 \\ &\leq s\mu^n(1 + s\mu + s^2\mu^2 + ... + (s\mu)^{m-n-1})d_0 \\ &\leq \frac{1}{1-s\mu}s\mu^n d_0 \\ &= \frac{s\mu^n}{1-s\mu}d_0. \end{aligned}$$

Hence,

$$d(u_n,u_m) \leq \frac{s\mu^n}{1-s\mu}d_0. \tag{3.16}$$

Letting $n \to \infty$, we find

$$lim_{n\to\infty} d(u_n, u_m) = 0. \tag{3.17}$$

Thus $\{u_n\}$ is a b-Cauchy sequence in hX.
Since function hX is complete, there exists $u^* \in hX$ such that

$$lim_{n\to\infty} d(u_n, u^*) = 0. \tag{3.18}$$

Since $u^* \in hX$ then there exists $x^* \in X$ such that $u^* = hx^*$.

Now we show that functions f, g and h have a coincidence point in X.

Here we show that $d(fx^*, u_n) \to 0$

$$\begin{aligned} d(u^*, u_n) &= d(hx^*, u_n) = d(hx^*, hx_{n+1}) \\ &\geq \alpha d(fx^*, hx_{n+1}) + \beta d(gx^*, gx_{n+1}) + \gamma d(fx^*, hx^*) + \eta d(fx_{n+1}, hx_{n+1}) \\ &\geq \alpha d(fx^*, u_n) + \beta d(gx^*, u_{n+1}) + \gamma d(fx^*, u^*) + \eta d(u_{n+1}, u_n) \\ &\geq \alpha d(fx^*, u_n) + \eta d(u_{n+1}, u_n). \end{aligned}$$

Therefore

$$\alpha d(fx^*, u_n) \leq d(u^*, u_n) - \eta d(u_{n+1}, u_n).$$

$$d(fx^*, u^*) \leq s[d(fx^*, u_n) + d(u_n, u^*)].$$

Letting $n \to \infty$ and using, we get

$$d(fx^*, u^*) = 0.$$

Therefore,

$$fx^* = u^*.$$

Thus,

$$fx^* = hx^* = u^*. \tag{3.19}$$

Hence, $\{f, h\}$ has a coincidence point.
Again now

$$\begin{aligned} d(u^*, u_{n-1}) &= d(hx^*, hx_n) \\ &\geq \alpha d(fx^*, hx_n) + \beta d(gx^*, gx_n) + \gamma d(fx^*, hx^*) + \eta d(fx_n, hx_n) \\ &\geq \alpha d(u^*, u_{n-1}) + \beta d(gx^*, u_n) + \gamma d(u^*, u^*) + \eta d(u_n, u_{n-1}) \\ &\geq \alpha d(u^*, u_{n-1}) + \beta d(gx^*, u_n) + \eta d(u_n, u_{n-1}). \end{aligned}$$

Therefore,

$$d(gx^*, u^*) \leq (1 - \alpha) d(u^*, u_{n-1}) - \eta d(u_n, u_{n-1}).$$

Letting $n \to \infty$, we get, $d(gx^*, u^*) = 0$. Therefore,

$$gx^* = u^*.$$

Thus,

$$gx^* = hx^* = u^*. \tag{3.20}$$

Hence, $\{f, h\}$ has a coincidence point.

Now from (3.19) and (3.20), we get $fx^* = hx^* = u^*$
and $gx^* = hx^* = u^*$.
Thus,

$$fx^* = gx^* = hx^* = u^*.$$

Hence functions f, g and h have common coincidence point u^* for $x^* \in X$.
Uniqueness of coincidence point If possible, suppose that v^* is another coincidence point of functions f, g, h then, we have

$$\begin{aligned} d(u^*, v^*) &= d(fx^*, gx^*) \\ &\geq \alpha d(fx^*, hx^*) + \beta d(gx^*, gx^*) + \gamma d(fx^*, hx^*) + \eta d(fx^*, hx^*) \\ &= (\alpha + \gamma + \eta) d(u^*, v^*). \end{aligned}$$

Therefore,

$$(\alpha + \gamma + \eta - 1) d(u^*, v^*) \leq 0.$$

Since $\alpha + \gamma + \eta - 1 > 0$. Thus $d(u^*, v^*) = 0$.
Hence,

$$u^* = v^*.$$

Now $fx^* = hx^*$ and $gx^* = hx^*$ and since $\{f, h\}$ weakly compatible, then we have $hfx^* = fhx^*$. Therefore

$$hu^* = hfx^* = fhx^* = fu^*.$$

Again since $\{g, h\}$ weakly compatible, then we have $hgx^* = ghx^*$.
Therefore,

$$hu^* = hgx^* = ghx^* = gu^*.$$

By using (3.14), we have

$$\begin{aligned} (u^*, hu^*) &= d(hx^*, hu^*) \\ &\geq \alpha d(fx^*, hu^*) + \beta d(gx^*, gu^*) + \gamma d(fx^*, hx^*) + \eta d(u^*, hu^*) \\ &= \alpha d(u^*, hu^*) + \beta d(u^*, hu^*) + \gamma d(u^*, u^*) + \eta d(u^*, hu^*). \end{aligned}$$

Therefore, $(\alpha+\beta+\eta-1)d(u^*, hu^*) \leq 0$.

Since $\alpha+\beta+\eta > 1$, then we get $d(u^*, hu^*) = 0$, implies that $hu^* = u^*$.

Thus, we have

$$fu^* = gu^* = hu^* = u^*.$$

Hence, functions f, g and h have common fixed point u^*.

Uniqueness of fixed point If possible, suppose that w^* is another fixed point of functions f, g and h such that $fw^* = gw^* = hw^* = w^*$. From (3.14), we have

$$\begin{aligned} d(u^*, w^*) &= d(hu^*, hw^*) \\ &\geq \alpha d(fu^*, hw^*) + \beta d(gu^*, gw^*) + \gamma d(fu^*, hu*) + \eta d(fw^*, hw^*) \\ &= \alpha d(u^*, w^*) + \beta d(u^*, w^*) + \gamma d(u^*, u^*) + \eta d(u^*, w^*). \end{aligned}$$

Therefore,

$$(\alpha+\beta+\eta-1))d(u^*, w^*) \leq 0.$$

Since $\alpha+\beta+\eta > 1$. Thus $d(u^*, w^*) = 0$.
Hence,

$$u^* = w^*.$$

This completes the proof. □

Corollary 3.3. *Let (X, d) be a b-metric space with $s \geq 1$ and f, h be two functions defined onto X itself and satisfies the following condition for all $x, y \in X$:*

$$d(hx, hy) \geq \alpha d(fx, hy) + \beta d(x, y) + \gamma d(fx, hx) + \eta d(fy, hy)$$

where $\alpha, \beta, \eta > 1$ and $0 < \gamma < 1$ such that $s(1-\gamma) < \beta + \eta$ and $\{f, h\}$ weakly compatible then f and h have common unique fixed point in X.

Proof. In *Theorem* 3.2 substitute $h = I$, where I is an identity function on X. □

4. Conclusion

In this paper, we proposed and proved some fixed point results in b-metric spaces for three functions on contraction mapping and expansive mapping. We extended the result of Budi Nurwahyu[7] on contraction mapping and expansive mapping in such spaces.

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